HEIGHTS OF AUTOMORPHIC FORMS AND MOTIVES

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ABSTRACT. The "height" of a cohomological automorphic form is, roughly speaking, the L^2 norm of an integrally normalized representative for the cohomology class. We conjecture this notion is closely related to the the Kato-Koshikawa arithmetic height of the associated motive, and examine how this conjecture is related to the Bloch-Kato conjecture for the adjoint L-function.

CONTENTS

1. INTRODUCTION: ARITHMETIC HEIGHT AND AUTOMORPHIC HEIGHT

Let π be a cohomological automorphic cusp form for some group G. This informally means a cohomology class on the associated locally symmetric space that "vanishes on the boundary," and is an eigenvector for Hecke operators. See §2.3 for a precise definition.

We shall attach an "automorphic height" to π. It is given by the L^2 norm of an integrally normalized cohomology class associated to π, taken in the minimal cohomological degree in which it occurs. This definition applies uniformly across groups: it is valid whether the associated locally symmetric space be a finite set of points, a Shimura variety, or a hyperbolic 3-manifold.

On the other hand, assuming that one may find a motive associated to π, as expected, Kato has attached an "arithmetic height" to this motive, in a fashion that closely parallels the definition of the Faltings height. The main conjecture we want to discuss is:

> *Height conjecture:* arithmetic and automorphic heights agree up to an L-value:

$$\frac{\text{automorphic height}}{\text{arithmetic height}} \sim L(1, \mathrm{Ad}\pi). \tag{1.1}$$

where "$\sim$" means that the equality holds up to a factor that depends on behavior at bad places and certain types of congruences (between π and Eisenstein forms, and congruences that change the level). From the analytic point of view L-functions at the edge of their critical strip behave in a very mild fashion – not too big, not too small – and so we should think of (1.1) as saying that the sizes of automorphic and arithmetic heights are comparable.

The conjecture above was introduced in [23]. More informally, and in the special case of Bianchi manifolds, this conjecture was also proposed in the earlier paper [1] (see remarks after Theorem 1.3), and also related to ideas appearing in [8]. In the context of the modular curve, similar ideas already appear in [16] to bound an arithmetic height by automorphic invariants, see also references cited below.

Although the word "height" occurs on both numerator and denominator of (1.1), a moment's thought with the definitions show them to be so different that there is no real reason to expect a relationship. *The goal of the present document is to make* (1.1) *somewhat plausible, by relating it to known conjectures.*

The relevant conjectures are the Bloch-Kato conjecture for the adjoint L-function, which relates the value of this L-function to the "volume" of an associated motivic cohomology group and, on the other hand, the conjectures of [24, 11, 19] asserting, informally speaking, that this same motivic cohomology group acts – by *degree-shifiting* endomorphisms – on the cohomology of the locally symmetric space in which π occurs.

We shall sketch a proof of the following assertion, referring to §4.3 for the precise version, including normalization of metrics.

> (*) *Main claim:* Fix a "good" prime p, as in §4.1. Assuming all expected properties of the "motivic" action on the cohomology of the arithmetic group, in particular compatibility between "real" and "p-adic" avatars of that action (see §4.2 for exact formulation), as well as the Bloch-Kato conjecture for the adjoint L-function, the two sides of (1.1) differ by $(2\pi)^r\sqrt{u}$ for $u \in \mathbf{Q}^\times$ a p-unit, and $r \in \mathbb{Z}$.

The definition of "good" prime is rigged to avoid the various subtleties that may occur – the more serious being congruences with Eisenstein series and level-lowering or level-raising congruences. The former lead to ambiguities in the choice of integral structure necessary to define height; the latter are known to affect the L^2-norms of integrally normalized representatives, as clearly illustrated by the work [18]. Of course, for a height to be usable for Diophantine applications, one needs control at all primes: it would be desirable to have a statement similar to the above one, up to an element of bounded p-valuation. The study of such "bad" primes is therefore a crucial issue.

We have phrased (*) in the form that seem simplest, a direct implication, but in fact the argument shows that the the height conjecture, Bloch-Kato conjecture, and "motivic" action conjecture are all related – given information about any two of these, we can hope to extract information about the third.

1.1. **An apology.**

It will be clear to this reader that this paper can only be considered a first step. It has an undeniably speculative nature, because, in order to set it up in any generality, we are forced to rely on many standard conjectures.

In many cases, at least the main conjecture of the paper could be given a very concrete form – e.g., if we start with a motive that is known to be modular, and indeed some of the motivating computations came from the study of Bianchi 3-folds [1]. So, despite the squishy ground of the current paper, we anticipate that there will be many specific settings where one can obtain much less conditional results.

To avoid getting completely bogged down in technicalities and notation I have freely made various simplifying assumptions (e.g. the coefficient field for all cohomology classes and motives that occur is $\mathbf{Q}$) to make life easier.

1.2. **About the proof.** Philosophically the idea of the proof is as follows (see also [23, §6] for a more conversational version):

- Heights arise from metrized line bundles; the line bundle corresponding to the arithmetic height arises from spaces of modular forms. These spaces are, in favorable situations, free over a Hecke algebra, and their determinant gives a line bundle, so a notion of height for $\mathbf{Q}$-points of the Hecke algebra.
- The motivic conjecture of [11, 24, 19] identifies this line bundle (up to some powers) with the "determinant of motivic cohomology," and the Bloch-Kato conjecture identifies this "determinant of motivic cohomology" line with a determinant of a certain piece of the Hodge filtration; the height associated to that latter line is Kato's height.

See §4.8 for an expanded sketch along these lines. In practice we implement the idea in a different way in §4, studying how Poincaré duality interacts with the derived deformation ring.

This outline may be more familiar in different terms. Speaking somewhat informally, the Bloch–Kato conjecture predicts that

$$L(1, \mathrm{Ad}, \pi) \sim mR\Omega,$$

where m is a "congruence number," Ω is a Hodge-theoretic period, and R is a regulator – the volume of an associated motivic cohomology group. As we shall see, Ω^{-1} is essentially the height of π defined by Kato. Therefore, the height conjecture then amounts to the statement that the automorphic height equals mR, up to fudge factors.

Computations of a similar nature, although with a different context, have been carried about by Tilouine and Urban [21]; their work has substantial overlap with the computations of §4. They have independently formulated and proved (4.27) in the case $\delta = 1$ and use it for related purposes.

1.3. **What's the point?** Given that our argument says that the height conjecture would follow from existing conjectures, one may ask what the point is of formulating it separately at all. There are several reasons why I think it is of some interest:

(a) The original impetus for this paper was Kato's beautiful article [13]. More precisely I was very struck that some of the formulas of [13] look like formulas in the apparently unrelated context of [19].

(b) Secondly, in the paper [1], it was proposed (in the language of the current paper) that the automorphic height of forms that appear on arithmetic hyperbolic 3-manifolds should be "small" in order to account for the experimentally observed phenomenon of torsion growth.

The height conjecture would explain this, or at least why it should typically be true – which is already not clear automorphically – by relating it to Diophantine heights. (Of course, to do this, one first needs to refine the conjecture to bound the fudge factor implicit in $\sim$.)

(c) In the reverse direction to (b): a more precise form of the conjecture gives rise to an *a priori* bound for arithmetic height in terms of the level of π. Such an *a* *priori* bound is quite surprising from the Diophantine point of view and has been fruitfully exploited in the case of the modular curve – see, for example, [17, 25].

1.4. **Notation.** We use the following notation throughout this paper:

For a field k, a k-line means a one-dimensional vector space over k. If L is a k-line we denote by L^m the mth tensor power of L for $m > 0$ and, for $m < 0$, we take the $-m$th tensor power of the dual line.

For a vector space S over a field k, the symbol

$$[S] = \bigwedge^{\dim S} S$$

is the determinant of S, that is to say, the top exterior power; it is a k-line.

Given a complex of vector spaces $S^0 \to S^1 \to \dots$ we denote by $[S^*]$ the determinant of cohomology, which is the alternating product of the $[S^i]$ or equivalently (up to a canonical isomorphism) the alternating product of the cohomology determinants $[H^i]$.

A "metric" on a $\mathbf{Q}$-vector space W will be, by definition, a positive definite inner product on $W \otimes \mathbf{R}$.

For A a module over a Dedekind ring, tf as subscript means the torsion free quotient, and tors means the torsion submodule, so that $A/A_{\mathrm{tors}} \simeq A_{\mathrm{tf}}$.

The symbol $\vee$ and $*$ will both be used to denote duality, in various contexts, for example if X is a k-vector space then $X^\vee$ or X^* can both be used for the dual space.

For G a split reductive group over $\mathbf{Q}$, we will denote by $\check{G}$ the Langlands dual group, considered as a split Chevalley group over $\mathbf{Z}$. We denote in particular by $\check{\mathfrak{g}}_{\mathbf{Z}}$ the Lie algebra of this split Chevalley group $\check{G}$ over $\mathbf{Z}$, and will use similar notation with $\mathbf{Z}$ replaced by other rings.

To avoid difficulties of square roots it is useful to instead use the *extended dual group* discussed in [6, Proposition 5.3.3]: we take the quotient of $\check{G} \times \mathbb{G}_m$ by the element $(\Sigma(-1), -1)$ with Σ the sum of positive roots. For the purpose of this paper we will write $\check{G}$ and we leave the straightforward task of altering the statements to use the extended dual group to the reader.

Acknowledgements: I would like to thank Shekhar Khare for many interesting conversations about the derived numerical criterion (4.27) and related topics. I also thank Frank Calegari for several insightful comments.

In the case $\delta = 1$, the computations that appear here are closely related to computations appearing in the work of Tilouine and Urban in [21] (our work was independent of theirs, and the emphasis of the paper is quite different).

These notes are based on talks given at the University of Chicago in 2015 and in Montréal in 2018. I think the audiences of those seminars for their helpful comments.

As noted above, the ideas here arose from collaborations over many years, in particular, with Bergeron, Calegari, Galatius, Prasanna, and Sengun. I thank them again.

As I was editing this paper I was struck by how deeply the ideas of Kato pervade every section. It has been a pleasure to learn about them and work with them.

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2. AUTOMORPHIC HEIGHT

2.1. Let us start with a tempered, cuspidal, cohomological automorphic form π for a split semisimple[1] group $\mathbf{G}$; thus π is realized in the cohomology of some locally symmetric space Y_K, corresponding to level structure K. For simplicity we suppose that the Hecke eigenvalues of π lie in $\mathbf{Q}$ and that π is the only element of its L-packet contributing cohomology at level K (see [19, §1.1] for a slightly more general setup).

Write $H^q(Y_K, \mathbf{Q})_\pi$ for those rational cohomology classes whose Hecke eigenvalues coincide with those of π. Then Poincaré duality on $H^q(Y_K, \mathbf{Q})$ induces a perfect pairing on the π-isotypical component $H^q(Y_K, \mathbf{Q})_\pi$.

Now π contributes to a range of cohomological degrees. The lowest degree q satisfies

$$2q + \delta = \dim Y_K, \tag{2.1}$$

with δ the difference between the real rank of $G(\mathbf{R})$ and the real rank of its maximal compact subgroup, and the largest degree is $q + \delta$. The integers q, δ are called q_0, ℓ_0 in the book [5].

A choice of invariant metric on the global symmetric space for $\mathbf{G}(\mathbf{R})$ gives rise to a Riemanian metric on Y_K. We assume such a choice has been made. The height depends on this, in an easily computable way.

2.2. **Automorphic height.** We follow [23] verbatim here. Write $H^q(Y_K, \mathbf{R})_\pi$ for those real cohomology classes whose Hecke eigenvalues coincide with those of π. The identification

$$H^q(Y_K, \mathbf{R})_\pi \simeq \text{harmonic } q\text{-forms on } Y_K, \text{ transforming according to } \pi \tag{2.2}$$

together with the Riemannian structure on Y_K, endows $H^q(Y_K, \mathbf{R})_\pi$ with the structure of a finite-dimensional inner product space (since we can take inner product of two harmonic forms, using the Riemannian metric). Let $h_q = \dim H^q(Y_K, \mathbf{R})_\pi$ be the dimension of this space; note that in our computations (§4) we will actually assume $h_q = 1$ to simplify, but we do not impose that here.

We put

$$H^q(Y_K, \mathbf{Z})_\pi := \text{integral classes inside } H^q(Y_K, \mathbf{R})_\pi.$$

i.e. the intersection of $H^q(Y_K, \mathbf{R})_\pi$ with the image of $H^q(Y_K, \mathbf{Z}) \to H^q(Y_K, \mathbf{R})$, or, said differently, classes in $H^q(Y_K, \mathbf{R})_\pi$ all of whose periods over singular cycles of dimension q are integral.

We now define $\mathrm{height}(\pi) \in \mathbf{R}_{>0}$, which we call "automorphic height," as follows:

$$\begin{aligned} \mathrm{height}(\pi)^{h_q} &:= \mathrm{volume}\left(H^q(Y_K, \mathbf{Z})_\pi\right)^2, && (2.3)\\ &= \det\langle e_i, e_j\rangle && (2.4) \end{aligned}$$

where $e_1, \ldots, e_{h_q}$ is a basis for the free $\mathbf{Z}$ module $H^q(Y_K, \mathbf{Z})_\pi$.

For example, when $h_q = 1$, we can choose a harmonic differential q-form ω generating the one-dimensional real vector space $H^q(Y_K, \mathbf{R})_\pi$. It has a lattice of periods: the collection of integrals of ω over q-cycles is a lattice in $\mathbf{R}$, and we scale ω so that this lattice is $\mathbf{Z}$. Then

$$\mathrm{height}(\pi) = \langle \omega, \omega \rangle.$$

[1]It is technically simpler to make $\mathbf{G}$ semisimple, but at various points we will anyway use $\mathbf{G} = \mathrm{GL}_n$ as an illustrative example.

For example, if Y_K is a 3-manifold, we have $H^1(Y_K) \simeq H_2(Y_K)$, and $\mathrm{height}(\pi)$ is in fact closely related to the minimal Thurston norm of an element of $H_2(Y_K, \mathbf{Z})$ that transforms according to π – see [1, §4] for this and further discussion.

2.2.1. *Example: elliptic curves over* $\mathbf{Q}$. We will give a familiar example, just to explain how the definition unfolds in a well-known case. (For an even simpler example, the reader can take the case of a definite quaternion algebra over $\mathbf{Q}$ when Y_K is a finite set of points.) Say π is the automorphic form on $\mathrm{PGL}(2)$ over $\mathbf{Q}$ associated to an elliptic curve E.

Take a modular parameterization $\phi : X_0(N) \to E$ of the associated "strong Weil curve", with degree m_E. Let ω be a Neron differential on E, so that $\phi^*\omega$ is the associated classical form $f(z)dz$. The space $H^1(Y_0(N), \mathbf{Z})_\pi$ is spanned, after tensoring with $\mathbf{C}$, by $\omega_1 = f(z)dz$ and $\omega_2 = \overline{f(z)dz}$.

Let γ_1, γ_2 be a basis of $H_1(E, \mathbf{Z})$. The property of the strong Weil curve is that the associated map $J_0(N) \to E$ has connected kernel, and therefore $H_1(X_0(N), \mathbf{Z}) \to H_1(E, \mathbf{Z})$ is surjective, so also $H_1(Y_0(N), \mathbf{Z}) \to H_1(E, \mathbf{Z})$ is surjective. In particular we can lift γ_i to $\widetilde{\gamma}_i \in H_1(Y_0(N), \mathbf{Z})$, so that $\phi_*\widetilde{\gamma}_i = \gamma_i$, and then the dual basis to $\widetilde{\gamma}_i$ generates $H^1(Y_0(N), \mathbf{Z})_\pi$.

The 2×2 "period matrix"

$$\mathrm{P} = \left(\int_{\widetilde{\gamma}_i} \omega_j \right) = \begin{pmatrix} \int_{\gamma_1} \omega & \int_{\gamma_2} \omega \\ \int_{\gamma_1} \overline{\omega} & \int_{\gamma_2} \overline{\omega} \end{pmatrix}$$

has determinant equal to the Néron area

$$\det \mathrm{P} = \mathrm{area}(E) := \left| \int_{E(\mathbf{C})} \omega \wedge \overline{\omega} \right| = m_E^{-1} \langle f, f \rangle.$$

(Note that $|dz \wedge \overline{dz}|$ gives twice the usual Lebesgue are on $\mathbb{C}$.)

Here, $\langle f, f \rangle = \int_{Y_0(N)} |f(z)|^2 |dz|^2$ is the same as the integral of $Y_0(N)$ of the *Riemannian* inner product $\langle \omega_1, \omega_1 \rangle$ with respect to the standard hyperbolic measure $\frac{dx\, dy}{y^2}$. Correspondingly, the squared volume $\mathrm{vol}(H^1(Y_0(N))_\pi)^2$ is given by

$$\frac{\det \begin{pmatrix} \langle \omega_1, \omega_1 \rangle & 0 \\ 0 & \langle \omega_2, \omega_2 \rangle \end{pmatrix}}{|\det \mathrm{P}|^2} = \frac{\langle f, f \rangle^2}{\mathrm{area}(E)^2} = m_E^2$$

so our definition (2.3) says that the automorphic height of π is the *square root* of m_E^2, i.e. equal to m_E.

Now $\mathrm{area}(E)^{-1}$ is, by definition, the (exponential) Faltings height of E (ignoring factors of 2π that are often included in the normalization). Also note that Bloch-Kato predicts that $L(1, \mathrm{Ad}, \pi) = \frac{m_E \mathrm{area}(E)}{N} c$, where the constant c is discussed in detail in [26]. So

$$\frac{\text{automorphic height}}{\text{(exponential) Faltings height}} = c^{-1} N \cdot L(1, \mathrm{Ad}, \pi).$$

3. Arithmetic height

We first give a definition of "arithmetic height" in a way that is adapted to our needs, then describe how it is related to the heights originally introduced by Kato [13, 14] and developed further by Koshikawa [15]. In short, it is the volume of the "F^1" piece of the adjoint motive associated to an automorphic form.

3.1. **Motives.** Let M be a pure (numerical) motive over $\mathbf{Q}$, with $\mathbf{Q}$ coefficients, of weight zero. In our later applications, it will be the adjoint motive (see §3.4) associated to a cohomological automorphic form. M has a de Rham realization $H_{\mathrm{dR}}(M)$, which carries a rational Hodge structure, that is to say, a decreasing filtration F^i which is compatible with a splitting

$$H_{\mathrm{dR}} \otimes \mathbf{C} = \bigoplus_{p,q} H_{\mathrm{dR}}^{(p,q)}.$$

We also have a Betti realization $H_B(M)$, another $\mathbf{Q}$-vector space, and if we fix a prime p an étale realization $H_{\mathrm{et}}(M)$ on a p-adic vector space.

3.2. **Metrics.** On the complex vector space $H_{\mathrm{dR}} \otimes \mathbf{C}$ there are multiple natural real structures and associated anti-holomorphic involutions: the "de Rham" involution c_{dR} which fixes de Rham cohomology of the real algebraic variety underlying M, and the "Betti" involution c_B which fixes rational Betti cohomology H_B with reference to the comparison isomorphism $H_{\mathrm{dR}} \otimes \mathbf{C} \simeq H_B \otimes \mathbf{C}$. We will often write $\overline{x}$ in place of $c_B(x)$.

We suppose that M is *symmetrically self-dual*, which means in particular that $H_{\mathrm{dR}}(M)$ comes with a symmetric perfect pairing (of pure Hodge structures)

$$[-,-] : H_{\mathrm{dR}}(M) \otimes H_{\mathrm{dR}}(M) \longrightarrow \mathbf{Q}.$$

where, on the right, $\mathbf{Q}$ is identified with the cohomology of the trivial variety. This pairing also respects Betti and de Rham conjugations.

For example, if E is an elliptic curve over $\mathbf{Q}$, $M := \mathrm{Sym}^2 H^1(E)(1)$ is a motive of this type; it inherits a pairing from the pairing $H^1(E) \otimes H^1(E) \to \mathbf{Q}(-1)$.

The pairing $[-,-]$ must necessarily pair nondegenerately the $H^{p,-p}$ and $H^{-p,p}$ subspaces of complex cohomology. This pairing then induces on $F^1 H_{\mathrm{dR}}(M) \otimes \mathbf{C}$ a nondegenerate Hermitian form:

$$\langle v_i, v_j \rangle = [v_i, \overline{v_j} = c_B(v_j)], \tag{3.1}$$

although we do not make any assertion that this is positive definite. Clearly $\langle v_i, v_j \rangle = \overline{\langle v_j, v_i \rangle}$. The pairing $\langle v_i, v_j \rangle$ is clearly real-valued on $H_B(M)$. Now c_B does not preserve the rational vector space $F^1 H_{\mathrm{dR}}$, but it does preserve $H_{\mathrm{dR}} \otimes \mathbf{R}$. Therefore $\langle -,- \rangle$ also defines a real-valued form on $F^1 H_{\mathrm{dR}}(M)$,

The determinant $[F^1 H_{\mathrm{dR}}(M)]$ – a one-dimensional $\mathbf{Q}$-vector space – also inherits a metric, given by

$$\|v_1 \wedge \cdots \wedge v_k\|^2 = |\text{determinant of } [v_i, \overline{v_j}]| \,.$$

3.3. **Integral structures.** Now suppose that the $\mathbf{Q}$-vector space $F^1 H_{\mathrm{dR}}(M)$ comes with an integral lattice. We will discuss later in our situation how to produce such a lattice, following Kato and Koshikawa, although in this paper we are only concerned with the easy "large p" case.

We can then compute its volume

$$\text{volume of } F^1 H_{\mathrm{dR},\mathbf{Z}} = \|\omega_1 \wedge \cdots \wedge \omega_k\| \tag{3.2}$$

$$= \left| \det \langle \omega_i, \overline{\omega_j} \rangle \right|^{1/2}, \tag{3.3}$$

where the ω_i are an integral basis for $F^1 H_{\mathrm{dR}}$.

This definition can be applied to any motive M as above. We will now explain how such a motive arises from π.

3.4. **The adjoint motive M_{Ad}.** Now suppose that π is as in §2.3. To proceed we must now assume the existence of

an "adjoint motive" M_{Ad} attached to π.

See [19, §4] for a discussion of the expected properties of this adjoint motive; for example, if π arises from an elliptic curve E over $\mathbf{Q}$, then $M_{\mathrm{Ad}} = \mathrm{Sym}^2 H^1(E)(1)$.

We will just summarize some of the properties here. The crucial property is that the étale realization of M_{Ad} is supposed to be the composition of the Galois representation associated to π (taking value in $\check{G}$) with the adjoint representation of $\check{G}$ on its Lie algebra.

M_{Ad}, if it exists, is a pure weight zero self-dual numerical motive over $\mathbf{Q}$ with $\mathbf{Q}$ coefficients, with trivial determinant, of dimension equal to $\dim(G)$, whose associated Galois representation is the adjoint Galois representation attached to π. Under the standard expectation that cohomological automorphic forms are attached to motives, one expects an adjoint motive *always* exists after possibly extending the coefficient field $\mathbf{Q}$; here, to simplify our life, we make the stronger assumption that it can be descended to $\mathbf{Q}$ coefficients. See [19, Appendix A] for discussion.

In particular, this motive M_{Ad} is equipped with an identification

$$H_B(M_{\mathrm{Ad}}) \otimes \overline{\mathbf{Q}} \simeq \check{\mathfrak{g}}_{\overline{\mathbf{Q}}} \tag{3.4}$$

which carries the (Betti) rational structure on the left to an inner twisting of the Chevalley rational structure on the right. Fixing an invariant quadratic form on $\check{\mathfrak{g}}$ induces a symmetric self-duality on M_{Ad}; we will assume this has been done.

If we fix a prime p, the Galois representation for π takes values in the inner twist of $\check{G}$ just noted. In particular, if p is such that this inner twist is split, the Galois representation is simply

$$\rho_p : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \check{G}(\mathbf{Q}_p) \tag{3.5}$$

(recall that, as mentioned in §1.4, we will allow $\check{G}$ to refer to the extended Langlands dual group when appropriate). Since π is cuspidal and tempered, one anticipates (and we will assume) that the image of ρ_p is not contained in any proper parabolic subgroup of the right-hand side.

3.5. **Integral structure for the adjoint motive.** We would like to produce a filtered lattice in the de Rham realization $H_{\mathrm{dR}} M_{\mathrm{Ad}}$ of M_{Ad}.

Kato and Koshikawa describe specific ways to produce such a lattice via p-adic Hodge theory, starting with a Galois-stable lattice in the étale realization; for large p one may simply use Fontaine–Laffaille theory. (See in particular [15, Definition 2.18]). So to produce a filtered integral lattice in $H_{\mathrm{dR}} M_{\mathrm{Ad}}$, we must produce a lattice L_p in the étale realization $H_{\mathrm{et}} M_{\mathrm{Ad}}$.

If the dual group were GL_n, then M_{Ad} arises as $N \otimes N^*$ for some other motive N, and then one can take L_p to be $N_p \otimes N_p^*$ where N_p is a Galois-stable lattice inside the étale realization of N_p.

For a general group we proceed thus. Consider, then, ρ_p as in (3.5) (if there is an inner twisting we just replace $\check{G}$ by that twist). Through it, the Galois group acts on the Bruhat-Tits building of $\check{G}(\mathbf{Q}_p)$ and its fixed set is bounded; if not, the image of the Galois group would be contained in a parabolic subgroup. For each fixed point x we get, by Bruhat-Tits theory [22, §3.4.1] an associated group scheme $\check{\mathcal{G}}_x$ over $\mathbf{Z}_p$ with generic fiber $\check{G}$; only finitely many such arise, because the fixed set is bounded. The integral form $\check{\mathcal{G}}$ of $\check{G}$ gives rise to an integral lattice in its Lie algebra, equivalently, a lattice $L_p \subset H_{\text{et}} M_{\text{Ad}}$. From each such choice of $\{L_p\}$, via the Kato-Koshikawa construction we obtain a corresponding latice in $H_{\text{dR}} M_{\text{Ad}}$ (equipped with a Hodge filtration).

The resulting class of lattices probably does not have too much intrinsic significance (so the precision of the discussion above was probably excessive, relative to its importance) and should be considered as a reasonable class "only up to bounded error" for finitely many primes, but in any case it gives at least a starting definition.

3.6. **Definition of the arithmetic height.** We may now define the arithmetic height of π, which we denote by $\text{height}_{\mathsf{K}}(\pi)$:

$$\text{arithmetic height of } \pi = \text{height}_{\mathsf{K}}(\pi) := \left(\text{volume of } F^1 H_{\text{dR},\mathbf{Z}}(M_{\text{Ad}})\,\right)^{-1}$$

where $F^1 H_{\text{dR},\mathbf{Z}}$ is an integral lattice in $F^1 H_{\text{dR}}$ arising from the process of §3.5, and, as discussed above, to define the right hand side we are implicitly using the self-duality on the adjoint motive arising from a fixed invariant quadratic form on $\check{\mathfrak{g}}$.

The resulting height depends on the choice of a lattice as in §3.5, but we will not make this explicit in the notation.

This is *not* the height defined by Kato and Koshikawa. However, we will outline in §3.7 why we expect it to be very close to it in the simply laced case.

3.7. **Canonical line of Hodge structures and the Kato-Koshikawa height.** In this section, we will recall the notion of the "canonical line" for a Hodge structure and describe the Kato-Koshikawa height. We will relate it to our definition in the next §.

Let N be a motive of weight w. The Hodge filtration for the de Rham realization $H_{\text{dR}}(N)$ of N can be written:

$$F^0 H_{\text{dR}} \supset F^1 H_{\text{dR}} \supset \cdots \supset F^w H_{\text{dR}}.$$

Write $G^p = F^p/F^{p+1}$ for the graded quotient. Define the "canonical line of the Hodge structure"

$$\mathsf{K} := \prod_i [G^i]^{i-w/2} = [G^w]^{w/2}[G^{w-1}]^{w/2-1}\dots[G^0]^{-w/2} \tag{3.6}$$

(Let us assume, to simplify, that w is even, but if not it causes only sign ambiguity that makes no difference to the resulting heights; alternately, one can "multiply by $[G]^{w/2}$, as does Kato.)

Now a crucial point is that K is equipped with a Hermitian metric *without using anything beyond the axioms of a real Hodge structure:*

As usual, the Hodge splitting permits us to identify each $G^p = F^p/F^{p+1}$ with a subspace $H^{p,w-p} \subset H^{\text{dR}}$, and correspondingly identifies $F^j \otimes \mathbf{C}$ to $\bigoplus_{p\geq j} H^{p,w-p}$. The complex conjugation carries $H^{p,w-p}$ to $H^{w-p,p}$, and accordingly induces a complex antilinear map $c_B : G^p \otimes \mathbf{C} \longrightarrow G^{w-p} \otimes \mathbf{C}$; and a similar antilinear bijection between the determinant lines. In particular, there is a metric on the one-dimensional $\mathbf{Q}$-vector space $[G^p][G^{w-p}]^{-1}$, explicitly given by the following formula ([14, 1.4.1]):

$$\|g^p \otimes (g^{w-p})^{-1}\| = \left| g^p/\overline{g^{w-p}} \right|. \tag{3.7}$$

We remark that given two Hermitian metrics h, h' on G^p and G^{w-p} that are compatible, in that c_B carries the first to the conjugate of the second, then the induced metric on $[G^p][G^{w-p}]^{-1}$ is seen to be given by (3.7), irrespective of exactly what h, h' were.

One can use this Hermitian metric and a choice of integral structure to measure heights; this is the idea of Kato–Koshikawa. A crucial feature of the resulting construction is that one does not need to worry too much about the choice of integral structure, in the sense that only finitely many possibilities for the height occur, at least in Koshikawa's definition: [15, Theorem 9.8].

The terminology used here comes from Griffiths, who, in his paper [12] considers a polarized variation of Hodge structure and calls

$$[G^w]^w[G^{w-1}]^{w-1}\dots[G^0]^0$$

the fundamental line. [2] In the situation of a variation of Hodge structure, this defines a line bundle on the base, and, when we endow each $[G^j]$ with the metric that arises from the polarization, Griffiths proves that it is in fact a non-negative line bundle. For example, in the case of the moduli space of elliptic curves, the Hodge filtration has steps $F^0 \supset F^1 \supset F^2 = 0$ and then $K \simeq \omega$. Now, from this point of view, the sequence of exponents $(e_w = w/2, w/2-1, \dots, e_0 = -w/2)$ of exponents that appear in (3.6) could be replaced by any decreasing sequence; but the symmetry $(e_i + e_{w-i} = w)$ of the sequence makes the metric definable without reference to the Hodge metric (see Remark after (3.7)) and moreover the specific choice of e_i is important in Koshikawa's proof of independence of lattice.

3.8. **Comparison of the Kato-Koshikawa height of the adjoint motive with the arithmetic height, for simply laced groups.** For *simply laced*, almost simple groups G we expect that the arithmetic height of the adjoint motive coincides (up to a power computed below) with its Kato-Koshikawa height at all good primes, and the difference can be bounded elsewhere. We will not prove this but rather a related identification of linear algebra: there are positive integers a, b and a canonical identification of $\mathbf{Q}$-lines

$$\text{canonical line for } H_{\mathrm{dR}}(M_{\mathrm{Ad}})^{\otimes a} \simeq [F^1 H_{\mathrm{dR}}(M_{\mathrm{Ad}})]^{\otimes b} \tag{3.8}$$

when M_{Ad} is an adjoint motive; $[\dots]$ is determinant and other notation is as in (3.6). We write down b/a in terms of the root system in (3.10). Likely, upon examination of integral structures, this will yield an associated relationship of heights up to bounded error.

The appearance of *simply laced* is a puzzle into which I have no insight.

A particularly transparent case is that of $G = \mathrm{GL}_n = G^\vee$. Here cohomological automorphic representations (for the trivial local system) give rise to motives N of weight $n-1$ with Hodge numbers $h^{n-1,0} = \dots = h^{0,n-1} = 1$. Then, writing G^p for the associated graded of the Hodge filtration for N,

$$[F^1 H_{\mathrm{dR}}(N \otimes N^*)] = \prod_{p>q}[G^p \otimes (G^q)^*] = [G^{n-1}]^{n-1}\dots[G^0]^{-(n-1)}$$

i.e this gives the canonical line for N squared.

We can generalize this example by some abstract nonsense, Lie-group style, which the reader should probably skip. The motive associated to a G-automorphic form is really a motives indexed by representations of the (extended) Langlands dual group $\check{G}$. The

[2] Griffiths uses the indexing where the Hodge filtration is increasing, so we have switched to match with our notation.

motive associated to π determines an inner form $\check{G}_*$ of $\check{G}$ and taking Betti cohomology of this system of motives gives rise to a tensor functor from representations of $\check{G}_*$ over $\mathbf{Q}$ to polarizable rational Hodge structures. The Hodge splitting comes from a complex character $\lambda : \mathbb{G}_m \to \check{G}_* \otimes \mathbf{C}$ and the Hodge filtration defines a parabolic subgroup P. In our case, if we pass to $\mathbf{C}$, λ is conjugate to the half-sum of positive coroots $\check{\rho}$ for the Borel $\check{B} \subset \check{G}$ and P is conjugate to $\check{B}$.

In particular, consider the adjoint representation of $\check{G}_*$ on its own Lie algebra $\check{\mathfrak{g}}_*$. Let $\check{\mathfrak{g}}_m$ be the mth weight space for λ, so that the Hodge filtration is given by $F^p = \bigoplus_{m \geq p} \check{\mathfrak{g}}_m$ and $\check{\mathfrak{g}}_m$ projects isomorphically to the mth graded piece. Our claim (3.8) amounts to an isomorphism

$$\bigotimes_{m \in \mathbb{Z}} \mathfrak{g}_m^{\otimes m} \simeq \left[\bigotimes_{m > 0} \mathfrak{g}_m \right]^{b/a} \tag{3.9}$$

where we interpret the fractional power as in (3.8). Now for the Chevalley group $\check{G}$, if we replace the role of λ by $\check{\rho}$, both sides have preferred generators up to sign, coming from the integral Chevalley basis, which defines an isomorphism (at least after squaring both sides, which is fine for us, because we can double a and b).

To transport the result to P, we choose an element of $\check{G}$ conjugating $\check{B}$ to P over $\mathbf{C}$. We need to verify that the resulting isomorphism is independent of conjugation; so we must verify that the $\check{B}$-weight on both sides of (3.9) is the same. This is the following equality:

$$\sum_{\alpha} \langle \check{\rho}, \alpha \rangle \alpha \stackrel{?}{=} \frac{b}{a} \sum_{\langle \lambda, \alpha \rangle > 0} \alpha$$

when the sum is taken over all roots α for $(\check{G}, \check{B})$ and all positive roots $\alpha > 0$ respectively; $\check{\rho}$ is the half-sum of positive coroots and the equality takes place in the dual $\mathfrak{t}^*$ to the Lie algebra of a maximal torus. To check this we fix an invariant bilinear form on $\mathfrak{t}$. We can then identify $\mathfrak{t} \simeq \mathfrak{t}^*$. On the left hand side the rule $\lambda \mapsto \sum \langle \lambda, \alpha \rangle \alpha$ gives a Weyl-invariant map $\mathfrak{t} \to \mathfrak{t}$ which must be a scalar multiple of the identity because the group G was assumed almost simple and in this case the Weyl representation is irreducible. Therefore, the left hand side is a scalar multiple of $\check{\rho} = \sum_{\beta > 0} \frac{\beta}{\langle \beta, \beta \rangle}$ (considered now in $\mathfrak{t}^*$) and the right hand side a multiple of $\rho = \frac{1}{2} \sum_{\beta > 0} \beta$. In the simply laced case, when there is only one possibility for $\langle \beta, \beta \rangle$, these are proportional. To compute the constant, we can pair both sides with $\check{\rho}$ to get

$$\frac{b}{a} = \frac{\sum_{\alpha} \langle \check{\rho}, \alpha \rangle^2}{\sum_{\alpha > 0} \langle \check{\rho}, \alpha \rangle}. \tag{3.10}$$

4. Proof of the main result

In this section we will formulate more precisely a version of statement (1.1) from the introduction and give an outline of a proof under various auxiliary technical assumptions. The setup takes the subsections §4.1–§4.2, we give the statement in §4.3, and then we proceed to the proof.

4.1. **General setup.** Let $Y = Y_K, \pi$ be, as before, a locally symmetric space attached to a split semisimple $\mathbf{Q}$-group G, and a cohomological automorphic representation contributing to its cohomology. We now impose the assumption that $h_q = 1$, i.e. the contribution of π to cohomology in minimal degree is one-dimensional.

We choose p to be "good," in the sense that:

(a) The inner form of $\check{G}$ determined by the adjoint motive is split at p.
(b) A Fontaine-Laffaille type condition: $2(\dim G)\langle\check{\rho}, \alpha\rangle < p-1$, notation as in §3.8; in particular p is odd.
(c) Avoiding Eisenstein congruences:
The Galois representation for π at p, which by (a) takes the form
$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \check{G}(\mathbf{Q}_p),$$
can be conjugated to be valued in the $\mathbf{Z}_p$-form of the split form of $\check{G}$, and its reduction
$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \check{G}(\mathbf{F}_p)$$
is not contained in any proper parabolic subgroup of the reduction $\check{G}$ over $\mathbf{F}_p$. We will denote by $\mathrm{Ad}\rho$ the associated adjoint representation on $\check{\mathfrak{g}}_{\mathbf{Q}_p}$, $\mathrm{Ad}\rho_{\mathbf{Z}}$ the integral form on $\check{\mathfrak{g}}_{\mathbf{Z}_p}$, and $\mathrm{Ad}\bar{\rho}$ the adjoint representation on $\check{\mathfrak{g}}_{\mathbf{F}_p}$.
(d) Avoiding level-raising or level-lowering:
At primes q dividing the level K, i.e. at which K is not hyperspecial, the local cohomology $H^*(\mathbf{Q}_q, \mathrm{Ad}\overline{\rho})$ vanishes. (In general we might otherwise expect a "local" contribution at q related to the existence of level-raising or level-lowering congruences; this needs to be further examined.)
(e) Prime-to-p level:
The open compact subgroup underlying K contains a hyperspecial subgroup of $G(\mathbf{Q}_p)$.

4.1.1. *Usual deformation rings.* Attached to $\bar{\rho}$ is Mazur's deformation ring R, where we will impose crystalline local conditions at p and no conditions at other primes dividing the level. In particular, assumption (b) and (d) imply that the local deformation rings at all such primes are smooth, and the liftings of $\overline{\rho}$ to $\mathbf{F}_p[x]/x^2$ are classified by the f-cohomology $H^1_f(\mathrm{Ad}\bar{\rho})$ of Bloch and Kato [2].

By considerations of Galois deformation theory, it is known that R has the structure of the quotient of the completed polynomial ring $\mathbf{Z}_p[[x_1, \dots, x_s]]$ (for some integer s) by $s+\delta$ relations. Here δ is the same integer appearing in (2.1); this computation is the basis for the Calegari-Geraghty version of obstructed modularity lifting [7].

The lift ρ to $\mathbf{Z}_p$ gives rise to a homomorphism
$$\mathrm{R} \longrightarrow \mathbf{Z}_p$$
Let $\mathfrak{p}$ be the prime ideal of R defined by the kernel of this augmentation. We will write
$$d_i = \#\mathrm{Der}^i_{\mathbf{Z}_p}(\mathrm{R}, \mathbf{Q}_p/\mathbf{Z}_p)$$
for the order of the Andre-Quillen cohomology groups [20], so that Der^0 is the usual derivations, and Der^1 classifies commutative algebra extensions – this $\mathrm{Der}^i_A(B, M)$ is denoted $D^i(B/A, M)$ in [20]. Here $\mathbf{Q}_p/\mathbf{Z}_p$ is a R-module via the augmentation above. In particular, d_0 is the order of $\mathfrak{p}/\mathfrak{p}^2$.

4.1.2. *Derived deformation rings.* Write $\mathcal{R}$ for the derived deformation ring for $\overline{\rho}$, at level K, which is constructed as in [11]: it is a pro-object in the category of simplicial commutative rings, which classifies, now, deformations of $\overline{\rho}$ with coefficients into various rings of this type.

This $\mathcal{R}$ is a pro-simplicial ring. One needs to know very little about the abstract theory of derived rings, and we will get by with the two points below. What matter for us are largely its homotopy groups $\pi_*\mathcal{R}$, which form a graded ring extending $\mathrm{R} = \pi_0\mathcal{R}$, the usual deformation ring from §4.1.1.

(a) Firstly, in the case at hand, the structure of $\pi_*\mathcal{R}$ is very simple, in a fashion generalizing the last paragraph of §4.1.1, the $\pi_*\mathcal{R}$ are isomorphic to the to the homology of a Koszul complex for the ring $S = \mathbf{Z}_p[[x_1, \dots, x_r]]$ with respect to various elements $f_1, \dots, f_s$ lying in the maximal ideal of S, with $s = r + \delta$, i.e. $\pi_j\mathcal{R}$ is the jth homology group of the complex

$$S \leftarrow S^{\oplus s} \leftarrow S^{\oplus \binom{s}{2}} \leftarrow \dots,$$

(where the left-most term lies in degree zero, and the differentials decrease degree).

(b) As noted above, the tangent space of R is related to $H^1_f(\mathrm{Ad}\rho)$. This generalizes to a relationship between the tangent complex of $\mathcal{R}$ and the whole cohomology $H^*_f(\mathrm{Ad}\rho)$. This leads, in particular, to an important relationship between $\pi_1(\mathcal{R})$ and Galois cohomology, in the form of a canonical morphism

$$H^2_f(\mathrm{Ad}\rho_{\mathbf{Q}_p}) \to \mathrm{Hom}_{\mathrm{R}}(\pi_1\mathcal{R}, \mathbf{Q}_p). \tag{4.1}$$

This has been discussed in §15 of [11] (see equation (15.7)).

4.1.3. *Notation for automorphic cohomology.* We will be assuming that

> The cohomology $H^*(Y_K, \mathbf{F}_p)_\rho$, the localization of the mod p cohomology at the character of the Hecke algebra determined by $\bar{\rho}$, is concentrated in degrees $[q, q+\delta]$ where q is as in (2.1).

For brevity we will omit coefficients entirely when talking about $\mathbf{Z}_p$ cohomology localized at $\bar{\rho}$:

$$H^j := H^j(Y, \mathbf{Z}_p)_{\bar{\rho}},$$

We write H^j_{tf} or the torsion-free quotient of H^j, and $H^j_{\mathbf{Q}}$ for its rationalization $H^j \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

We write $H^*_{\mathrm{tf}}[\mathfrak{p}]$ and $H^*_{\mathbf{Q}}[\mathfrak{p}]$ for the respective $\mathfrak{p}$-kernels and $H^*_{\mathrm{tf}}/\mathfrak{p}$ and $H^*_{\mathbf{Q}}/\mathfrak{p}$ for the $\mathfrak{p}$-cokernel; then the evident maps

$$H^j_{\mathrm{tf}}[\mathfrak{p}] \otimes \mathbf{Q} \to H^*_{\mathbf{Q}}[\mathfrak{p}] \to H^*_{\mathbf{Q}}/\mathfrak{p}$$

are isomorphisms.

In words $H^*_{\mathbf{Q}}[\mathfrak{p}]$ is the p-adic completion of the space of cohomology classes with the same Hecke eigenvalues as π, whereas $H^*_{\mathbf{Q}}$ includes classes congruent to π. We denote by

$$[-,-] : H^*_{\mathbf{Q}} \times H^*_{\mathbf{Q}} \to \mathbf{Q}_p$$

the Poincaré duality pairing on cohomology. This induces a $\mathbf{Z}_p$-valued pairing on torsion free quotients of integral cohomology.

We write

$$H^j_{\mathbf{R}} = H^j(Y_K, \mathbf{R})_\pi,$$

the component of real cohomology that transforms according to the Hecke eigenvalues of π. This is not quite compatible with the above notation, since, above, $H^*_{\mathbf{Q}}$ denotes $\mathbf{Q}_p$ coefficients rather than $\mathbf{Q}$ coefficients; but our use of the $\mathbf{R}$ notation will be very minimal and hopefully will not cause any confusion. We also write $\star : H^*_{\mathbf{R}} \to H^*_{\mathbf{R}}$ the Hodge $\star$ operation.

4.2. **Motivic cohomology and its conjectural action.** We assume an adjoint motive associated to π exists, in the sense of §3.4. Under our assumptions from §4.1 there is a unique conjugate of ρ that is valued in $\check{G}(\mathbf{Z}_p)$. We then take the integral structure on p-adic étale cohomology $H_{\mathrm{et}} M_{\mathrm{Ad}}$ of the adjoint motive M_{Ad} to arise from the Chevalley integral form of $\check{\mathfrak{g}}$. This is "compatible" with the discussion of §3.5.

Associated to this M_{Ad} one obtains a motivic cohomology group Λ which we should informally regard as the extension group $\mathrm{Ext}_{\mathcal{M}}(\mathbf{Q}, M_{\mathrm{Ad}}(1))$, where the extensions are taken in a category $\mathcal{M}$ of "mixed motives over $\mathbf{Z}$." Assuming that M comes from a Chow motive one can give a rigorous definition. For discussion and a precise formulation of what we are assuming, including of the issue of independence on the lift to a Chow motive, see [19], where the group Λ is called L.

This Λ is a $\mathbf{Q}$-vector space, equipped with regulators

$$\Lambda \otimes \mathbf{Q}_p \longrightarrow H^1_f(\mathrm{Ad}\rho(1)) \text{ and } \Lambda \otimes \mathbf{R} \longrightarrow \mathfrak{a}, \tag{4.2}$$

where $\mathfrak{a}$ is a real vector space arising as a Deligne cohomology group discussed in [19]. In the paper [19] we also construct a free *isometric* action of $\wedge^* \mathfrak{a}^*$ on the π-component $H^*(Y_K, \mathbf{R})_\pi$ of automorphic cohomology, for a suitable metric on $\mathfrak{a}^*$ (see [19, §3.5] for the metric).

The conjectures we will assume about these action are:

(a) Both of the maps of (4.2) are isomorphisms. This is a a standard conjecture about motivic cohomology.
(b) The archimedean conjecture of [19], see [19, §5.4]: The action of Λ^* – considered as a $\mathbf{Q}$-vector subspace of $\mathfrak{a}^*$ – on $H^*(Y_K, \mathbf{R})_\pi$ preserves the image of rational cohomology; thus the rational cohomology becomes a free module under the exterior algebra $\bigwedge^* \Lambda^*$.
(c) The main theorem of [11] holds in the case at hand:

 We assume that homology carries the structure of a free $\mathcal{R}$-module, extending its structure of R-module factoring through a map $\mathrm{R} \to \mathrm{T}$; in particular, as we suppose that $h_q = 1$, there is an isomorphism of $\pi_* \mathcal{R}$-modules:

$$H^{q+\delta-j} \simeq H_{q+j} \simeq \pi_j \mathcal{R} \tag{4.3}$$

 Note that the action of $\pi_* \mathcal{R}$ on homology, raising homological degree, transfers by Poincaré duality to one on cohomology, now lowering cohomological degree.

 In [11] we show that, in favorable circumstances, such a statement follows using the Taylor-Wiles method given the expected properties of Galois representations.
(d) Compatibility between archimedean and p-adic conjectures:

 The actions of (b) and (c) are compatible with reference to the isomorphisms

$$\wedge^* \Lambda \otimes \mathbf{Q}_p \overset{(4.2)}{\simeq} \bigwedge^* H^1_f(\mathrm{Ad}\rho(1))^* \overset{(4.1)}{\to} \pi_* \mathcal{R} \otimes_{\pi_0 \mathcal{R}} \mathbf{Q}_p$$

> The second map is induced by taking exterior power of the dual of (4.1); since we suppose that (4.2) is an isomorphism it follows also that (4.1) is an isomorphism, since both sides act freely on homology $\otimes_{\mathrm{R}} \mathbf{Q}_p$.

Our assumptions mean that $\mathrm{R} \otimes \mathbf{Q}_p$ maps isomorphically to the $\mathbf{Q}_p$-algebra generated by Hecke operators acting on degree $q+\delta$ cohomology, and so is a sum of fields. Moreover, $\pi_* \mathcal{R} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \simeq \pi_* \mathcal{R}[p^{-1}]$ is a $(R \otimes \mathbf{Q}_p)$-module and its localization at the $\mathbf{Q}_p$ factor corresponding to π is an exterior algebra. As with (c) it is plausible this can be checked directly with modularity lifting theorems under assumptions, as happens in [11].

4.3. **Formulation of the result.** Here is the precise statement of (1.1) from the introduction.

Let $Y = Y_K$, π and the prime p be as before, and assume that the conditions enunciated in §4.1 and §4.2 hold true.

To normalize heights we choose bilinear forms on G and its dual:

(a) (as in §3 of [19]) an invariant quadratic form on the real Lie algebra of $G(\mathbf{R})$, giving rise to a metric on Y_K;
(b) (as in Definition 4.2.1, (3) of [19]) a $\mathbf{Q}$-valued invariant quadratic form on $\check{\mathfrak{g}}_{\mathbf{Q}}$ which is integral of unit discriminant on $\check{\mathfrak{g}}_{\mathbf{Z}_p}$.

We require these to be compatible, a condition spelled out later in §4.16 which amounts to them inducing the same volume form on $\mathfrak{a}$; this is readily arranged by explicit computation e.g. [19, §7.3] gives explicit examples of compatible forms for classical groups and $p \neq 2$.

With this background, here is our main statement:

> Assume the Bloch-Kato conjecture for the adjoint L-function $L(1, \mathrm{ad}, \pi)$ is valid up to p-units. Assume also the conditions of §4.1 and §4.2 hold true, in particular, the validity of the motivic conjecture and compatibility between its archimedean and p-adic forms.
>
> Then, for some integer d depending only on G, the height conjecture holds up to p-units:

$$\frac{\text{automorphic height of } \pi}{\text{arithmetic height of } \pi} = \sqrt{q}(2\pi)^d L(1, \mathrm{Ad}, \pi). \tag{4.4}$$

> where $q \in \mathbf{Q}^*$ is a p-unit.

The power of 2π arises from Tate twists in the argument – we did not try to keep track of it, but it should be straightforward to do so. To avoid having to write these powers of 2π we adopt the notation:

$$A \sim B \tag{4.5}$$

for two real numbers A, B means that

$$\frac{A}{B} = (2\pi)^r \sqrt{q}$$

with $q \in \mathbf{Q}^\times$ a p-unit. We use the same notation if A, B are elements of the same one-dimensional $\mathbf{R}$-vector space. If $A, B \in \mathbf{Q}_p$, the notation $A \sim B$ will simply mean $A/B \in \mathbf{Z}_p^\times$.

4.4. **The Bloch-Kato conjecture for an adjoint motive.** We recall what the Bloch-Kato conjecture says, in the elegant formulation of Fontaine and Perrin-Riou, and then we translate it, in the case of interest, to an explicit formula (4.11).

4.4.1. *Bloch-Kato, after Fontaine and Perrin-Riou.* The paper of Flach [10] is an excellent short survey of this material and we will freely refer to it.

For any Chow motive M (over $\mathbf{Q}$ and with $\mathbf{Q}$ coefficients) we have an associated p-adic étale realization, which gives $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{Q}_p)$, and this has, after Bloch and Kato, f-cohomology groups $H^*_f(\mathbf{Q}, \rho)$. Note we define f-cohomology here as a mapping cone, so it is supported in degrees $[0, 3]$ and has good duality properties.

These are p-adic vector spaces, but the fact that ρ arose from a motive gives the possibility of constructing rational structures: One can define motivic cohomology groups $H^0_{\mathrm{mot}}(M), H^1_{\mathrm{mot}}(M)$, which are rational vector spaces, together with regulators

$$H^i_{\mathrm{mot}}(M) \otimes \mathbf{Q}_p \to H^i_f(\mathbf{Q}, \rho)$$

which are *conjecturally* isomorphisms ([10, p83]). To formulate Bloch-Kato one needs to assume these are isomorphisms, and we will do so.

To produce a corresponding rational structure for $i = 2, 3$, we use duality in Galois cohomology, i.e. we formally set $H^i_{\mathrm{mot}}(M) := H^{3-i}_{\mathrm{mot}}(M^\vee(1))^\vee$ for $i \in \{2, 3\}$ (and also assume that the corresponding maps for $M^\vee(1)$ are isomorphisms). We emphasize then that the notation $H^i_{\mathrm{mot}}(M)$ for $i = 2, 3$ is purely formal, and does not reflect an actual cohomology theory. Finally, we (unconventionally) define $[H^*_{\mathrm{mot}}(M)]$, the "determinant of motivic cohomology", as the $\mathbf{Q}$-line given by the alternating product

$$[H^*_{\mathrm{mot}}(M)] := \bigotimes_{i=0}^{3} [H^i_{\mathrm{mot}}(M)]^{(-1)^i}, \tag{4.6}$$

recalling that square brackets $[\dots]$ mean determinants.

Define the "fundamental line" of Fontaine and Perrin-Riou as the one-dimensional $\mathbf{Q}$-line

$$\Psi := \mathrm{Hom}([H^*_{\mathrm{mot}}(M)], [H_B(M)^+] \otimes [H_{\mathrm{dR}}/F^0 H_{\mathrm{dR}}(M)]^{-1})$$

Here $H_B(M), H_{\mathrm{dR}}(M)$ are Betti and de Rham cohomology; these are $\mathbf{Q}$-vector spaces. $+$ refers to fixed points of the involution of Betti cohomology induced by the antiholomorphic conjugation of the complex points of the variety underlying M.

So, Ψ is a one-dimensional $\mathbf{Q}$-vector space. For discussion we refer to [10, §2]; our line Ψ is the inverse of Flach's line, so some quantities are inverted in our discussion. Moreover, the Beilinson-Deligne regulator and a p-adic regulator respectively define morphisms

$$\Psi \otimes \mathbf{R} \to \mathbf{R}, \tag{4.7}$$

$$\Psi \otimes \mathbf{Q}_p \to \text{determinant of compactly supported Galois cohomology of } \rho. \tag{4.8}$$

which we suppose to be isomorphisms; again, this is necessary to formulate the Bloch-Kato conjecture. As a reference again see Flach [10, §3] and the references quoted therein; we will also briefly review in §4.4.4 and §4.4.5.

The conjecture of Bloch and Kato now says that:

> The integrally normalized generator for Ψ (defined so that it maps to an p-integral element of the determinant of compactly supported Galois cohomology for every p, thus normalized up to sign) maps, under the

Beilinson-Deligne regulator, to the L-function of M evaluated at the point 0.

4.4.2. *Our setting.* We will apply this to the case $M := M_{\mathrm{Ad}}(1)$, the adjoint motive twisted by 1; so we have $F^0 H_{\mathrm{dR}}(M) = F^1 H_{\mathrm{dR}}(M_{\mathrm{Ad}})$. In this case the determinant of motivic cohomology is $[H^*_{\mathrm{mot}} M] = [\Lambda]^{-1}$ – only the $i = 2$ term of (4.6) contributes here. After fixing standard bases for de Rham and Betti realizations of $\mathbf{Q}(1)$ we get:

$$\begin{aligned}[\Psi] \otimes [\Lambda]^{-1} &\simeq [H_B^+(M)] \otimes [H_{\mathrm{dR}}/F^0 H_{\mathrm{dR}}(M)]^{-1} \\ &= ([H_B^-(M_{\mathrm{Ad}})] \otimes [H_{\mathrm{dR}}(M_{\mathrm{Ad}})]^{-1}) \otimes [F^1 H_{\mathrm{dR}}(M_{\mathrm{Ad}})].\end{aligned} \tag{4.9}$$

In our situation we have a preferred choice of p-integral structure on H_B^- and H_{dR} after tensoring with $\mathbf{Q}_p$.

4.4.3. Λ *and determinant of Galois cohomology.* Our choice of an integral model for ρ gives an integral model for Λ (at least at p). Explicitly, the integral lattice in Λ is the pullback of the image of

$$H^1_f(\mathrm{Ad}\rho_{\mathbf{Z}}(1)) \to H^1_f(\mathrm{Ad}\rho(1))$$

under the regulator map (4.2). The resulting integral lattice in $\Lambda \otimes \mathbf{Q}_p$ will be denoted $\Lambda_{\mathbf{Z}_p}$. Note this is constructed using Galois cohomology and *not* using any theory of motivic cohomology with integral coefficients.

In any case $\Lambda_{\mathbf{Z}_p}$ gives rise to a point (normalized up to p-units) in the determinant line: $\nu \in [\Lambda]$, which will be any point that maps to the determinant of $\Lambda_{\mathbf{Z}_p}$ inside $[\Lambda \otimes \mathbf{Q}_p]$. Later we will denote by

$$\nu^\vee \in [\Lambda^\vee]$$

a class dual to ν, i.e., normalized so that $\langle \nu, \nu^\vee \rangle$ is a p-unit. Thus ν and $\nu^\vee$ are elements of the $\mathbf{Q}$-lines $[\Lambda]$, $[\Lambda^\vee]$, determined up to p-units.

We have an isomorphism (recall we write $[W]$ for the determinant of the vector space W and $[H^*]$ for the determinant of cohomology):

$$[\Lambda^\vee] \otimes \mathbf{Q}_p \simeq [H^1_f(\mathrm{Ad}\rho(1))]^{-1} \simeq [H^*_f(\mathrm{Ad}\rho(1))]$$

where the last comes from vanishing of the cohomology outside degree 1. Accordingly, $\nu^\vee$ defines a point in the determinant $[H^*_f(\mathrm{Ad}\rho(1))]$ of Galois f-cohomology.

However, the "natural" integral structure on the determinant of Galois f-cohomology $[H^*_f(\mathrm{Ad}\rho(1))]$ is generated not by $\nu^\vee$, but (up to p-units) by $d_0^{-1}\nu^\vee$ (since, as usual in such matters, one takes into account torsion).

4.4.4. Return now to (4.9), an isomorphism of $\mathbf{Q}$-lines

$$[\Psi] \simeq [\Lambda] \otimes ([H_B^-] \otimes [H_{\mathrm{dR}}]^{-1}) \otimes [F^1 H_{\mathrm{dR}}]. \tag{4.10}$$

where on the right all the groups refer to the adjoint motive M_{Ad}. Now the following p-integral structures match under the isomorphism (4.10):

- On the fundamental line Ψ, we put the integral normalization discussed after (4.8);
- On $[\Lambda]^{-1}$, we take the integral normalization arising from the identification with determinant of Galois f-cohomology, i.e. the normalization generated by $d_0^{-1}\nu^\vee$.
- On $[F^1 H_{\mathrm{dR}}]$ we take the lattice $[F^1 H_{\mathrm{dR},\mathbf{Z}}]$ arising from Fontaine-Laffaille theory from the integral structure on H_B corresponding to the integral Chevalley basis for $\mathfrak{g}$.

- On $[H_B^-], [H_{\mathrm{dR}}]$ we take the p-integral structures arising from taking the integral Chevalley basis for $\mathfrak{g}$.

This compatibility of p-integral structures comes from tracing through the definitions. The integral structure on Ψ comes from that on compactly supported Galois cohomology, the integral structure on Λ come from Galois f-cohomology. The difference between these is of local nature, and can be analyzed prime by prime; this is done in equation (3.1) of [10]. The contribution from primes at ∞, p is precisely matching that of $[H_B^-]$ and $[F^1 H_{\mathrm{dR}}][H_{\mathrm{dR}}]^{-1}$ respectively.

4.4.5. We can now show that Bloch-Kato conjecture for the adjoint L-function (up to p-units) is equivalent to the statement that

$$L(1, \mathrm{Ad}, \pi) \sim d_0 \|\nu\| \mathrm{height}_{\mathrm{K}}(\pi)^{-1} \tag{4.11}$$

where $\|\nu\|$ refers to the norm of ν for the Hermitian metric on $\mathfrak{a}$ to which Λ maps (mentioned after (4.2)). See (4.5) for the notation $\sim$.

So, (4.11) says that the adjoint L-value is the product of a class number d_0, a motivic regulator $\|\nu\|$, and a Hodge-theoretic period $\mathrm{height}_{\mathrm{K}}(\pi)$, and the remainder of our proof of (1.1) will be to relate $d_0\|\nu\|$ to the automorphic height.

Proof. We return again to (4.10):

$$[\Psi] \simeq [\Lambda] \otimes ([H_B^-] \otimes [H_{\mathrm{dR}}]^{-1}) \otimes [F^1 H_{\mathrm{dR}}].$$

The regulator $\Psi \otimes \mathbf{R} \simeq \mathbf{R}$ arises from the regulator $\Lambda \otimes \mathbf{R} \to \mathfrak{a}$ and the exact sequence (where everything refers to the adjoint motive)

$$H_B^- \otimes \mathbf{R} \to H_{\mathrm{dR}}/F^1 H_{\mathrm{dR}} \otimes \mathbf{R} \to \mathfrak{a} \tag{4.12}$$

(the map r of [10, §2] or [19, (2.1.11)], applied to the adjoint motive twisted by 1).

Our prior formulation of Bloch-Kato is that the integrally normalized element of Ψ is sent to $L(1, \mathrm{Ad})$ under the regulator $\Psi \otimes \mathbf{R} \to \mathbf{R}$. Unwinding using §4.4.4, this says that, under the map of determinants

$$[H_B^- \otimes \mathbf{R}] \otimes [\mathfrak{a}] \simeq [H_{\mathrm{dR}}/F^1 H_{\mathrm{dR}} \otimes \mathbf{R}]$$

induced by (4.12), the integral structures described in §4.4.4 line up, up to p-units. (The integral structure on $[\mathfrak{a}]$ comes from $d_0\nu \in [\Lambda]$.)

We would prefer to work with the dual sequence to (4.12), namely, (see [19, (2.2.10)])

$$F^1 H_{\mathrm{dR}}(\mathrm{Ad}M)_{\mathbf{R}} \to H_B^+(\mathrm{Ad}(M))_{\mathbf{R}} \to \mathfrak{a}. \tag{4.13}$$

where the first map is projection onto the fixed space of Betti conjugation, $x \mapsto \frac{x+\bar{x}}{2}$.

The relationship between the two sequences is related to the functional equation of the L-function, and, as in the computations of the compatibility of the functional equation with Beilinson's conjecture (see [9, §5] for the critical case) there is a power of $(2\pi i)$ and a period determinant which intervene in passing between them. In the current case, the determinant of the comparison map $H_B \otimes \mathbf{C} \to H_{\mathrm{dR}} \otimes \mathbf{C}$ is a p-unit: it is a rational number because the determinant of M_{Ad} is trivial, and moreover the comparison map preserves the pairings on both sides, and the integral lattices in each are self-dual at p (cf. [4, §4] for interaction of Fontaine-Laffaille functor with duality) so it follows that this rational number is a p-unit.

In particular, the compatibility up to p-units of the volumes on (4.12) induced by the various p-integral structures means that the same holds true for (4.13) also, in the sense that

$$[F^1 H_{\mathrm{dR},\mathbf{Z}}] \otimes d_0\nu \sim [H^+_{B,\mathbf{Z}}] \cdot L(1, \mathrm{Ad}, \pi). \tag{4.14}$$

This equality takes place inside an $\mathbf{R}$-line, the determinant of the middle group of (4.13). Recall that the notation $\sim$ allows a power of (2π) as well as $\sqrt{u}$ for u a p-unit.

To complete the proof, we will compute a self-pairing on both sides of (4.14):

The middle term of (4.13) has a real-valued quadratic form, arising from the symmetric self-pairing on the adjoint motive $\mathrm{Ad}(M)$. This induces (by restriction) a quadratic form on the source and (identifying with an orthogonal complement) a quadratic form on the target also. We will compute the self-pairing of each term in (4.14) with reference to this form.

- For $\omega_i, \omega_j \in F^1 H_{\mathrm{dR}}$ their pairing in the middle group of (4.13) is
$$[\frac{\omega_i + \overline{\omega_i}}{2}, \frac{\omega_j + \overline{\omega_j}}{2}] = \frac{1}{2}\mathrm{Re}\langle \omega_i, \overline{\omega_j}\rangle = \frac{1}{2}\langle \omega_i, \overline{\omega}_j\rangle, \tag{4.15}$$
This coincides with the Hermitian norm that we have defined previously in (3.1) (up to a power of 2, which does not count as we work up to p-units). Therefore the self-pairing of $[F^1 H_{\mathrm{dR},\mathbf{Z}}]$ gives $\mathrm{height}_K(\pi)^{-2}$.
- In the middle of (4.13), the integral structure is self-dual, and so the self-pairing of $[H^+_B]$ is a p-unit.
- On the right of (4.13), the squared covolume of the integral structure is $(d_0\|\nu\|)^2$. This requires some discussion. The quadratic form on $\mathfrak{a}$ arising from (4.13) is not the same as the metric previously discussed; however we say that the metric on the symmetric space and the invariant bilinear form on $\check{\mathfrak{g}}$ are *compatible* if
$$\text{they induce forms on the determinant of } \mathfrak{a} \text{ which differ by a rational } p\text{-unit} \tag{4.16}$$
If we have compatibility, in this sense, we can compute the self-pairing of $d_0\nu \in \mathfrak{a}$ using the metric on $\mathfrak{a}$ mentioned after (4.2), giving by definition $d_0^2\|\nu\|^2$.

Taken together, these computations show that Bloch-Kato amounts to the equality (4.11). □

4.5. **Proof of the main statement.** We now prove the main result, as enunciated in §4.3, or rather reduce it to a computation related to the action of the derived deformation ring, which we treat in the remainder of the paper.

We choose integrally normalized classes:

$$f \in H^q_{\mathbf{Q}}[\mathfrak{p}], f^\vee \in H^{q+\delta}_{\mathbf{Q}}[\mathfrak{p}] \tag{4.17}$$

so that f is an integral generator, i.e. the image of a generator under $H^q_{\mathrm{tf}}[\mathfrak{p}] \hookrightarrow H^q_{\mathbf{Q}}[\mathfrak{p}]$; and $f^\vee$ is dual to f under Poincaré duality (i.e. $[f, f^\vee] = 1$).

More explicitly, let $\overline{f^\vee} \in H^{q+\delta}_{\mathrm{tf}}/\mathfrak{p}$ be an integral generator. Then $f^\vee$ is, up to a p-unit, the preimage of $\overline{f^\vee}$ under the isomorphism $H^*_{\mathbf{Q}}[\mathfrak{p}] \to H^*_{\mathbf{Q}}/\mathfrak{p}$. To see that this has the desired property under Poincaré duality note that the perfect duality pairing $H^q_{\mathrm{tf}} \times H^{q+\delta}_{\mathrm{tf}} \to \mathbf{Z}_p$ gives rise to a perfect duality pairing of rank one $\mathbf{Z}_p$-modules

$$H^q_{\mathrm{tf}}[\mathfrak{p}] \times H^{q+\delta}_{\mathrm{tf}}/\mathfrak{p} \to \mathbf{Z}_p \tag{4.18}$$

(Too see the second group is free of rank one, note that $H^{q+\delta}_{\mathrm{tf}}/\mathfrak{p}$ is quotient of $H^{q+\delta}/\mathfrak{p} = \mathbf{Z}_p$ by the image of torsion.) Visibly any homomorphism from the latter group to $\mathbf{Z}_p$ comes via pairing with the former, so we get a perfect pairing as claimed.

We claim that, up to p-units,

$$\nu^\vee \cdot f \sim d_0 f^\vee. \tag{4.19}$$

(on the left we have the action of motivic cohomology Λ on cohomology of Y_K, and $\nu^\vee$ is as in (4.4.3) a point in the determinant of Λ, thus shifting degrees by q.) We verify this in the subsequent subsections; the main point is that it amounts to a purely algebraic statement about the derived deformation ring and Galois cohomology. Let us see why (4.19) completes the proof of (4.4).

We may as well choose f (a priori a p-adic cohomology class) to be represented by a rational cohomology class $\tilde{f}$, which is in turn represented by a harmonic form ω. Then we have

$$\langle \tilde{f}, \tilde{f} \rangle = \int_{Y_K} \langle \omega, \omega \rangle d\mathrm{vol} = \int_{Y_K} \omega \wedge (\star \omega),$$

with $\star$ the Hodge star; that is to say $\langle \tilde{f}, \tilde{f} \rangle = [\tilde{f}, \star \tilde{f}]$ where the latter pairing is Poincaré duality. The isometricity of the $\mathfrak{a}^*$ action, mentioned in §4.2, implies that (writing $\|\cdot\|$ for the L^2-norm on cohomology) $\|\nu^\vee \cdot \tilde{f}\| = \|\nu^\vee\| \cdot \|\tilde{f}\|$ and so

$$\nu^\vee \cdot \tilde{f} = \|\nu^\vee\| (\star \tilde{f}), \tag{4.20}$$

up to sign, because both sides lie in the same one-dimensional Hecke eigenspace of real cohomology and have the same norm. Therefore, if we assume the validity of the Bloch-Kato conjecture,

$$\text{automorphic height} = \langle \tilde{f}, \tilde{f} \rangle \overset{(4.20)}{=} \frac{[\tilde{f}, \nu^\vee \cdot \tilde{f}]}{\|\nu^\vee\|} \overset{(4.19)}{\sim} d_0 \|\nu^\vee\|^{-1} \overset{(4.11)}{\sim} \mathrm{height}_K(\pi) L(1, \mathrm{Ad}, \pi).$$

which implies the validity of (4.4). So it remains to explain why (4.19) is valid.

4.6. **Some useful computations in $\mathcal{R}$.** Now $\pi_* \mathcal{R} \otimes_{\mathrm{R}} \mathbf{Q}_p \simeq \wedge^* \mathbf{Q}_p^\delta$ contains a lattice, given by the image of $\pi_* \mathcal{R}$:

$$\mathrm{R}^{\mathrm{tf}}_* := \text{image of } \pi_* \mathcal{R} \to \pi_* \mathcal{R} \otimes_{\mathrm{R}} \mathbf{Q}. \tag{4.21}$$

Of course $\mathrm{R}^{\mathrm{tf}}_*$ need not be an integral exterior algebra and the discrepancy will be important to us later. Clearly, $\mathrm{R}^{\mathrm{tf}}_*$ acts on $H^* \otimes_{\mathrm{R}} \mathbf{Q}_p = H^*_{\mathbf{Q}}/\mathfrak{p} \simeq H^*_{\mathbf{Q}}[\mathfrak{p}]$.

Lemma 4.1. *Let $\overline{f^\vee}$ be, as after* (4.17)*, an integral generator for $H^{q+\delta}_{\mathrm{tf}}/\mathfrak{p}$. Then the map $r \mapsto r\overline{f^\vee}$ gives an isomorphism*

$$\mathrm{R}^{\mathrm{tf}}_* \xrightarrow{\sim} \textit{image of integral cohomology inside } H^*_{\mathbf{Q}}/\mathfrak{p}.$$

where degree k on the left goes to degree $q+\delta-k$ on the right.

With reference to this isomorphism, the lattice $c_0 \mathrm{R}^{\mathrm{tf}}_\delta$ (on the left, in degree δ) is carried to $\mathbf{Z}_p f$ (on the right, in cohomological degree q). Here c_0 is the congruence number for the torsion-free quotient of R*, at the prime ideal $\mathfrak{p}$, that is to say, the index of the image of* $\mathrm{R}^{\mathrm{tf}}[\mathfrak{p}] \to \mathrm{R}^{\mathrm{tf}}/\mathfrak{p} \simeq \mathbf{Z}_p$.

Proof. For the first statement, the map in question goes between free $\mathbf{Z}_p$-modules and is an isomorphism tensored with $\mathbf{Q}$, so it is enough to verify surjectivity. This follows from the fact that $\overline{f^\vee}$ was chosen to be an integral generator of $H^{q+\delta}/\mathfrak{p}$ (and, of course, the assumption from §4.2 that integral homology is free over $\mathcal{R}$).

We verify the second statement. By what we just said, it is enough to show the equality, inside $H^q_{\mathbf{Q}}/\mathfrak{p}$:

$$c_0 \cdot \text{image of } H^q = \mathbf{Z}_p f. \tag{4.22}$$

Consider now the diagram

$$\begin{tikzcd} H^q[\mathfrak{p}] \arrow[r] & (H^q/\mathfrak{p})^{\mathrm{tf}} \\ H^{q+\delta}_{\mathrm{tf}}/\mathfrak{p} & H^{q+\delta}_{\mathrm{tf}}[\mathfrak{p}] \arrow[l] \end{tikzcd} \tag{4.23}$$

of free rank one $\mathbf{Z}_p$-modules. (The upper left group is torsion-free because, otherwise, the localized cohomology group H^{q-1} would be nonzero, contradicting our assumptions.)

We claim that the vertical pairs are in perfect pairing via Poincaré duality, and the horizontal maps are adjoint to one another with reference to this pairing. On the left, the perfect pairing was seen in (4.18), taking account $H^q = H^q_{\mathrm{tf}}$ as just noted. On the right any $\mathbf{Z}_p$-valued functional on $(H^q/\mathfrak{p})^{\mathrm{tf}}$ comes from a functional on $H^q = H^q_{\mathrm{tf}}$ and so arises from some element of $H^{q+\delta}_{\mathrm{tf}}$, necessarily killed by $\mathfrak{p}$, and the result follows by comparing ranks.

Now, $H^{q+\delta}$ is isomorphic, as an R-module, to R itself. It follows that the cokernel of the bottom map has order c_0. It follows that the same is true of the top map also; in particular f (which is an integral generator for $H^q[\mathfrak{p}]$) is carried to c_0 times an integral generator for $(H^q/\mathfrak{p})^{\mathrm{tf}}$. That shows (4.22). $\square$

We will also need:

Lemma 4.2. *Taking the determinant of the isomorphism* (4.1), $\pi_1\mathcal{R}\otimes_R\mathbf{Q}_p = H^1_f(\mathrm{Ad}\,\rho(1))$, *the integral structure generated by the image* $\mathrm{R}_1^{\mathrm{tf}}$ *of* $\pi_1\mathcal{R}$ *on the left is carried to the integral structure generated by* $d_1\nu$.

Proof. This comes taking the long exact sequence of André–Quillen cohomology from the sequence $\mathbf{Z}_p \to \mathcal{R} \to \pi_0\mathcal{R}$ (taking direct limits as appropriate in the pro-simplicial context); with $\mathbf{F}_p$ coefficients the corresponding sequence appears in [11, Corollary 4.4]:

$$\mathrm{Der}^1_{\mathbf{Z}_p}(\mathrm{R}, M) \hookrightarrow \underbrace{\mathrm{Der}^1_{\mathbf{Z}_p}(\mathcal{R}, \mathbf{Z}_p)}_{H^2_f(\mathrm{Ad}\rho_{\mathbf{Z}})} \xrightarrow{\delta} \mathrm{Hom}_{\mathrm{R}}(\pi_1\mathcal{R}, \mathbf{Z}_p) \to \mathrm{Der}^2_{\mathbf{Z}_p}(\mathrm{R}, \mathbf{Z}_p) \to \underbrace{\mathrm{Der}^2_{\mathbf{Z}_p}(\mathcal{R}, \mathbf{Z}_p)}_{0} \tag{4.24}$$

where we use $\mathrm{Der}^2(\mathrm{R}/\mathcal{R}, \mathbf{Z}_p) \simeq \mathrm{Hom}_{\mathrm{R}}(\pi_1\mathcal{R}, \mathbf{Z}_p)$ and also $\mathrm{Der}^1_{W(k)}(\pi_0\mathcal{R}, \mathbf{Q}_p/\mathbf{Z}_p) \simeq \mathrm{Der}^{i+1}_{W(k)}(\pi_0\mathcal{R}, \mathbf{Z}_p)$ to see that the order of the fourth group is d_1. This shows that the image of $H^2_f(\mathrm{Ad}\rho_{\mathbf{Z}})$ in $\mathrm{Hom}_{\mathrm{R}}(\pi_1\mathcal{R}, \mathbf{Z}_p)$ has index d_1, and, dualizing, we obtain the claimed assertion. $\square$

4.7. **Proof of** (4.19)**.** The top exterior power $\det(\pi_1\mathcal{R} \otimes_{\mathrm{R}} \mathbf{Q}_p)$ lowers cohomological degree by δ, i.e. we have

$$H^{q+\delta}_{\mathbf{Q}}[\mathfrak{p}] \otimes \det(\pi_1\mathcal{R} \otimes_{\mathrm{R}} \mathbf{Q}_p) \to H^q[\mathfrak{p}], \tag{4.25}$$

and by duality

$$H^q_{\mathbf{Q}}[\mathfrak{p}] \otimes \det \mathrm{Hom}_{\mathrm{R}}(\pi_1, \mathbf{Q}_p) \simeq H^{q+\delta}_{\mathbf{Q}}[\mathfrak{p}] \tag{4.26}$$

Now we have

$$\mathrm{Hom}_{\mathrm{R}}(\pi_1, \mathbf{Q}_p) \overset{(4.1)}{\simeq} H^2_f(\mathrm{Ad}\rho) \simeq H^1_f(\mathrm{Ad}\rho(1))^*,$$

and therefore the determinant appearing in (4.26) is just the determinant of $H^1_f(\mathrm{Ad}\rho(1))^*$. In particular, $\nu^\vee$ defines a point of this determinant; by the assumed compatibility (§4.2 point (d)) the desired statement (4.19) is then equivalent to the assertion that $f \otimes \nu^\vee$ maps to to $d_0 f^\vee$ under the map (4.26), which is equivalent to the dual assertion:

> *Claim:* $d_0 f^\vee \otimes \nu$ map to f, up to p-units, under (4.25).

By Lemma 4.2 this is equivalent to verifying that under (4.25)

$$f^\vee \otimes \frac{d_0}{d_1}(\det \mathrm{R}_1^{\mathrm{tf}}) = \mathbf{Z}_p f.$$

Writing c_1 for the index of the image of $\wedge^\delta \mathrm{R}_1^{\mathrm{tf}}$ in $\mathrm{R}_\delta^{\mathrm{tf}}$, this is equivalent to verifying that

$$f^\vee \otimes \frac{d_0}{d_1} c_1 \, \mathrm{R}_\delta^{\mathrm{tf}} = \mathbf{Z}_p f.$$

By Lemma 4.1 this is equivalent to $\frac{d_0}{d_1} c_1 = c_0$ up to p-units, that is to say, it follows from the following statement:

Proposition 4.1. *(Fakhruddin–Khare, Proposition A.6 of the Appendix to* [3]*).*

$$\frac{d_0}{d_1} = \frac{c_0}{c_1} \tag{4.27}$$

up to p-units, where

- *c_0 is the congruence number of the* torsion-free quotient *of the ring $\mathrm{R} = \pi_0 \mathcal{R}$, i.e. the index of the image of $\mathrm{R}^{\mathrm{tf}}[\mathfrak{p}] \to \mathrm{R}^{\mathrm{tf}}/\mathfrak{p} \simeq \mathbf{Z}_p$.*
- *c_1 is the index of the image of $\wedge^\delta \mathrm{R}_1^{\mathrm{tf}}$ in $\mathrm{R}_\delta^{\mathrm{tf}}$, where $\mathrm{R}_*^{\mathrm{tf}}$ was the image of $\pi_* \mathcal{R}$ inside the exterior algebra $\pi_* \mathcal{R} \otimes_{\mathrm{R}} \mathbf{Q} \simeq \wedge^* \mathbf{Q}_p^\delta$. (So this measures the deviation from $\mathrm{R}_*^{\mathrm{tf}}$ being an integral exterior algebra.)*
- *d_0, d_1 denote the order of the André-Quillen cohomology groups $\mathrm{Der}^i_{\mathbf{Z}_p}(\mathrm{R}, \mathbf{Q}_p/\mathbf{Z}_p)$.*

In the underived case case $\delta = 0$ this is the equality $c_0 = d_0$ that plays an important role in Wiles' work. I originally conjectured this formula in 2015 after computing a few interesting cases, and a general proof for (4.27) was given shortly after by Fakhruddin–Khare. Some notes on translating the formula [3] to the above situation:

- what we call c_0, c_1 are called c_1, c_2 in [3].
- Our ring $\pi_* \mathcal{R}$ can be presented as the derived quotient of a finite flat complete intersection over $\mathbf{Z}_p$ by δ relations. This can be proved by an argument with prime avoidance. It is in this form that Fakhruddin-Khare analyze the situation.
- In the case $\delta = 1$ the result has been independently formulated and proved by Tilouine and Urban [21, Proposition 2.11]. Here there is no factor c_1.

4.8. **Rephrasing via determinant lines.** The proof that we have given can be formulated very succinctly in terms of determinant lines, as we will now briefly sketch. We follow the notation of §4. But we do not need to assume that $h_q := \dim H^q(Y_K, \mathbf{Q})_\pi$ is equal to 1 any more.

Let Λ be the motivic cohomology group mentioned in §4.2, attached to the adjoint motive. The motivic conjectures recalled in §4.2 imply that that the highest exterior power

$[\Lambda^*]$ of the $\mathbf{Q}$-dual Λ^* indexes an isomorphism between bottom and top cohomological degree:

$$H^q(Y_K, \mathbf{Q})_\pi \otimes [\Lambda^*] \cong H^{q^*}(Y_K, \mathbf{Q})_\pi$$

and now, taking determinants, we get $[H^q(Y_K, \mathbf{Q})_\pi] \cdot [\Lambda^*]^{h_q} \simeq [H^{q^*}(Y_K, \mathbf{Q})_\pi]$ Taking account of Poincaré duality we arrive at an isomorphism of one-dimensional $\mathbf{Q}$-vector spaces:

$$[\Lambda]^{h_q} \simeq [H^q(Y_K, \mathbf{Q})_\pi]^2. \tag{4.28}$$

Now let Ψ be the fundamental line of the adjoint motive, as defined in the previous section; from (4.4.4) we get $\Psi = [\Lambda] \otimes [F^1 H_{\mathrm{dR}}(M)] \otimes (\dots)$ and then

$$[H^q(Y, \mathbf{Q})_\pi]^2 \overset{(4.28)}{\simeq} [\Lambda]^{h_q} \overset{(4.10)}{\simeq} (\dots) \otimes \Psi^{h_q} \otimes [F^1 H_{\mathrm{dR}}]^{-h_q} \tag{4.29}$$

The terms $(\dots)$ do not affect the analysis, being self-dual. The automorphic and arithmetic heights measure the size of integral structures (respectively) the far-left and the far-right $\mathbf{Q}$-lines. Our previous analysis can be summarized in the statements that integral structures and metrics match up under the map above (where we give Ψ the metric coming from the Deligne regulator; this is what gives rise to the factor $L(1, \mathrm{Ad}, \pi)$ in the height comparison). It seems to me that one can treat this conceptually assuming a satisfactory theory of coherent duality for derived rings and this also gives another approach to (4.27), but I don't know a published reference for such a theory.

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