

ENTROPY BOUNDS AND QUANTUM UNIQUE ERGODICITY FOR HECKE EIGENFUNCTIONS ON DIVISION ALGEBRAS

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ABSTRACT. We prove the arithmetic quantum unique ergodicity (AQUE) conjecture for non-degenerate sequences of Hecke eigenfunctions on quotients $\Gamma \backslash G/K$, where $G \simeq \mathrm{PGL}_d(\mathbb{R})$, K is a maximal compact subgroup of G and $\Gamma < G$ is a lattice associated to a division algebra over \mathbb{Q} of prime degree d .

More generally, we introduce a new method of proving positive entropy of quantum limits, which applies to higher-rank groups. The result on AQUE is obtained by combining this with a measure-rigidity theorem due to Einsiedler-Katok, following a strategy first pioneered by Lindenstrauss.

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1. INTRODUCTION

1.1. **Result.** In this paper, we shall show (a slightly sharper version of) the following statement. For precise definitions we refer to §6 especially §6.2.

Theorem 1.1. *Let Γ be a lattice in $\mathrm{PGL}_d(\mathbb{R})$, with d prime, associated to a division algebra¹, and ψ_i a non-degenerate sequence of Hecke-Maass eigenfunctions on $Y := \Gamma \backslash \mathrm{PGL}_d(\mathbb{R})/\mathrm{PO}_d(\mathbb{R})$, normalized to have L^2 -norm 1 w.r.t. the Riemannian volume $d\mathrm{vol}$.*

Then the measures $|\psi_i|^2 d\mathrm{vol}$ converge weakly to the Haar measure, i.e. for any $f \in C(Y)$,

$$(1.1) \quad \lim_{i \rightarrow \infty} \int_Y |\psi_i|^2 f d\mathrm{vol} = \int_Y f d\mathrm{vol}.$$

¹ This means that Γ is the image of \mathcal{O}^\times in $\mathrm{PGL}_d(\mathbb{R})$, where \mathcal{O} is an order in a \mathbb{Q} -division algebra so that $\mathcal{O} \otimes \mathbb{R} = M_d(\mathbb{R})$. We also impose a class number one condition, see §6.

In words, Theorem 1.1 asserts that the eigenfunctions ψ_i become *equidistributed* – that they do not cluster too much on the manifold Y .

This theorem is a contribution to the study of the “Arithmetic Quantum Unique Ergodicity” problem. A detailed introduction to this problem may be found in our paper [20]. While it is hard to dispute that the spaces Y are far too special to provide a reasonable model for the physical problem of “quantum chaos,” both the statement (1.1) and the techniques we use to prove it seem to the authors to be of interest because of the scarcity of results concerning analysis of higher rank automorphic forms. In particular, we believe our techniques will find applications to analytic problems beside QUE.

Our strategy follows that of Lindenstrauss in his proof of the arithmetic QUE conjecture for quotients of the hyperbolic plane (the case $d = 2$ of the Theorem above) and has three conceptually distinct steps. Let $A \subset \mathrm{PGL}_d(\mathbb{R})$ be the subgroup of diagonal matrices.

- (1) *Microlocal lift*: notation as above, any weak limit (as $i \rightarrow \infty$) of $|\psi_i|^2$ dvol may be lifted to an A -invariant measure σ_∞ on $X := \Gamma \backslash \mathrm{PGL}_d(\mathbb{R})$, in a way compatible with the Hecke correspondence.
- (2) *Mass of small tubes*: If σ_∞ is as in (1), then the σ_∞ -mass of an ϵ -ball in $\Gamma \backslash \mathrm{PGL}_d(\mathbb{R})$ is $\ll \epsilon^{d-1+\delta}$, for some $\delta > 0$; note that the bound $\ll \epsilon^{d-1}$ is trivial from the A -invariance.²
- (3) *Measure rigidity*: Any A -invariant measure satisfying the auxiliary condition prescribed by (2) must necessarily be a convex combination of algebraic measures. In our setting (Γ associated to a division algebra of prime degree) this means it must be Haar measure.

* * *

In the context of Lindenstrauss’ proof, the analogues of steps 2 and 3 are due, respectively, to Bourgain–Lindenstrauss [1] and Lindenstrauss [11]. The analogue of step 1 is due to [9] (based on constructions of Schnirel’man [21], Zelditch [26, 27], Colin de Verdière [4] and Wolpert [24]).

We shall concern ourselves with the higher rank case ($d > 2$), where step 1 – under a nondegeneracy condition – has been established by the authors in [20], while step 3 was established by Einsiedler–Katok–Lindenstrauss in [6].

The contribution of the present paper is then the establishment of step 2.

1.2. Bounding mass of tubes – vague discussion. As was discussed in the previous Section, the main point of the present paper is to prove upper bounds for the mass of eigenfunctions in small tubes.

A *correspondence* on a manifold X , for our purposes, will be a subset $S \subset X \times X$ such that both projections are topological coverings. Such a correspondence induces an endomorphism of $L^2(X)$: pull back to Y and push forward to X . We also think of a correspondence as a “multi-valued” or “set-valued” function h_S from X to X . In the latter view a correspondence induces a natural convolution action on functions on X , given by $(f * h_S)(x) = \sum_{y \in h_S(x)} f(y)$.

Two correspondences can be composed in a natural way and resulting algebra is, in general, non-commutative. However, the manifolds of interest to us ($X = \Gamma \backslash \mathbf{G}$ with Γ an arithmetic lattice in the semisimple Lie group G) come equipped with

²This asserts, then, that σ_∞ has some “thickness” transverse to the A -direction; for instance, it immediately implies that the dimension of the support of σ_∞ is strictly larger than $d - 1$.

a large algebra of *commuting* correspondences, the Hecke algebra \mathcal{H} , which acts on $L^2(X)$ by normal operators. We will be interested in possible concentration of simultaneous eigenfunctions of the Hecke algebra.

As a concrete example, for $X = \mathrm{PGL}_d(\mathbb{Z}) \backslash \mathrm{PGL}_d(\mathbb{R})$ the Hecke correspondences are induced by *left* multiplication with $\mathrm{PGL}_d(\mathbb{Q})$: given $\gamma \in \mathrm{PGL}_d(\mathbb{Q})$ and a coset $x \in X$, we consider the set of products γg as g varies over representatives in $\mathrm{PGL}_d(\mathbb{R})$ for x . It turns out that these products generate a finite set of cosets $h_\gamma(x) \subset X$. It is easy to check that the adjoint of h_γ is $h_{\gamma^{-1}}$, but the commutativity of the Hecke algebra is more subtle. An important feature of the Hecke correspondences on X is their equivariance w.r.t. the action of $G = \mathrm{PGL}_d(\mathbb{R})$ on X on the right.

Returning to the general $X := \Gamma \backslash G$, let $\mathcal{T}(\epsilon)$ be a small subset of G , with its size in certain directions on the order of ϵ (for us $\mathcal{T}(\epsilon)$ will be a tube of width ϵ around a compact piece of a Levi subgroup of G). Our goal will be to prove a statement of the following type, for some fixed $\eta > 0$ depending only on G :

$$(1.2) \quad \text{Each } \mathcal{H}\text{-eigenfunction } \psi \in L^2(X) \text{ satisfies } \mu_\psi(x\mathcal{T}(\epsilon)) \ll \epsilon^\eta.$$

Here $\mu_\psi := |\psi|^2 d\mathrm{vol}$ is the product of the $\mathrm{PGL}_d(\mathbb{R})$ -invariant measure and the function $|\psi|^2$, normalized to be a probability measure. (1.2) asserts that the eigenfunction ψ cannot concentrate on a small tube. This is proven, in the cases of interest for this paper, in Theorem 5.3.

We will sketch here our approach to the proof. A basic form of the idea appeared in the paper [17] of Rudnick and Sarnak. If ψ is an eigenfunction of a correspondence $h \in \mathcal{H}$, and ψ were large at some point x , it also tends to be quite large at points belonging to the orbit $h.x$. We can thereby “disperse” the local question of bounding the mass of a small tube to a global question about the size of ψ throughout the manifold.

Say that the image of the point x under $h \in \mathcal{H}$ is the collection of N points $h.x = \{x_i\}$. Equivariance implies that the image of the tube $x\mathcal{T}(\epsilon)$ under h is the collection of tubes $\{x_i\mathcal{T}(\epsilon)\}$. For any $t \in \mathcal{T}(\epsilon)$, we have, then

$$\lambda_h \psi(xt) = \sum_{i=1}^N \psi(x_i t),$$

where λ_h is so that $h.\psi = \lambda_h \psi$.

Squaring, applying Cauchy–Schwarz and integrating over $t \in \mathcal{T}(\epsilon)$ gives:

$$(1.3) \quad \begin{aligned} \mu_\psi(x\mathcal{T}(\epsilon)) &\leq \frac{N}{|\lambda_h|^2} \sum_{i=1}^N \mu_\psi(x_i\mathcal{T}(\epsilon)) \\ &\leq \frac{N}{|\lambda_h|^2} \max_i \# \{j | x_i\mathcal{T}(\epsilon) \cap x_j\mathcal{T}(\epsilon) \neq \emptyset\}. \end{aligned}$$

If the tubes $x_i\mathcal{T}(\epsilon)$ are *disjoint*³ and, furthermore, $|\lambda_h|$ is “large” (w.r.t. N), (1.3) yields a good upper bound for $\mu_\psi(x\mathcal{T}(\epsilon))$.

The issue of choosing h so that λ_h is “large” turns out to be relatively minor. The solution is given in the appendix.

³ It suffices for the number of tubes intersecting a given one to be uniformly bounded independently of ϵ .

The more serious problem is that the tubes $x_i\mathcal{T}(\epsilon)$ might not be disjoint. It must be emphasized that this issue is not a technical artifact of the proof but related fundamentally to the analytic properties of eigenfunctions on arithmetic locally symmetric spaces: “returns” of the Hecke correspondence influence the sizes of eigenfunctions. For an instance of this phenomenon see the Rudnick–Sarnak example of a sequence of eigenfunctions on a hyperbolic 3-manifold with large L^∞ -norms and the more recent work of Milicevic [15, 16].

Our approach to this difficulty is as follows: we prove a variant of (1.3) where the “worst-case” intersection number (i.e. \max_j) is replaced by an *average* intersection number (an average over j). This variant is presented in Lemma 3.4. The rest of the paper is then devoted to giving upper bounds for this average intersection number, which turns out to be much easier than bounding the worst-case intersection number.

Remark 1.2. Let us contrast this approach to prior work. An alternate idea would be to choose a subset of translates $\{x_i\}$ for which we can prove an analogue of (1.3) and such that the tubes $x_i\mathcal{T}(\epsilon)$ are disjoint. Versions of this were used in the prior work of Rudnick–Sarnak and Bourgain–Lindenstrauss, with the quantitative version of Bourgain–Lindenstrauss requiring a sieving argument to find non-intersecting correspondences. Our original proof was based on a further refinement of this technique, which avoided sieves entirely by using some geometry of buildings. A presentation of that proof may be found in the PhD thesis of the first author, [18]. The technique of this paper seems to us to be yet more streamlined.

Remark 1.3. In order to “disperse” the eigenfunction we require the use of Hecke operators at many primes. The recent work [2] shows that in the case of hyperbolic surfaces, it suffices to use the Hecke operators at a single place. Generalizing that result to higher rank would be an interesting problem.

1.3. Spectrum of quotients. Significance of division algebras. More generally, the second technique can be interpreted as an implementation of the following philosophy, related to the work of Burger and Sarnak:

The analytic behavior of Hecke eigenfunctions on $\Gamma\backslash G$ along orbits of a subgroup $H \subset G$ is controlled by the spectrum of quotients $L^2(G_p/H_p)$.

Here G_p is the p -adic group corresponding to G , and $H_p \subset G_p$ is a p -adic Lie subgroup “corresponding” to H in a suitable sense. In the main situation of this paper, $G_p = \mathrm{PGL}_d(\mathbb{Q}_p)$ for almost all p , H will be a real Levi subgroup, and H_p will be a torus.

In this context, the possibilities for the subgroup H_p that can occur are closely related to the \mathbb{Q} -structure of the group underlying G . In general, the fewer \mathbb{Q} -subgroups \mathbf{G} has, the fewer the possibilities for H_p . For this reason we can only reach Theorem 1.1 for quotients $\Gamma\backslash G$ arising from division algebras of prime rank: the corresponding \mathbb{Q} -groups have very few subgroups. As one passes to general $\Gamma\backslash G$, the possibilities for H_p become wilder, and eventually the methods of this paper do not seem to give much information.

1.4. Organization of this paper. In Section 2 we describe our setup in the general setting of algebraic groups. Further notation regarding our special case of division algebras is discussed in Section 4.4.

Section 3 contains the derivation of our first technical result, Lemma 3.4, giving a bound for the integral on a small set of the squared modulus of an eigenfunction

of an equivariant correspondence. This is a version of (1.3) which can be used for non-disjoint translates. The bound we obtain depends on the average multiplicity of intersection among the translates of the tube as well as on covering properties of the tubes (easily understood in natural applications).

In Section 4 we define the kind of tubes we shall be interested in and study the intersection patterns of their translates by elements of the Hecke algebra. We give two treatments of the analysis, one that is applicable to general \mathbb{R} -split groups, and another, more concrete, specific to division algebras of prime degree. In both cases we use a diophantine argument to show that under suitable hypotheses the intersection pattern is controlled by a torus in the underlying \mathbb{Q} -algebraic group.

Section 5 then obtains the desired power-law decay of the mass of small tubes. The considerations of this section are again fairly general.

Finally in Section 6 we recall our previous result (“step 1” of the strategy) and prove our main Theorems.

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2. NOTATION

We shall specify here the “general” notation to be used throughout the paper. Later sections (after Section 3) will use, in addition to these notations, certain further setup about division algebras. This will be explained in §4.4.

Let \mathbf{G} be a semisimple group over \mathbb{Q} . We choose an embedding $\rho : \mathbf{G} \rightarrow \mathrm{SL}_n$. Let $G = \mathbf{G}(\mathbb{R})$, $G = NAK$ a Cartan decomposition. Set $\mathbf{G}(\mathbb{Z}_p) = \rho^{-1}(\mathrm{SL}_n(\mathbb{Z}_p))$, and let $K_f \subset \mathbf{G}(\mathbb{A}_f)$ be an open compact subgroup contained in $\prod_p \mathbf{G}(\mathbb{Z}_p)$. We set $X = \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_f$, $Y = X/K$. We let $dvol$ be the natural probability measures on either X or Y : in both cases, the projection of the $\mathbf{G}(\mathbb{A})$ -invariant probability measure on $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})$. We shall sometimes denote $dvol_X$ by dx and $dvol_Y$ by dy . We also normalize the Haar measure on K to be a probability measure.

For any subset $B \subset G$ or $B \subset \mathbf{G}(\mathbb{A})$, we denote by \overline{B} the image of B in X under the natural projections $G \rightarrow X$ and $\mathbf{G}(\mathbb{A}) \rightarrow X$.

We say that a prime p is *good* if $\mathbf{G}(\mathbb{Q}_p)$ is unramified and $K_p = \mathbf{G}(\mathbb{Z}_p) \subset \mathbf{G}(\mathbb{Q}_p)$ is a good maximal compact subgroup which is contained in K_f (via the natural embedding $\mathbf{G}(\mathbb{Q}_p) \hookrightarrow \mathbf{G}(\mathbb{A}_f)$). Then all but finitely many p are *good*. If p is not good, we say it is *bad*.

We denote by \mathcal{H} the Hecke algebra of $\prod_{p \text{ good}} K_p$ bi-invariant functions on $\prod_{p \text{ good}} \mathbf{G}(\mathbb{Q}_p)$. It forms a commutative algebra under convolution. It acts in a natural way on functions on X . We can identify an element of \mathcal{H} with a K_f -invariant

function on $\mathbf{G}(\mathbb{A}_f)/K_f$ which is, moreover, supported on $K_f \cdot \prod_{p \text{ good}} \mathbf{G}(\mathbb{Q}_p)$. We shall abbreviate the latter condition to: *supported at good primes*.

If $\mathbf{H} \subset \mathbf{G}$ is a semisimple \mathbb{Q} -subgroup, we say a prime p is \mathbf{H} -good if it is good for \mathbf{G} and moreover $\mathbf{H}(\mathbb{Q}_p) \cap K_p$ is a maximal compact subgroup of $\mathbf{H}(\mathbb{Q}_p)$.

We fix compact subsets $\Omega_\infty \subset G, \Omega \subset \mathbf{G}(\mathbb{A})$, and let $X_1 \subset X$ denote the (compact) projection of Ω to X .

Definition 2.1. We call $\psi \in L^2(X)$ a *Hecke eigenfunction* if it is a joint eigenfunction of the Hecke algebra \mathcal{H} . We set $\mu_\psi = c|\psi|^2 d\text{vol}$, where the constant c is chosen so that μ_ψ is a probability measure on X .

In §3 we will deal with an abstract \mathbf{G} as above. In §4 — §6 we sometimes specialize to the case of \mathbf{G} arising from a division algebra of prime rank, D . The extra notation necessary for this specialization will be set up in §4.4.

Notational convention: We shall allow the implicit constants in the notation \ll and $O(\cdot)$ to depend on \mathbf{G} and the data $\rho, K_f, \Omega, \Omega_\infty$ *without explicit indication*. In other words, the notation $A \ll B$ means that there exists a constant c , which may depend on all the data specified in this section, so that $A \leq cB$. Later we introduce further data concerning division algebras, and, as we specify there, we shall allow implicit constants after that point to depend on these extra data also.

3. BOUNDS ON THE MASS OF TUBES

3.1. A covering argument. Let $B_0 \subset \Omega_\infty$ be an open set containing the identity. We set

$$(3.1) \quad B := B_0 \cdot B_0^{-1}, B_2 = B \cdot B, B_3 = B \cdot B \cdot B, \dots$$

In this section, we shall discuss estimating from above $\mu_\psi(\overline{xB})$ for $x \in G$.⁴ We will repeatedly use the following fact. If $S \subset G$ is a compact subset and $x \in X_1$ is arbitrary, the cardinality of the fibers of the map $S \mapsto X$ defined by $s \mapsto x.s$ is bounded in a fashion that depends on S but not on x . This is an immediate consequence of the compactness of X_1 : indeed, there exists an open neighbourhood U of the identity in G so that $u \in U \mapsto xu \in X$ is injective for any $x \in X_1$. In particular, we may apply this remark when $S = B, B_2, B_3$. The size of the relevant fibers will therefore be bounded by a constant depending on Ω .

Lemma 3.1. *There exists a covering of X by translates $x_\alpha B$ so that any set zB , for $z \in X$, intersects at most $\frac{\text{vol} B_3}{\text{vol} B_0}$ of the $x_\alpha B$.*

Proof. Choose a maximal subset $\{x_\alpha\} \subset X$ with the property that $x_\alpha B_0$ are disjoint.

Given $z \in X$, we must have $zB_0 \cap x_\alpha B_0 \neq \emptyset$ for some α . This means precisely that $X = \bigcup x_\alpha B$.

Next, fix $z \in X$. For any α so that $x_\alpha B \cap zB \neq \emptyset$, we may choose $\varpi_\alpha \in B \cdot B^{-1}$ so that $x_\alpha = z\varpi_\alpha$. Then the sets $z\varpi_\alpha B_0 \subset X$ are all disjoint.

A necessary condition for this is that the sets $\varpi_\alpha B_0$, considered as subsets of G , are disjoint. Since each $\varpi_\alpha B_0$ belongs to B_3 , their number is bounded by $\frac{\text{vol} B_3}{\text{vol} B_0}$. \square

⁴In fact, all our estimates really work for $\mu_\psi(xB)$ for $x \in X$, as the notation is meant to suggest. We prove our results only for $x \in G$ simply to keep notation to a minimum. If G acts with a single orbit on X , as we assume in our final applications anyway, there is no difference at all.

Lemma 3.2. *Let ν be a probability measure on X and $y_1, \dots, y_r \in X$.*

Then

$$\sum_{i=1}^r \nu(y_i B)^{1/2} \ll \frac{\text{vol} B_3}{\text{vol} B_0} (\#\{(i, j) : y_i B_2 \cap y_j B_2 \neq \emptyset\})^{1/2}$$

Proof. Choose a collection x_α as in Lemma 3.1. Each set $y_i B$ is covered by at most $\frac{\text{vol} B_3}{\text{vol} B_0}$ sets $x_\alpha B$. Clearly $\nu(y_i B)^{1/2} \leq \sum_{\alpha: x_\alpha B \cap y_i B \neq \emptyset} \nu(x_\alpha B)^{1/2}$, and so

$$(3.2) \quad \left(\sum_i \nu(y_i B)^{1/2} \right)^2 \leq \left(\sum_\alpha \nu(x_\alpha B)^{1/2} \sum_{i: y_i B \cap x_\alpha B \neq \emptyset} 1 \right)^2 \\ \leq \sum_\alpha \nu(x_\alpha B) \sum_\alpha \left(\sum_{i: y_i B \cap x_\alpha B \neq \emptyset} 1 \right)^2$$

Now, $\sum_\alpha \nu(x_\alpha B) \leq \frac{\text{vol} B_3}{\text{vol} B_0}$ because each $z \in X$ belongs to at most $\frac{\text{vol} B_3}{\text{vol} B_0}$ of the $x_\alpha B$.

Moreover,

$$\sum_\alpha \left(\sum_{i: y_i B \cap x_\alpha B \neq \emptyset} 1 \right)^2 = \#\{i, j, \alpha : y_i B \cap x_\alpha B \neq \emptyset, y_j B \cap x_\alpha B \neq \emptyset\}$$

For given i, j , the pertinent set of α is nonzero only if $y_i B_2 \cap y_j B_2 \neq \emptyset$. If it is nonempty, it has size at most $\text{vol}(B_3)/\text{vol}(B_0)$. \square

3.2. General bound. Let ψ be a Hecke eigenfunction on X , μ_ψ the associated probability measure.

Let $h \in \mathcal{H}$, which we can think of as a function $s \mapsto h_s$ on $\mathbf{G}(\mathbb{A}_f)/K_f$, supported on good primes. Let $S \subset \mathbf{G}(\mathbb{A}_f)/K_f$ be the support of h .

Because ψ is a Hecke eigenfunction, there is $\Lambda_h \in \mathbb{C}$ so that:

$$(3.3) \quad \Lambda_h \psi(x) = \sum_{s \in S} h_s \psi(x.s) \quad (x \in G)$$

Note that $x.s \in \mathbf{G}(\mathbb{A})/K_f$ and therefore $\psi(x.s)$ makes sense. The following Lemma bounds $\mu_\psi(\overline{xB})$ in terms of the average mass of certain Hecke translates of $\overline{xB_2}$.

Lemma 3.3. *Let $x \in X_1$. Then*

$$\mu_\psi(\overline{xB}) \ll \frac{(\sum_{s \in S} |h_s| \mu_\psi(\overline{xB}.s)^{1/2})^2}{|\Lambda_h|^2}$$

Proof. This follows by squaring out equation (3.3), integrating over B and applying Cauchy-Schwarz. We use the fact that $b \in B \mapsto x.s.b \in X$ has fibers whose cardinality is bounded in terms of Ω_∞ (see discussion in §3.1). \square

The next Lemma clarifies that the only necessary input to bound $\mu_\psi(\overline{xB})$ is an estimate for the average intersection multiplicity of Hecke translates of \overline{xB} .

Lemma 3.4. *Suppose that h is supported on $S \subset \mathbf{G}(\mathbb{A}_f)/K_f$ and $|h| \leq 1$.*

Then for $x \in G$

$$(3.4) \quad \mu_\psi(\overline{xB}) \ll \left(\frac{\text{vol}B_3}{\text{vol}B_0} \right)^2 |\Lambda_h|^{-2} (\#\{s, s' \in S : \overline{xB_2s} \cap \overline{xB_2s'} \neq \emptyset\})$$

Proof. This follows from the prior Lemma and Lemma 3.2 (applied to the set $\{y_i\} = \{xs : s \in S\} \subset X$). \square

Lemma 3.4 is our key technical Lemma. Setting $N = \#S$ as in the introduction we may rewrite the right-hand-side as:

$$\left(\frac{\text{vol}B_3}{\text{vol}B_0} \right)^2 \frac{N}{|\Lambda_h|^2} \cdot \frac{1}{N} \sum_{s \in S} \#\{s' \in S : \overline{xB_2s} \cap \overline{xB_2s'} \neq \emptyset\}.$$

This expression should be compared with the right-hand-side of equation (1.3). The key difference is that the bound now only depends on an “average” intersection number of Hecke translates (the average over s of the number of s' such that the translates by s and s' intersect), whereas the bound in (1.3) depended on a “worst case” intersection number (the supremum over s of the number of such s'). Our antecedent [1] relied on controlling this latter quantity, which imposed greater restrictions on the use of Hecke operators.

4. DIOPHANTINE LEMMATA

Recall that we have fixed a maximal \mathbb{R} -split torus $A \subset G$. Given a nontrivial $a \in A$, we fix a compact neighbourhood of the identity $C \subset Z_G(a)$ in the centralizer a inside G (the choice is immaterial; we will later take C to be small enough). Now let $B = B(C, \varepsilon)$ be an ε -neighbourhood of C inside G (“a tube around a piece of a Levi subgroup”). We intend to bound the μ_ψ -mass of sets of the form $\overline{xB(C, \varepsilon)} \subset X$.

In the previous Section we saw that such bounds require control on the intersection pattern of translates of these sets. This Section is devoted to two results giving this control.

We first analyze the case where \mathbf{G} is \mathbb{R} -split and $a \in A$ is *regular*, that is when $Z_G(a)$ is a maximal torus. It turns out that the intersection is controlled by a \mathbb{Q} -subtorus of \mathbf{G} . Roughly, we show that all the $\gamma \in \mathbf{G}(\mathbb{Q})$ that lie “very close” to an \mathbb{R} -torus (where “very close” really means “very close relative to the denominator of γ ”) must all lie on a single \mathbb{Q} -torus. Using this we will show that the intersection pattern of Hecke translates of tubes of this type are controlled by the torus.

Secondly, more control is possible in the simplest case: when \mathbf{G} arises from a division algebra of prime degree. In that case \mathbf{G} has very few \mathbb{Q} -subgroups and one may remove the regularity assumption: elements close to any Levi subgroup (a subgroup of the form $Z_G(a)$) must lie on a \mathbb{Q} -torus.

An argument of this type (in the case $\dim_{\mathbb{Q}} D = 4$) already appears in [1]; this argument does not suffice for the higher rank case, however, because [1] uses commutativity of the relevant Levi subgroups in an important way. Also, it is somewhat awkward in the case of algebras which are not division algebras. However, this approach is more transparent. Our second argument takes this *ad hoc* approach, and is a generalization specific to the case of division algebras.

Remark 4.1. Here is a toy model of the type of reasoning we use: Let ℓ be a line segment in \mathbb{R}^2 of length 1. Suppose $P_i = (x_i, y_i)$, for $1 \leq i \leq 3$, are points in \mathbb{R}^2 with rational coordinates all of which lie within ε of ℓ , and let M be an upper

bound for the denominators of all x_i, y_i . Then, if $\varepsilon < \frac{1}{10}M^{-6}$, the P_i are themselves co-linear. Indeed, the area of the triangle formed by P_1, P_2, P_3 is a rational number with denominator $\leq 2M^6$. On the other hand the area of this triangle is $\leq 2\varepsilon$, whence the conclusion.

Remark 4.2. Recall that we have fixed compact subsets $\Omega \subset \mathbf{G}(\mathbb{A})$ and $\Omega_\infty \subset G$. We will later take C and ε small enough to ensure that $B(C, \varepsilon) \subset \Omega_\infty$. In particular, $C \subset \Omega_\infty$. Consequently, in view of the convention discussed in §2, we shall not explicitly indicate that implicit constants in \ll or $O(\dots)$ depend on C .

4.1. Denominators on an adelic group. As in the toy example, in our diophantine analysis it is convenient to use a notion of *denominator* rather than a notion of height. We define this notion progressively for scalars, for elements of $\mathrm{SL}_n(\mathbb{A}_f)$, and then for elements of $\mathbf{G}(\mathbb{A}_f)$. We write \mathbb{N} for the set of natural numbers, where our denominator function will be valued.

Given $x \in \mathbb{Q}_p$ we let $d(x)$ denote the minimal non-negative power of p such that $d(x) \cdot x \in \mathbb{Z}_p$. In other words, p -adic integers have no denominator while a p -adic number of the form u/p^e , $u \in \mathbb{Z}_p^\times$, has denominator p^e . It is easy to check that the function $d: \mathbb{Q}_p \rightarrow \mathbb{N}$ is invariant under translation by \mathbb{Z}_p and is hence uniformly continuous. Next, for $x = (x_p)_{p < \infty} \in \mathbb{A}_f$ we set $d(x) = \prod_{p < \infty} d(x_p)$ (almost all the factors are equal to 1). This gives a locally constant function $d: \mathbb{A}_f \rightarrow \mathbb{N}$. The diagonal embedding of \mathbb{Q} in \mathbb{A}_f allows us to restrict d to \mathbb{Q} and it is easy to check that if $a, b \in \mathbb{Z}$ are relatively prime with $b > 0$ then $d(\frac{a}{b}) = b$. In other words, the restriction is the usual notion of the denominator of a rational number.

For $g = (g_p) \in \mathrm{SL}_n(\mathbb{A}_f)$ set $d(g)$ to be the least common multiple of the denominators of the matrix entries. For $k_p \in \mathrm{SL}_n(\mathbb{Z}_p)$, or more generally for $k \in \prod_p \mathrm{SL}_n(\mathbb{Z}_p)$ we then have $d(k_p) = d(k) = 1$, and in fact $d: \mathrm{SL}_n(\mathbb{A}_f) \rightarrow \mathbb{N}$ is left- and right-invariant by $\prod_p \mathrm{SL}_n(\mathbb{Z}_p)$.

Finally, for $g \in \mathbf{G}(\mathbb{A}_f)$, we set $d(g) = d(\rho(g))$. This function is then bi- K_f -invariant. By restriction this induces a notion of denominator for elements $\gamma \in \mathbf{G}(\mathbb{Q})$, the least common multiple of the denominators of the matrix elements of $\rho(\gamma) \in \mathrm{SL}_n(\mathbb{Q})$.

For the infinite place we fix a left-invariant Riemannian metric dist on $\mathrm{SL}_n(\mathbb{R})$. Pulling the metric tensor back via ρ induces a left-invariant Riemannian metric dist_G on G . Since it maps geodesics on G to curves of the same length in $\mathrm{SL}_n(\mathbb{R})$, ρ is a nonexpansive map with respect to these metrics.

The “ ε neighbourhoods” $B(C, \varepsilon)$ are taken with respect to the metric dist_G .

Lemma 4.3. *Let $g, g' \in \mathbf{G}(\mathbb{A}_f)$. Then $d(gg') \leq d(g)d(g')$ and $d(g^{-1}) \leq d(g)^{n-1}$.*

Proof. Replacing g, g' by their images under ρ we may assume $g, g' \in \mathrm{SL}_n(\mathbb{A}_f)$. For the first claim we use an alternate characterization of $d(g)$: it is the smallest positive integer m for which $m \cdot g \in \mathrm{GL}_n(\prod_p \mathbb{Z}_p)$. This implies

$$d(g)d(g')gg' \in \mathrm{GL}_n\left(\prod_p \mathbb{Z}_p\right),$$

and the claim follows. For the second note that the matrix entries of g^{-1} are polynomials of degree $n - 1$ and integer coefficients in the matrix entries of g . \square

4.2. A Diophantine Lemma for \mathbb{R} -split groups. Assume now that \mathbf{G} is \mathbb{R} -split, and let $a \in A$ be a *regular* element. In that case $Z_G(a) = MA$ is a maximal torus in $\mathbf{G}(\mathbb{R})$, and $\rho(MA)$ is an \mathbb{R} -split torus in $\mathrm{SL}_n(\mathbb{R})$.

The following is the basic diophantine result.

Lemma 4.4. *For $c > 0$ sufficiently large (in fact, depending only on \mathbf{G}, ρ), and $c' > 0$ sufficiently small (depending on $\mathbf{G}, \rho, A, \Omega_\infty$), for any $g \in \Omega_\infty$ the set of $\gamma \in \mathbf{G}(\mathbb{Q})$ such that*

$$(4.1) \quad \inf\{\mathrm{dist}_G(\gamma, t) \mid t \in g(MA \cap \Omega_\infty \Omega_\infty^{-1})g^{-1}\} \leq \varepsilon, \quad d(\gamma) \leq M$$

is contained in a \mathbb{Q} -subtorus $\mathbf{T} \subset \mathbf{G}$, provided that

$$(4.2) \quad \varepsilon M^c \leq c'$$

Let S be the set of γ defined in equation (4.1). We establish two preparatory results before the Lemma itself. The first is based on the analysis of the case of division algebras (see Lemma 4.9) where the idea is more fully exploited.

Lemma 4.5 (enough to check subsets of fixed size). *A set $S \subset \mathbf{G}(\mathbb{Q})$ is contained in a \mathbb{Q} -torus iff the same holds for every subset S' of S of size $n + 1$.*

Proof. For each S' of size $n + 1$ let $A_{S'} \subset M_n(\mathbb{C})$ be the \mathbb{C} -algebra generated in $M_n(\mathbb{C})$ by the image set $\rho(S')$, and choose S' such that $A' = A_{S'_0}$ has maximal dimension (as a vector space over \mathbb{C}).

Since $\rho(S')$ is contained in a subtorus of SL_n , the subalgebra A' is conjugate to a subalgebra of the algebra of diagonal matrices. In particular, it has dimension at most n . In a proper containment of \mathbb{C} -algebras, the two algebras have different dimensions. Thus there exists a subset $S_0 \subset S'$ of size at most n such that $A_{S_0} = A'$.

Further, for any $\gamma \in S$, $A_{S_0 \cup \{\rho(\gamma)\}}$ contains A' and has at most the same dimension. It follows that they are equal, and hence that $A_S = A'$. Let \mathbf{T} be the intersection with \mathbf{G} of the \mathbb{Q} -subalgebra of $M_n(\mathbb{Q})$ generated by $\rho(S)$. This is a \mathbb{Q} -subgroup of \mathbf{G} containing S . This subgroup is contained in A' . That \mathbf{T} is a torus follows from the fact that its \mathbb{C} -points are contained in A' , hence conjugate to a set of diagonal matrices. \square

Lemma 4.6 (checking can be done algebraically). *There exists (finite) sets of polynomials $P_{ij}, Q_{ik} \in \mathbb{Z}[x_1, \dots, x_{(n+1)n^2}]$ (to be thought of functions of $n + 1$ matrices of size n), such that $\{\gamma_l\}_{l=1}^{n+1} \subset \mathbf{G}(\mathbb{Q})$ all lie on a \mathbb{Q} -torus iff for some i , $P_{ij}(\rho(\vec{\gamma})) = 0$ and $Q_{ik}(\rho(\vec{\gamma})) \neq 0$ hold for all j, k .*

Proof. The analysis of the previous Lemma shows that the desired statement about $\rho(\vec{\gamma})$ is equivalent to the following formula in the language of fields:

There exist matrices $D, A \in \mathrm{GL}_n(\mathbb{C})$ such that D is diagonal with distinct entries and such that each $\rho(\gamma_k)$ commutes with ADA^{-1} .

By Chevalley's Theorem on elimination of quantifiers for the theory of algebraically closed field, this formula is equivalent (over \mathbb{C}) to a one without quantifiers, which we may assume to be in the normal form

$$\bigvee_i \left(\bigwedge_j (P_{ij}(\rho(\vec{\gamma})) = 0) \bigwedge_k (Q_{ik}(\rho(\vec{\gamma})) \neq 0) \right)$$

We reiterate the point that the γ_k generate a \mathbb{Q} -torus iff they generate a torus over \mathbb{C} , and that the language of algebraically closed fields has no names for field elements other than 0, 1 so that atomic formulas in it are equalities and inequalities of rational polynomials, which without loss of generality may be assumed integral. \square

Proof of the Diophantine Lemma 4.4. We note first that the polynomials P_{ij}, Q_{ik} were only constructed from \mathbf{G} and ρ . Taking $c' < 1$ we may assume that $\varepsilon < 1$, so that all the γ are drawn from the 1-neighbourhood of the compact set $\Omega_\infty(MA \cap \Omega_\infty \Omega_\infty^{-1})\Omega_\infty^{-1}$. Let L be a bound for the Lipschitz constant of the smooth functions $P_{ij} \circ \rho, Q_{ik} \circ \rho$ in this domain. Again by compactness, there exists $b > 0$ such that for each $g \in \Omega_\infty$ and each $\vec{t} \in (MA \cap \Omega_\infty)^{n+1}$ for the i for which our system of constraints holds for $g\vec{t}g^{-1}$, and each k , $|Q_{ik}(g\vec{t}g^{-1})| > b$.

Fixing $g \in \Omega_\infty$, now let $\vec{\gamma}$ from S^{n+1} , and let $\vec{t} \in (MA \cap \Omega_\infty)^{n+1}$ so that γ_i is ε -close to $gt_i g^{-1}$. By the analysis above, there exists i such that $P_{ij}(\rho(\vec{t})) = 0$ while $Q_{ik}(\rho(\vec{t}))$ are at least b in magnitude. It follows that the magnitude $Q_{ik}(\rho(\vec{\gamma}))$ is at least $b - L\varepsilon$, which is positive as long as we ensure $c' < b/L$. It also follows that the magnitude of $P_{ij}(\rho(\vec{\gamma}))$ is at most $L\varepsilon$. Let c bound the total degree each P_{ij} . Then the denominator of each rational number $P_{ij}(\rho(\vec{\gamma}))$ is at most M^c . If $c' < 1/L$ this ensures that these rational numbers vanish. \square

4.3. The intersection pattern of translates of tubes around Levi subgroups.

Proposition 4.7. *Let \mathbf{G} be \mathbb{R} -split. Fix a relatively compact open neighbourhood of the identity $C \subset A$ and assume $C \subset \Omega_\infty$. There are $c_2, c_4 > 0$, depending only on the isomorphism class of \mathbf{G} and $c_1, c_3 = O_C(1)$, so that*

For any $x = (x_\infty, x_f) \in \Omega$ and any $0 < \varepsilon < c_3$ there exists a \mathbb{Q} -subtorus $\mathbf{T} \subset \mathbf{G}$ so that:

- (1) *If $s, s' \in \mathbf{G}(\mathbb{A}_f)$ both have denominator $\leq c_1 \varepsilon^{-c_2}$ and are so that*

$$\overline{xB(C, \varepsilon)s} \cap \overline{xB(C, \varepsilon)s'} \neq \emptyset \text{ in } \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})/K_f$$

then there exists $\gamma \in \mathbf{T}(\mathbb{Q}) \subset \mathbf{G}(\mathbb{Q})$ with

$$(4.3) \quad \gamma xB(C, \varepsilon)s \cap xB(C, \varepsilon)s' \neq \emptyset \text{ in } \mathbf{G}(\mathbb{A})/K_f.$$

- (2) *There at most $O(1 - c_4 \log \varepsilon)$ primes which are \mathbf{T} -bad.*

Proof. Let $s, s' \in \mathbf{G}(\mathbb{A}_f)$ satisfy the intersection condition of item (1). By assumption there is – after replacing s, s' by suitable elements of sK_f and $s'K_f$ respectively – an element $\gamma \in \mathbf{G}(\mathbb{Q})$ so that $\gamma x_f s = x_f s'$ (equality in $\mathbf{G}(\mathbb{A}_f)$) and, moreover, $\gamma \in x_\infty B(C, \varepsilon) B(C, \varepsilon)^{-1} x_\infty^{-1}$ (equality in $\mathbf{G}(\mathbb{R})$).

The former equality implies, in particular, that $d(\gamma) \ll c'_1 \varepsilon^{-c'_2}$, where c'_i depends on c_i , and $c'_i \rightarrow 0$ as $c_i \rightarrow 0$ for $i = 1, 2$.

Since conjugation by elements of a compact set is a map of bounded Lipschitz constant w.r.t. the metric dist_G , the latter inclusion shows that γ lies within $L\varepsilon$ of $x_\infty(MA \cap \Omega_\infty \Omega_\infty^{-1})x_\infty^{-1}$, where L depends only on \mathbf{G} and Ω_∞ .

Let R be the set of such γ , let $\mathbf{T} \subset \mathbf{G}$ be the closed subgroup they generate, and let E be the subalgebra of $M_n(\mathbb{Q})$ generated by $\rho(R)$. If c'_1, c'_2 are sufficiently small – this occurs, in particular, if c_1, c_2 are sufficiently small – then Lemma 4.4 shows that \mathbf{T} is a torus, and E is its linear span, a semisimple abelian subalgebra

of $M_n(\mathbb{Q})$. Analyzing the reasoning shows that the exponent c_2 may be taken to depend only on the isomorphism class of \mathbf{G} , whereas $c_1 = O(1)$. This proves (4.3).

For a \mathbf{G} -good prime p to be \mathbf{T} -good, it suffices that $E_p \cap M_n(\mathbb{Z}_p)$ is a maximal compact subring of $E_p = E \otimes \mathbb{Q}_p \subset M_n(\mathbb{Q}_p)$. For this it suffices to have generators $\{\gamma'_i\}_{i=1}^n \subset M_n(\mathbb{Z})$ for E as a \mathbb{Q} -algebra such that $\mathbb{Z}_p[\gamma'_1, \dots, \gamma'_d] = E_p \cap M_n(\mathbb{Z}_p)$. That will happen as long as p does not divide the discriminant of the characteristic polynomial of each γ'_i .

By the proof of Lemma 4.5, there exist $\{\gamma_i\}_{i=1}^n \subset R$ which generate E , and let $\gamma'_i = d(\gamma_i) \cdot \gamma_i$. Then $\gamma'_i \in M_n(\mathbb{Z})$, still generate E as a \mathbb{Q} -algebra. Next, since $\gamma_i \in \Omega_\infty \Omega_\infty \Omega_\infty^{-1} \Omega_\infty^{-1}$ and as ρ is continuous, the matrix entries of γ'_i are $O(\epsilon^{-c'_2})$. Finally, the discriminant of γ'_i is a polynomial in the coefficients of its characteristic polynomial, themselves polynomials in the matrix entries of γ'_i .

It follows that the set of \mathbf{G} -good but \mathbf{T} -bad primes is contained in the set of prime divisors of an integer bounded by $O(\epsilon^{-O(1)})$. \square

Generalizations. Proposition 4.7 is a statement of the following type:

Let $H \subset G$ be a closed subgroup. Then given $x \in \Omega_\infty$ and a tubular neighbourhood $B(C, \epsilon)$ of a piece $C \subset H$, there exists a \mathbb{Q} -subgroup $\mathbf{T} \subset \mathbf{G}$ such that intersection of Hecke translates of small denominator of $xB(C, \epsilon)$ are controlled by \mathbf{T} , in the sense that if xBs and xBs' intersect in X , there exists $\gamma \in \mathbf{T}(\mathbb{Q})$ such that $\gamma xBsK_f = xBs'K_f$ holds in $\mathbf{G}(\mathbb{A})/K_f$.

We have established this for \mathbf{G} which is \mathbb{Q} -anisotropic and \mathbb{R} -split and H a maximal \mathbb{R} -split torus. Specializing further to the case of \mathbf{G} associated to a division algebra of prime degree, we establish a result of this type for any Levi subgroup $H \subset G$ (that is, for a subgroup of the form $H = Z_G(a)$, $a \in A$). It turns out that the subgroup \mathbf{T} remains a torus.

In general one would expect that points lying near pieces of orbits of Levi subgroups defined over \mathbb{R} to lie on some like an orbit of a Levi subgroup defined over \mathbb{Q} . This is not quite correct, but precise versions of this intuition can be proved; see the work [14, §4].

4.4. Extra notations for the case of division algebras. Let D be a division algebra over \mathbb{Q} of prime degree d , and fix a lattice $D_{\mathbb{Z}} \subset D$ (i.e. a free \mathbb{Z} -submodule of maximal rank) and a Euclidean norm $\|\cdot\|$ on $D \otimes_{\mathbb{Q}} \mathbb{R}$. In other words, we have chosen a norm on $D \otimes_{\mathbb{Q}} \mathbb{Q}_v$ for all v , finite or infinite. Since the only central division algebras over \mathbb{R} are \mathbb{R} itself and Hamilton's quaternions, assuming $d \geq 3$ ensures that $D \otimes \mathbb{R}$ is the full matrix algebra.

We will consider the case where \mathbf{G} is the projectivized group of units (=invertible elements) of D , so (for $d \geq 3$) $G = \mathbf{G}(\mathbb{R})$ is isomorphic to $\mathrm{PGL}_d(\mathbb{R})$. The Lie algebra of G is identified with a quotient of $D \otimes \mathbb{R}$; as such, the norm on $D \otimes \mathbb{R}$ gives rise to a norm on the Lie algebra of G and thus to a left-invariant Riemannian metric on G .

We fix extra data $(\rho, K_f, \Omega, \Omega_\infty)$ for the group \mathbf{G} , as discussed in §2. In the rest of this paper, when discussing this case the implicit constants in the notations \ll and $O(\cdot)$ will be allowed to depend on $D, D_{\mathbb{Z}}$, the norm $\|\cdot\|$ and this extra data, without explicitly indicating this.

It should be noted that we do not assume that $D_{\mathbb{Z}}.D_{\mathbb{Z}} \subset D_{\mathbb{Z}}$; on the other hand, clearly there is an integer $K = O(1)$ so that $D_{\mathbb{Z}}.D_{\mathbb{Z}} \subset K^{-1}D_{\mathbb{Z}}$.

4.5. A diophantine lemma for \mathbb{Q} -algebras. In this section only we shall use an additional notion of denominator, special to the case of \mathbb{Q} -algebras. We fix a central simple \mathbb{Q} -algebra D of dimension d^2 and a \mathbb{Z} -lattice $D_{\mathbb{Z}} \subset \mathbb{Q}$. Given $x \in D$, we set

$$(4.4) \quad \tilde{d}(x) := \inf\{m \in \mathbb{N} : mx \in D_{\mathbb{Z}}\}.$$

Recall that for $\gamma \in \mathbf{G}(\mathbb{Q}) = D^{\times}/\mathbb{Q}^{\times}$, we also have the denominator $d(\gamma)$ defined in Section 4.1. We first clarify the relation between the two notions.

Lemma 4.8. *Let $\gamma \in \mathbf{G}(\mathbb{Q}) = D^{\times}/\mathbb{Q}^{\times}$ satisfy $d(\gamma) \leq M$ and belong to a compact subset $E \subset \mathbf{G}(\mathbb{R})$. Then there exists $\alpha \in D^{\times}$ lifting γ so that:*

$$\tilde{d}(\alpha) \ll_E M^c, \|\alpha^{-1}\|, \|\alpha\| \ll_E 1$$

where c is a constant depending only on the isomorphism class of \mathbf{G} .

Proof. In fact, let $\tilde{\mathbf{G}}$ be the algebraic group corresponding to D^{\times} , i.e. $\tilde{\mathbf{G}}(R) = (D \otimes_{\mathbb{Q}} R)^{\times}$ if R is a ring containing \mathbb{Q} .

Then $\tilde{\mathbf{G}}$ and \mathbf{G} are affine algebraic groups. We first show that the map $\tilde{\mathbf{G}} \rightarrow \mathbf{G}$ admits an algebraic section over a Zariski-open set $U \subset \mathbf{G}$.

Let $\mathbf{G}^{(1)}$ denote the group of elements of norm 1 in D^{\times} . It is a (geometrically) irreducible variety, because SL_d is an irreducible variety. The map $\mathbf{G}^{(1)} \rightarrow \tilde{\mathbf{G}}$ is a covering map (i.e. étale) and its kernel is the group of d th roots of unity. Let E be the function field of \mathbf{G} , considered as \mathbb{Q} -variety. The generic point in $\eta \in \mathbf{G}(E)$ does not lift to a point of $\mathbf{G}^{(1)}(E)$, but it does at least to lift to a point of $\tilde{\eta} \in \mathbf{G}^{(1)}(\tilde{E})$ for some finite extension \tilde{E}/E , which we may assume to be Galois and to contain the d th roots of unity. Then $\sigma \mapsto \tilde{\eta}^{\sigma}/\tilde{\eta}$ defines a 1-cocycle of $\mathrm{Gal}(\tilde{E}/E)$ valued in the group of d th roots of unity. By Hilbert's theorem 90, there exists $\tilde{e} \in \tilde{E}$ so that this cocycle is $\sigma \mapsto \tilde{e}^{\sigma}/\tilde{e}$. Adjusting $\tilde{\eta}$ by \tilde{e} gives a \tilde{E} -valued point of $\tilde{\mathbf{G}}$, which is invariant under $\mathrm{Gal}(\tilde{E}/E)$ and therefore is indeed an E -valued point of $\tilde{\mathbf{G}}$. This gives the desired section.

One may, by translating U , find a finite collection of open sets U_1, \dots, U_h which cover \mathbf{G} , and so that $\tilde{\mathbf{G}} \rightarrow \mathbf{G}$ admits a section $\theta_j : U_j \rightarrow \tilde{\mathbf{G}}$ over each U_j .

It follows from this that there exists $\alpha \in D^{\times}$ lifting γ so that

$$\tilde{d}(\alpha), \tilde{d}(\alpha^{-1}) \ll M^c$$

where c is a constant depending only on the choice of sets U_j and the sections, i.e. only the isomorphism class of \mathbf{G} .

From this bound, it follows in particular that $M^{-c} \ll_E \|\alpha\| \ll_E M^c$. The lower bound is clear; for the upper bound, we use the fact that α projects to the compact subset $E \subset \mathbf{G}(\mathbb{R})$.

Let p/q be a rational number satisfying $\|\alpha\| < p/q < 2\|\alpha\|$. We may choose p, q so that $\max(p, q) \ll M^c$. Replacing α by $q\alpha/p$, we obtain a representative α for γ that satisfies:

$$\tilde{d}(\alpha) \ll_E M^{2c}, \|\alpha\| \asymp_E 1$$

We increase c as necessary.

Finally, the bound for $\|\alpha^{-1}\|$ follows from the bound for $\|\alpha\|$ together with the fact that α projects to the compact set $E \subset \mathbf{G}(\mathbb{R})$. \square

The following should be compared with Lemma 4.4.

Lemma 4.9. *Let $S \subset D \otimes \mathbb{R}$ be a proper \mathbb{R} -subalgebra.*

For $c > 0$ sufficiently large (in fact, depending only on d) and for $c' > 0$ sufficiently small (in fact, depending only on $D, D_{\mathbb{Z}}, \|\cdot\|$), the set of $x \in D$ satisfying

$$(4.5) \quad \|x\| \leq R, \inf_{s \in S} \|x - s\| \leq \varepsilon, \tilde{d}(x) \leq M$$

is contained in a proper subalgebra $F \subset D$ as long as

$$(4.6) \quad \varepsilon R^c M^c < c'$$

In other words: points of D near a proper subalgebra of $D \otimes \mathbb{R}$ lie on a proper \mathbb{Q} -subalgebra of D . This proof will not use the fact that D is a *division* algebra, nor the fact that it is of prime rank.

Proof. We use here $f_1(d), f_2(d), \dots$ to denote positive quantities that depend on the rank d alone.

Let $s = \dim(S) + 1$. Then there is a polynomial function $G : D^s \rightarrow \mathbb{Q}$, with integral coefficients with respect to $D_{\mathbb{Z}}$, so that $G(\alpha_1, \dots, \alpha_s) = 0$ exactly when $\alpha_1, \dots, \alpha_s$ span a linear space of dimension $\leq s - 1$. For example one may use the sum of the squares of the minors of a suitable matrix. The degree of G is $f_1(d)$ and the size of its coefficients is $O(1)$.

Take x_1, \dots, x_s belonging to the set defined by (4.5). There are $y_1, \dots, y_s \in S$ so that $\|x_i - y_i\| \leq \varepsilon$. Then $G(x_1, \dots, x_s) \ll R^{f_2(d)} \varepsilon$. On the other hand, if $G(x_1, \dots, x_s) \neq 0$ then, because $d(x_i) \leq M$, we must have $G(x_1, \dots, x_s) \gg M^{-f_3(d)}$. It follows that, if a condition of the type (4.6) holds for suitable c, c' as stated, then x_1, \dots, x_s span a \mathbb{Q} -linear space of dimension $s - 1$.

Now let X be the \mathbb{Q} -algebra spanned by those x satisfying (4.5). It is clear that X is, in fact, spanned by monomials in such x of length at most $\dim_{\mathbb{Q}} D$. Each such monomial y satisfies $\|y\| \ll R^{f_4(d)}, \inf_{s \in S} \|y - s\| \ll R^{f_5(d)} \varepsilon, d(y) \gg M^{f_6(d)}$. It follows that – increasing c and decreasing c' in (4.6) as necessary – it follows that the \mathbb{Q} -subalgebra generated by all solutions to (4.5) has dimension $\leq s - 1$, in particular, is a proper subalgebra of D . \square

Proposition 4.10. *Let \mathbf{G} be the projectivized group of units of a division algebra D/\mathbb{Q} of prime degree d . There are $c_2, c_4 > 0$, depending only on the isomorphism class of \mathbf{G} and $c_1, c_3 = O_C(1)$, so that*

For any $x = (x_{\infty}, x_{\mathfrak{f}}) \in \Omega$ and any $0 < \varepsilon < 1/2$ there exists a subfield $F \subset D$ so that:

- (1) *If $s, s' \in \mathbf{G}(\mathbb{A}_{\mathfrak{f}})$ both have denominator $\leq c_1 \varepsilon^{-c_2}$ and are so that*

$$\overline{x B(C, \varepsilon) s} \cap \overline{x B(C, \varepsilon) s'} \neq \emptyset \text{ in } \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_{\mathfrak{f}}$$

then there exists $\gamma \in F^{\times} / \mathbb{Q}^{\times} \subset \mathbf{G}(\mathbb{Q})$ with

$$(4.7) \quad \gamma x B(C, \varepsilon) s \cap x B(C, \varepsilon) s' \neq \emptyset \text{ in } \mathbf{G}(\mathbb{A}) / K_{\mathfrak{f}}.$$

- (2) *F is generated by $\alpha \in D^{\times}$, so that $\alpha D_{\mathbb{Z}} + D_{\mathbb{Z}} \alpha \subset D_{\mathbb{Z}}$, and with $\|\alpha\| \leq c_3 \varepsilon^{-c_4}$.*

Proof. Let $s, s' \in \mathbf{G}(\mathbb{A}_{\mathfrak{f}})$ satisfy the intersection condition of item (1). As in the general case before we find – after replacing s, s' by suitable elements of $s K_{\mathfrak{f}}$ and $s' K_{\mathfrak{f}}$ respectively – an element $\gamma \in \mathbf{G}(\mathbb{Q})$ so that $\gamma x_{\mathfrak{f}} s = x_{\mathfrak{f}} s'$ (equality in $\mathbf{G}(\mathbb{A}_{\mathfrak{f}})$) and, moreover, $\gamma \in x_{\infty} B(C, \varepsilon) B(C, \varepsilon)^{-1} x_{\infty}^{-1}$ (equality in $\mathbf{G}(\mathbb{R})$).

Again, this implies $d(\gamma) \ll c'_1 \varepsilon^{-c'_2}$, where c'_i depends on c_i , and $c'_i \rightarrow 0$ as $c_i \rightarrow 0$ for $i = 1, 2$.

The latter inclusion shows that γ lies in a fixed compact subset of $\mathbf{G}(\mathbb{R})$ depending only on Ω, C .

The element γ belongs to $D^\times/\mathbb{Q}^\times$. We may choose a representative $\alpha \in D^\times$ for γ as in Lemma 4.8. In that case α lies in a fixed compact subset of $(D \otimes \mathbb{R})^\times$ – depending only on Ω, C – and $\tilde{d}(\alpha) \ll c''_1 \varepsilon^{-c''_2}$, where (for $i = 1, 2$) c''_i depends on c_i and $c''_i \rightarrow 0$ as $c'_i \rightarrow 0$.

Let E be the subalgebra of $D \otimes \mathbb{R}$ that centralizes $x_\infty a x_\infty^{-1}$. The assertion that $\gamma \in x_\infty B(C, \varepsilon) B(C, \varepsilon)^{-1} x_\infty^{-1}$ shows that α is “close” to E ; in fact, it is clear that

$$\inf_{e \in E} \|\alpha - e\| \ll \varepsilon$$

and moreover $\|\alpha\| \ll 1$ (because α lies in a fixed compact subset of $(D \otimes \mathbb{R})^\times$).

By Lemma 4.9 we see that, if c''_1, c''_2 are sufficiently small – this occurs, in particular, if c_1, c_2 are sufficiently small – then all such α necessarily belong to a proper \mathbb{Q} -subalgebra of D ; because D has prime degree, this must be a field F . Analyzing this reasoning shows that c_2 may be taken to depend only on the isomorphism class of \mathbf{G} , whereas $c_1 = O(1)$. This proves (4.7).

Also, there exists $K \in \mathbb{N}$ so that $D_{\mathbb{Z}}.D_{\mathbb{Z}} \subset K^{-1}D_{\mathbb{Z}}$. Then $\alpha.D_{\mathbb{Z}} \subset K^{-1}.d(\alpha)^{-1}.D_{\mathbb{Z}}$ and similarly for $D_{\mathbb{Z}}.\alpha$.

Replacing α with $\alpha' = K.d(\alpha).\alpha$, we see that $\alpha'.D_{\mathbb{Z}} + D_{\mathbb{Z}}\alpha' \subset D_{\mathbb{Z}}$ and $\|\alpha'\| \leq c_3 \varepsilon^{-c_4}$, where $c_3 = O_C(1)$ and c_4 depends only on the isomorphism class of \mathbf{G} . \square

We also need to know that there are only a few bad primes.

Lemma 4.11. *Let $F \subset D$ be a subfield, and let $\mathbf{T}_F \subset \mathbf{G}$ be the torus defined by F , i.e. the centralizer of F in \mathbf{G} . Suppose that F is generated, over \mathbb{Q} , by an element α satisfying $\alpha D_{\mathbb{Z}} + D_{\mathbb{Z}}\alpha \subset D_{\mathbb{Z}}$. Then the number of primes which fail to be \mathbf{T}_F -good is at most $O(1 + \log \|\alpha\|)$.*

Proof. Let K'_f be the stabilizer⁵ of $D_{\mathbb{Z}}$ inside $\mathbf{G}(\mathbb{A}_f)$.

Then $\mathbf{T}_F(\mathbb{Q}_p) \cap K'_f$ is maximal compact inside $\mathbf{T}_F(\mathbb{Q}_p)$ as long as the maximal compact subring of $(F \otimes \mathbb{Q}_p)$ preserves $D_{\mathbb{Z}} \otimes \mathbb{Z}_p$ under both left and right multiplication. This will be so, in particular, at any prime where the maximal compact subring of $(F \otimes \mathbb{Q}_p)$ equals $\mathbb{Z}_p[\alpha]$. This will always be the case if p does not divide the discriminant of the ring $\mathbb{Z}[\alpha]$.

From this, one deduces that there are at most $O(1 + \log \|\alpha\|)$ primes for which $\mathbf{T}_F(\mathbb{Q}_p) \cap K'_f$ fails to be maximal compact in $\mathbf{T}_F(\mathbb{Q}_p)$. But, for all but $O(1)$ primes, the intersection $\mathbf{G}(\mathbb{Q}_p) \cap K'_f$ coincides with $\mathbf{G}(\mathbb{Q}_p) \cap K_f$. So there are at most $O(1 + \log \|\alpha\|)$ primes for which $\mathbf{T}_F(\mathbb{Q}_p) \cap K_f$ fails to be maximal compact in $\mathbf{T}_F(\mathbb{Q}_p)$. \square

5. BOUNDS ON THE MASS OF TUBES, II

We define “tubes” $B_0 := B(C, \varepsilon)$ as in Section 4. When \mathbf{G} is merely assumed \mathbb{R} -split we take $C \subset MA$. In the special case of division algebras of prime degree C may be taken to lie in any Levi subgroup of G .

⁵ Note that $\mathbf{G}(\mathbb{A}_f)$ acts naturally on lattices inside D , in a fashion derived from the conjugation action of \mathbf{G} on D . Indeed, if V is a \mathbb{Q} -vector space, the group $\mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{A}_f)$ acts naturally on lattices inside V .

Set $B = B_0 B_0^{-1} \supset B(C, \varepsilon)$. There exists a compact subset $C' \subset Z_G(a)$ and a constant $M = O(1)$ such that B , $B_2 (= B_1 \cdot B_1)$ and B_3 are subsets of $B(C', M\varepsilon)$. Here notations are as (3.1). Also, $\frac{\text{vol} B_3}{\text{vol} B_0} \ll 1$.⁶

5.1. Sets S for which intersections are controlled by tori. Let Q be so that $Q/2$ is larger than any bad prime for \mathbf{G} , and let ℓ be fixed. (In practice, $Q \rightarrow \infty$ as B becomes small, whereas ℓ is fixed depending only on \mathbf{G}).

Set $S_p = \{g_p \in \mathbf{G}(\mathbb{Q}_p)/K_p : d_p(g_p) \leq p^\ell\}$. Initially we shall consider the set of translates given by $\bigcup_{p \in [Q/2, Q]} S_p$, where we identify $\mathbf{G}(\mathbb{Q}_p)/K_p$ with a subset of $\mathbf{G}(\mathbb{A}_f)/K_f$ in the natural way.

Part (1) of the conclusions of Propositions 4.7 and 4.10 can now be rephrased⁷ as establishing (for $Q^\ell \ll \varepsilon^{-c_2}$) the following condition. In words, it states that *intersections between Hecke translates of B_2 by $\bigcup S_p$ all arise from a \mathbb{Q} -torus*:

$\star = \star_{B, Q, \ell}$: For any $x \in \Omega_\infty$ there is a \mathbb{Q} -torus $\mathbf{T} \subset \mathbf{G}$ so that

$$\overline{x B_2 s} \cap \overline{x B_2 s'} \neq \emptyset \quad \text{in } X$$

with $s, s' \in \bigcup_{p \in [Q/2, Q]} S_p$ only if there is $\gamma \in \mathbf{T}(\mathbb{Q})$ so that for these s, s' ,

$$\gamma x B_2 s \cap x B_2 s' \neq \emptyset \quad \text{in } \mathbf{G}(\mathbb{A})/K_f.$$

Lemma 5.1. *Suppose that condition $\star_{B, Q, \ell}$ is satisfied. Take $x \in \Omega_\infty$ and let \mathbf{T} be the torus specified by $\star_{B, Q, \ell}$. Assume that $S \subset \bigcup_p S_p \subset \mathbf{G}(\mathbb{A}_f)/K_f$, with the union taken over $p \in [Q/2, Q]$ which are \mathbf{T} -good. Then:*

$$\#\{s, s' \in S : \overline{x B_2 s} \cap \overline{x B_2 s'} \neq \emptyset\} \ll_\ell Q^2 + |S|$$

Proof. Consider any intersection $\overline{x B_2 s} \cap \overline{x B_2 s'} \neq \emptyset$ in X when $s, s' \in S$. This means that there is $\gamma \in \mathbf{T}(\mathbb{Q})$ so that $\gamma x B_2 s \cap x B_2 s' \neq \emptyset$ in $\mathbf{G}(\mathbb{A})/K_f$. Then $s \in S_p, s' \in S_q$ when p, q are \mathbf{T} -good primes in the range $[Q/2, Q]$; we distinguish two cases according to whether $p = q$ or not.

(1) $q = p$. In this case,

$$\gamma \in (x B_2 B_2^{-1} x^{-1}) \cdot \mathbf{T}(\mathbb{Q}_p) \cdot K_f.$$

For any fixed p , the number of K_f -cosets contained in $\mathbf{T}(\mathbb{Q}_p) \cdot K_f$ satisfying $d_p \leq \ell$ is $O_\ell(1)$: since p is \mathbf{T} -good, the quotient $\mathbf{T}(\mathbb{Q}_p)/\mathbf{T}(\mathbb{Q}_p) \cap K_f$ is a free abelian group of rank $\leq \dim(\mathbf{T})$. Pick generators t_1, \dots, t_r for this quotient; they generate a discrete subgroup. We need to show that the number of $(e_1, \dots, e_r) \in \mathbb{Z}^r$ so that $d_p(t_1^{e_1} \dots t_r^{e_r}) \leq p^\ell$ is $O_\ell(1)$. To see this, pass to an extension of \mathbb{Q}_p where $\rho(t_i)$ become diagonalizable.

Therefore, γ is an element of $\mathbf{G}(\mathbb{Q})$ so that:

(a) considered as an element of $\mathbf{G}(\mathbb{R})$, γ belongs to

$$x B_2^{-1} B_2 x^{-1},$$

which in turn is contained in a compact set depending only on Ω_∞ ;

(b) considered as an element of $\mathbf{G}(\mathbb{A}_f)$, γ belongs to $O_\ell(1)$ right K_f -cosets.

⁶ In this and in the statement $M = O(1)$, the implicit constant certainly depends on C ; however, C was assumed to belong to Ω_∞ , and we have permitted implicit constants to depend on Ω_∞ without explicit mention.

⁷ Note that, in what follows, we are applying the Proposition with $B(C, \varepsilon)$ replaced by the larger set $B_2 \subset B(C', M\varepsilon)$. It is easy to see that the stated result holds even though this larger set may not be contained in Ω_∞ .

The number of possibilities for γ is therefore $O_\ell(1)$. Since $(sK_f, s'K_f)$ is determined by (sK_f, γ) , it follows the number of possibilities for (s, s') in the case “ $p = q$ ” is at most $O_\ell(|S|)$.

(2) $p \neq q$.

In this case, $s \in \mathbf{T}(\mathbb{Q}_p).K_p$ and $s' \in \mathbf{T}(\mathbb{Q}_q).K_q$. By an argument already given, the number of K_p -cosets contained in $\mathbf{T}(\mathbb{Q}_p).K_p$ and satisfying $d_p \leq \ell$ is $O_\ell(1)$, and similarly with q replacing p . It follows that the number of possibilities for (s, s') is $O_\ell(1)$ for *given* p, q .

The total number of possibilities for (s, s') in the case “ $p \neq q$ ” is therefore $O_\ell(Q^2)$. □

5.2. Conclusion. We shall apply Lemma A.1 to our setting. Take ℓ as in that Lemma.

Lemma 5.2. *Suppose that condition $\star_{B, Q, \ell}$ is satisfied. Take $x \in \Omega_\infty$ and let \mathbf{T} be the torus specified by $\star_{B, Q, \ell}$, \mathcal{P} the set of \mathbf{T} -good primes in $[Q/2, Q]$.*

For any Hecke eigenfunction ψ on X .

$$\mu_\psi(\overline{xB}) \ll \frac{Q^{1.01}}{|\mathcal{P}|^2}.$$

Proof. For $p \in \mathcal{P}$ let h_p be the K_p -bi-invariant function on G_p furnished in Lemma A.1. Now, if we consider h_p as a function on G_p/K_p , we have upper and lower bounds $p \ll \#\text{supp}(h_p) \ll p^{c'}$, where c depends only on \mathbf{G}, ρ .

Therefore, by a dyadic decomposition argument, there exists a subset $\mathcal{P}_1 \subset \mathcal{P}$ and $A \gg Q$ so that $\#\text{supp}(h_p) \in [A, 2A]$ ($p \in \mathcal{P}_1$). Set $h = \sum_{p \in \mathcal{P}_1} h_p$. Then, by what we have proven in Lemma 3.4 and Lemma 5.1,

$$\mu_\psi(\overline{xB}) \ll \left(\frac{\text{vol}B_3}{\text{vol}B_0} \right)^2 \frac{Q^2 + \sum_p \#\text{supp}(h_p)}{\left(\sum_p \#\text{supp}(h_p)^{1/2} \right)^2} \ll \left(\frac{\text{vol}B_3}{\text{vol}B_0} \right)^2 Q^{0.001} \left(\frac{Q}{|\mathcal{P}|^2} + g^{-1} \right).$$

Finally, as observed at the start of this section, $\frac{\text{vol}B_3}{\text{vol}B_0} \ll 1$. □

We now combine this Lemma with the results of Section 4, which give conditions under which $\star_{B, Q, \ell}$ is true.

Theorem 5.3. *Let \mathbf{G} be a semisimple group defined over \mathbb{Q} which splits over \mathbb{R} . Let ψ be a Hecke eigenfunction on $X = \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_f$.*

Let $\Omega_\infty \subset G = NAK$ be compact. Further, let $B(C, \varepsilon) \subset \Omega_\infty$ be a tube as in Section 4 such that either

- (1) $C \subset MA$; or,
- (2) \mathbf{G} is the projectivized group of units of a division algebra D of prime degree over \mathbb{Q} .

Then there is $c > 0$, depending only on the isomorphism class of \mathbf{G} so that, uniformly over $x \in \Omega_\infty$,

$$\mu_\psi(xB(C, \varepsilon)) \ll \varepsilon^c.$$

Proof. Let ℓ be as in Lemma A.1.

Recall, as remarked at the start of the present section, that $B_3 \subset B(C', M, \varepsilon)$, for a suitable compact set C' and a suitable constant M .

Proposition 4.7 (for case (1)) or 4.10 (for case (2)), applied to $B(C', M.\varepsilon)$ ⁸, shows that property $\star_{B,Q,\ell}$ holds so long as

$$(5.1) \quad Q^\ell \leq a\varepsilon^{-b},$$

where a, b depend only the isomorphism class of \mathbf{G} .

If Q satisfies this constraint, the previously quoted Propositions, in combination with Lemma 4.11, show that the number of primes that are not \mathbf{T} -good is $O(\log \varepsilon)$. (Here \mathbf{T} is the torus occurring in the definition of $\star_{B,Q,\ell}$.)

This shows, notation as in Lemma 5.2, that $|\mathcal{P}| \gg \frac{Q}{\log Q}$. Thus $\mu_\psi(xB(C, \varepsilon)) \ll Q^{-0.98}$. Choosing Q as large as allowable under (5.1) yields the desired result. \square

6. THE AQUE PROBLEM AND THE APPLICATION OF THE ENTROPY BOUND.

We now return to the AQUE problem discussed in the Introduction, recall our previous work on this problem, and explain how our main theorem concerning AQUE is deduced.

6.1. Quantum unique ergodicity on locally symmetric spaces.

Problem 6.1. (QUE on locally symmetric spaces; Sarnak) Let G be a connected semi-simple Lie group with finite center. Let K be a maximal compact subgroup of G , $\Gamma < G$ a lattice, $X = \Gamma \backslash G$, $Y = \Gamma \backslash G/K$. Let $\{\psi_n\}_{n=1}^\infty \subset L^2(Y)$ be a sequence of normalized eigenfunctions of the ring of G -invariant differential operators on G/K , with the eigenvalues w.r.t. the Casimir operator tending to ∞ in absolute value. Is it true that $\bar{\mu}_{\psi_n} := |\psi_n|^2 d\text{vol}$ converge weak-* to the normalized projection of the Haar measure to Y ?

In the paper [20] we have obtained Theorem 6.2, recalled below, constructing the microlocal lift in this setting. We needed to impose a non-degeneracy condition on the sequence of eigenfunctions (the assumption essentially amounts to asking that all eigenvalues tend to infinity, at the same rate for operators of the same order.) For the precise definition of *non-degenerate*, we refer to [20, Section 3.3].

With K and G as in Problem 6.1, let A be as in the Iwasawa decomposition $G = NAK$, i.e. $A = \exp(\mathfrak{a})$ where \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . For $G = \text{PGL}_d(\mathbb{R})$ and $K = \text{PO}_d(\mathbb{R})$, one may take A to be the subgroup of diagonal matrices with positive entries. Let $\pi: X \rightarrow Y$ be the projection. We denote by dx the G -invariant probability measures on X , and by dy the projection of this measure to Y .

Theorem 6.2. *Let $\psi_n \in L^2(Y)$ be a non-degenerate sequence of normalized eigenfunctions, whose eigenvalues approach ∞ . Then, after replacing ψ_n by an appropriate subsequence, there exist functions $\tilde{\psi}_n \in L^2(X)$ and distributions μ_n on X such that:*

- (1) (Lift) *The projection of μ_n to Y coincides with $\bar{\mu}_n$, i.e. $\pi_*\mu_n = \bar{\mu}_n$.*
- (2) *Let σ_n be the measure $|\tilde{\psi}_n(x)|^2 dx$ on X . Then, for every $g \in C_c^\infty(X)$, we have $\lim_{n \rightarrow \infty} (\sigma_n(g) - \mu_n(g)) = 0$.*
- (3) (Invariance) *Every weak-* limit σ_∞ of the measures σ_n (necessarily a positive measure of mass ≤ 1) is A -invariant.*

⁸ instead of $B(C, \varepsilon)$; it is easy to see the proof works verbatim even though $B(C', M.\varepsilon)$ need not be contained in Ω_∞

- (4) (*Equivariance*). Let $E \subset \text{End}_G(C^\infty(X))$ be a \mathbb{C} -subalgebra of bounded endomorphisms of $C^\infty(X)$, commuting with the G -action. Noting that each $e \in E$ induces an endomorphism of $C^\infty(Y)$, suppose that ψ_n is an eigenfunction for E (i.e. $E\psi_n \subset \mathbb{C}\psi_n$). Then we may choose $\tilde{\psi}_n$ so that $\tilde{\psi}_n$ is an eigenfunction for E with the same eigenvalues as ψ_n , i.e. for all $e \in E$ there exists $\lambda_e \in \mathbb{C}$ such that $e\psi_n = \lambda_e\psi_n$, $e\tilde{\psi}_n = \lambda_e\tilde{\psi}_n$.

We first remark that the distributions μ_n (resp. the measures σ_n) generalize the constructions of Zelditch (resp. Wolpert). Although, in view of (2), they carry roughly equivalent information, it is convenient to work with both simultaneously: the distributions μ_n are canonically defined and easier to manipulate algebraically, whereas the measures σ_n are patently positive and are central to the arguments of the present paper.

The existence of the microlocal lift already places a restriction on the possible weak-* limits of the measures $\{\tilde{\mu}_n\}$ on Y . For example, the A -invariance of μ_∞ shows that the support of any weak-* limit measure $\tilde{\mu}_\infty$ must be a union of maximal flats. Following Lindenstrauss, we term the weak-* limits σ_∞ of the lifts σ_n *quantum limits*.

More importantly, Theorem 6.2 allows us to pose a new version of the problem:

Problem 6.3. (QUE on homogeneous spaces) In the setting of Problem 6.1, is the G -invariant measure on X the unique non-degenerate quantum limit?

The main result of this paper is the resolution of Problem 6.3 for certain higher rank symmetric spaces, in the context of *arithmetic* quantum limits. We refer to [20, Section 1.4] for a further discussion of the significance of these spaces and how the introduction of arithmetic helps to eliminate degeneracy.

6.2. Results: Arithmetic QUE for division algebra quotients. For brevity, we state the result in the language of automorphic forms; in particular, \mathbb{A} is the ring of adèles of \mathbb{Q} .

Let \mathbf{G} be a semisimple group over \mathbb{Q} , and let $G = \mathbf{G}(\mathbb{R})$. Let K_f be an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $X = \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_f$ consists of a single G -orbit (this condition is mainly cosmetic: see Remark 6.3.1 in §6.3). Then there exists a discrete subgroup $\Gamma < G$ such that $X = \Gamma \backslash G$. Let \mathcal{H} be the Hecke algebra, as defined in Section 2. It acts on $L^2(X)$. Set $Y = \Gamma \backslash G / K$ the associated locally symmetric space, where K is the a maximal compact subgroup inside G . A will denote a maximal \mathbb{R} -split torus of in G compatible with K .

In the special case, let D/\mathbb{Q} be a division algebra of prime degree d , and let \mathbf{G} be the associated projective general linear group, i.e. the quotient of the group of units in D by its center. Assume that \mathbf{G} is \mathbb{R} -split, i.e. $G = \mathbf{G}(\mathbb{R}) \simeq \text{PGL}_d(\mathbb{R})$ (when $d \geq 3$ this is always the case). For this group let K be the standard maximal compact subgroup, A the group of diagonal matrices with positive entries (up to scaling).

Theorem 5.3 implies:

Theorem 6.4. Let $\tilde{\psi}_n \in L^2(X)$ be a sequence of \mathcal{H} -eigenfunctions on X such that the associated probability measures $\sigma_n := |\tilde{\psi}_n(x)|^2 dx$ on X converge weak-* to an A -invariant probability measure σ_∞ . Then every regular $a \in A$ acts on every A -ergodic component of σ_∞ with positive entropy. When \mathbf{G} is associated to a division algebra, the same holds for any $a \in A \setminus \{1\}$.

Proof. This is essentially a rephrasing of Theorem 5.3, where the uniformity of the estimate means it carries over to weak-* limits.

For a proof that the bound on measures of tubular neighbourhood of $Z_G(a)$ implies that $a \in A$ acts with positive entropy see [10, Sec. 8]. While written for the case of quaternion algebras ($d = 2$), that discussion readily generalizes to our situation by modifying its “Step 2” to account for the action of a on the Lie algebra. \square

Using results on measure-rigidity due to Einsiedler and Katok, this has the following implication for the QUE problem:

Theorem 6.5. *Let \mathbf{G} be the projectivized unit group of a division algebra of prime degree, and maintain the other notations as above. Let $\{\psi_n\}_{n=1}^\infty \subset L^2(Y)$ be a non-degenerate sequence of eigenfunctions for the ring of G -invariant differential operators on G/K (cf. [20, Sec. 3.3]) which are also eigenforms of the Hecke algebra \mathcal{H} (cf. Section 2). Such ψ_n are also called Hecke-Maass forms.*

Then the associated probability measures $\bar{\mu}_n$ converge weak- to the normalized Haar measure on Y , and their lifts μ_n (see Theorem 6.2) converge weak-* to the normalized Haar measure dx on $X = \Gamma \backslash \mathrm{PGL}_d(\mathbb{R})$.*

Proof. The case $d = 2$ is Lindenstrauss’s theorem, and we will thus assume $d \geq 3$.

Passing to a subsequence, let $\psi_n \in L^2(Y)$ be a non-degenerate sequence of Hecke-Maass forms on Y such that $\bar{\mu}_n \rightarrow \bar{\mu}_\infty$ weakly. Passing to a subsequence, let $\tilde{\psi}_n$ and σ_n be as in Theorem 6.2 such that $\sigma_n \rightarrow \sigma_\infty$ weakly and σ_∞ lifts $\bar{\mu}_\infty$. Then σ_∞ is a non-degenerate arithmetic quantum limit on X . By Theorem 6.4, σ_∞ is an A -invariant probability measure on X such that every $a \in A \setminus \{1\}$ acts on almost every A -ergodic component of σ_∞ with positive entropy. Remark also that the measure σ_∞ is invariant under the finite group $Z_K(A)$, the centralizer of A in K , by construction (see [20, Remark 1.7, (5)]).

Then [5, Thm. 4.1(iv)] shows that σ_∞ has a unique ergodic component, μ_{Haar} . \square

Our methods also apply to the case of a split central simple \mathbb{Q} -algebra, that is when $\mathbf{G}(\mathbb{Q}) = \mathrm{PGL}_d(\mathbb{Q})$. The result is somewhat weaker, however:

Theorem 6.6. *Let $G = \mathrm{PGL}_d(\mathbb{R})$ (d prime), and let $\Gamma < G$ be a lattice of the form ${}^9 \mathrm{PGL}_d(\mathbb{Z}) \cap \gamma \mathrm{PGL}_d(\mathbb{Z}) \gamma^{-1}$ for some $\gamma \in \mathrm{PGL}_d(\mathbb{Q})$. Let $X = \Gamma \backslash G$, $Y = X/K$. Let $\{\psi_n\}_{n=1}^\infty \subset L^2(Y)$ be a non-degenerate sequence of Hecke-Maass forms.*

Then the associated probability measures $\bar{\mu}_n$ converge weak- to the a Haar measure on Y , and their lifts μ_n (see Theorem 6.2) converge weak-* to a Haar measure $c dx$ on $X = \Gamma \backslash \mathrm{PGL}_d(\mathbb{R})$, where $c \in [0, 1]$.*

Proof. Let $\mathbf{G} = \mathrm{PGL}_d/\mathbb{Q}$. For any prime p let \mathcal{O}_p be the (“Eichler”) order $M_n(\mathbb{Z}_p) \cap \gamma M_n(\mathbb{Z}_p) \gamma^{-1}$ of $M_n(\mathbb{Q}_p)$. Let $K_p = \mathcal{O}_p^\times$, $K_f = \prod_p K_p$. Then $\Gamma = \mathrm{PGL}_d(\mathbb{Q}) \cap K_f$, so that $X = \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_f$.

Passing to a subsequence, let μ be a weak-* limit of a sequence of lifts. Theorem 6.4 shows that there exist $a \in A$ which act with positive entropy on almost every A -ergodic component of μ . The measure rigidity results of [6] together with the orbit classification results of [12] (we use here the fact that d is prime) show that

⁹ Again, the result is best expressed for an “adelic” quotient

μ is a Haar measure on X . Since X is not compact, this method does not control the total mass c of μ . \square

6.3. Remarks on generalizations.

6.3.1. *Class number one.* The assumption imposed that G act with a single orbit in $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_f$ is, as we remarked, cosmetic. In general, if we remove this assumption, one would still know – making analogous definitions – that quantum limits remain G -invariant. However, this would not quite be a complete answer since the space of G -invariant measures on $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_f$ is now finite dimensional, and we would not know the relative measures of the different components.

6.3.2. *Nondegeneracy.* The second author has obtained a version of Theorem 6.2 without the non-degeneracy assumption, see [19]. In that case the lifts are asymptotically invariant under (non-trivial) subgroups of A . The bounds on the mass of tubes obtained in this paper are, at their foundation, purely statements about Hecke eigenfunctions, and thus carry over to degenerate limits. However, the interpretation of these bounds as lower bounds of the *entropy* only applies to tubes associated to elements $a \in A$ by which the measure is invariant.

In the degenerate case, a-priori one only has invariance by *singular* elements $a \in A$. Thus our methods only show that those elements act with positive entropy in the case of division algebras.

An analogue of Theorem 6.5 would accordingly follow from a result classifying measures on X which are only invariant by a (potentially one-dimensional) subgroup of the Cartan subgroup, assuming a Hecke recurrence condition a-la [11]. An advance in this direction is necessary in order to show that every sequence of Hecke-Maass forms is equidistributed; in the rest of the remarks we only consider the case of non-degenerate limits.

It is worth noting that the mass of tubes corresponding to the flow of a regular element would also be small, but since the measure is not invariant by the regular element it is not clear how to incorporate this information into the measure-classification result.

6.3.3. *Escape of mass.* When the quotient X is not compact (for example, in the split case of Theorem 6.6), there is an additional potential obstruction to equidistribution: weak-* limits are not necessarily probability measures – they may even be the zero measure. This possibility is known as “escape-of-mass”.

In the case of congruence lattices in $\mathrm{SL}_2(\mathbb{Z})$, escape-of-mass was ruled out by Soundararajan [22]. This was generalized to the congruence lattices in $\mathrm{SL}_2(\mathcal{O})$, \mathcal{O} the ring of integers of a number field, in the M.Sc. Thesis [25]. The extension to higher-rank groups is the subject of current research.

In the particular case of congruence lattices in $\mathrm{GL}_d(\mathbb{Z})$ the normalization of the measure is already controlled by the degenerate Eisenstein series. Hence a sub-convexity result for the Rankin–Selberg L -function would control the escape, just as in the better-known case of GL_2 (though, notably, Soundararajan’s argument does not rely on an L -function bound). In fact, such a sub-convexity result would also prove that the limits have *positive entropy* (this is demonstrated in [8]).

In the rest of our remarks we ignore the issue as well; the reader may assume the group \mathbf{G} to be anisotropic.

6.3.4. *The case when \mathbf{G} is not associated to a division algebra.* We expect the techniques developed for the proof of Theorem 6.5 will generalize at least to some other locally symmetric spaces, the case of division algebras of prime degree being the simplest; but there are considerable obstacles to obtaining a theorem for *any* arithmetic locally symmetric space at present. A brief discussion of some of these difficulties follows.

First, the intersection patterns of Hecke translates will be controlled by subgroups more complicated than tori. Except for the simplest case, in general the best one can hope for is that intersections be controlled by Levi subgroups defined over \mathbb{Q} . Lemma 4.9 already establishes this for unit groups of semisimple \mathbb{Q} -algebras. Such subgroups will have *exponential* volume growth (say in terms of their orbit on the building of $\mathbf{G}(\mathbb{Q}_p)$), compared with the polynomial behaviour of tori. Even the purely local question of whether an eigenfunction on a building can concentrate appreciably on the orbit of such a large subgroup is difficult. To see where such an issue can arise note that when G is not \mathbb{R} -split, even the centralizer of the maximal \mathbb{R} -split torus is not a torus.

Dealing with intersections created by larger \mathbb{Q} -subgroups is essential for a more fundamental reason. The best possible outcome of the type of measure classification results one would use here (for state of the art see [7]) is that the measure is, in some sense, *algebraic*: it is a linear combination of measures supported on orbits of subgroups. From this point of view, to show that the limit measure is the G -invariant measure should at least require showing that the mass of orbits of these subgroups is zero a-la Rudnick–Sarnak. In our terms, this means showing that the mass concentrated in an ϵ -neighbourhood of the orbit goes to zero with ϵ even if it is not necessary to achieve power-law decay of the mass (i.e. positive entropy). In fact, for this reason it is hard to imagine an application of the current techniques that would rule out these intermediate measures without also establishing that all elements of A act with positive entropy.

APPENDIX A. PROOF OF LEMMA A.1: HOW TO CONSTRUCT A HIGHER RANK AMPLIFIER

To readers familiar with the usage of “amplification” in analytic number theory (as represented, for instance, in the work of Duke, Friedlander and Iwaniec): the Lemma A.1 in effect represents a way to construct an amplifier in higher rank.

Let \mathbf{G} be a semisimple algebraic group over \mathbb{Q} , $\rho: \mathbf{G} \rightarrow \mathrm{SL}_N$ an embedding. For each prime p let $G_p := \mathbf{G}(\mathbb{Q}_p)$. There is p_0 such that for $p > p_0$, $K_p = \rho^{-1}(\mathrm{SL}_N(\mathbb{Z}_p))$ is a special maximal compact subgroup of the unramified group G_p .

We will allow the implicit constant in the symbols \gg, \ll to depend on N (and hence, also on dimension of \mathbf{G}), but not on anything else.

Lemma A.1. *Possibly increasing p_0 there exist integers ℓ, ℓ' depending only on N such that for any $p > p_0$ and for any character Λ of the Hecke algebra of G_p with respect to K_p , there is a K_p -bi-invariant function h_p with the following properties:*

- (1) *Its support satisfies $p^{\ell'} \gg \#\mathrm{supp}(h_p) \gg p$, where we think of h_p as a function on G_p/K_p ;*
- (2) *$|h_p| \in \{0, 1\}$;*
- (3) *$\Lambda(h_p)$ is positive and $\Lambda(h_p) \gg \#\mathrm{supp}(h_p)^{1/2}$*
- (4) *For any $g \in \mathrm{supp}(h_p)$, the denominator of $\rho(g)$ is $\ll_N p^\ell$.*

A.1. Notation on p -adic groups. We shall use certain standard properties of semisimple algebraic groups over p -adic fields. Standard references are [23] and [3].

Increasing p_0 we may suppose that G_p quasi-split and unramified, and that K_p hyperspecial. Let \mathbf{A} be a maximal \mathbb{Q}_p -split torus in G_p so that the corresponding apartment in the building of G_p contains the point fixed by K_p , and let $A_p = \mathbf{A}_p(\mathbb{Q}_p)$ be the corresponding subgroup of G_p .

Let $X_* = \text{Hom}(\mathbb{G}_m, \mathbf{A})$ and $X^* = \text{Hom}(\mathbf{A}, \mathbb{G}_m)$. Let $\Phi \subset X^*$ be the set of roots for the action of \mathbf{A} on the Lie algebra of \mathbf{G} . We note that, as p varies, there will be at most finitely many distinct root systems. In particular, our bounds may depend on Φ .

Fix a positive system of roots for A_p , let N_p be the subgroup corresponding to all the positive roots. We have Iwasawa decomposition $G_p = N_p A_p K_p$. Let $\delta : A_p \rightarrow \mathbb{R}^\times$ be the character corresponding to the half-sum of positive roots, composed with $\|\cdot\|_p$ on \mathbb{Q}_p .

Let $\mathfrak{a} := A_p/(A_p \cap K_p)$, a free abelian group of rank equal to the rank of $\mathbf{G}(\mathbb{Q}_p)$. Then \mathfrak{a} is identified with X_* : for, given $a \in \mathfrak{a}$, there exists a unique homomorphism $\theta : \mathbb{G}_m \rightarrow \mathbf{A}_p$ so that $\theta(p)$ and a lie in the same $A_p \cap K_p$ coset.

Next, let $V = X_* \otimes_{\mathbb{Z}} \mathbb{R}$, $V^* = X^* \otimes_{\mathbb{Z}} \mathbb{R}$, and $V_{\mathbb{C}}^* = V^* \otimes_{\mathbb{R}} \mathbb{C}$. Then to any $\nu \in V_{\mathbb{C}}^*$, and any $\theta \in X_*$ with $a = \theta(p)$ we set $a^\nu = p^{(\theta, \nu)}$ with the obvious pairing. In particular this gives an identification between the unramified characters of A_p and the torus $\mathfrak{a}_{\text{temp}}^* = iV^*/(2\pi i \log p X^*)$.

Let W be the Weyl group of A_p . It acts on \mathfrak{a} . Moreover, we fix a W -invariant inner product on $\mathfrak{a} \otimes \mathbb{R}$ in the following way: the elements of Φ , considered as belonging to $\text{Hom}(\mathfrak{a}, \mathbb{Z})$, define elements of a root system. We require the longest root for each simple factor of \mathbf{G} to have length 1. This uniquely normalizes a W -invariant inner product on the dual to $\mathfrak{a} \otimes \mathbb{R}$, so also on $\mathfrak{a} \otimes \mathbb{R}$.

This normalization has the following property: if $\alpha \in \Phi$ is any root and $a \in \mathfrak{a}$, then “ $|\alpha(a)|$ ” – we implicitly identify a with an element of X_* – is bounded above by $p^{\|\alpha\|}$.

Finally, let $M_p = Z_{K_p}(A_p)$ so that $N_p A_p M_p$ is a Borel subgroup of G_p .

A.2. Plancherel formula. To any character $\nu : \mathfrak{a} \rightarrow \mathbb{C}^\times$ (parametrized by an element of $V^*/(2\pi i \log p X^*)$), we associate the spherical representation¹⁰ $\pi(\nu)$ of G_p obtained by extending $\nu\delta$ to $N_p A_p M_p$ trivially on $N_p M_p$, inducing to G_p , and taking the unique spherical subquotient.

For any K_p -bi-invariant function k on G_p , let $\hat{k}(\nu)$ be the scalar by which k acts on the spherical vector in $\pi(\nu)$. There is a unique K_p -bi-invariant function Ξ_ν on G_p (the spherical function with parameter ν) so that:

$$(A.1) \quad \hat{k}(\nu) = \int_{g \in G_p} k(g) \Xi_\nu(g) dg, \quad k(g) = \int_{\nu \in \mathfrak{a}_{\text{temp}}^*} \hat{k}(\nu) \Xi_\nu(g) d\mu(\nu).$$

where the first integral is taken w.r.t. the Haar measure that assigns mass 1 to K_p , and the second integral is taken w.r.t. the Plancherel measure μ on $\mathfrak{a}_{\text{temp}}^*$. In our normalization, μ is a probability measure.

The map $k \mapsto \hat{k}(\nu)$ is an isomorphism between the space of compactly supported, K_p -bi-invariant functions on $\mathbf{G}(\mathbb{Q}_p)$, and the space of W -invariant “trigonometric

¹⁰Recall a spherical (irreducible) representation of G_p is one that possesses a one dimensional space of K_p -invariants

polynomials” on \mathfrak{a} ; here “trigonometric polynomial” means “finite linear combination of characters.” Also $k \mapsto \hat{k}$ is an isometry:

$$(A.2) \quad \int_{\mathbf{G}(\mathbb{Q}_p)} |k(g)|^2 dg = \int_{\nu \in \mathfrak{a}_{\text{temp}}^*} |\hat{k}(\nu)|^2 d\mu(\nu)$$

The explicit form of the Plancherel measure is known [13]. From it we extract the following fact: “ $\mu_p = \mu_\infty + O(p^{-1/2})$.” More precisely, there exists a measure μ_∞ on $V^*/2\pi X^*$ (whose Fourier transform is supported in $\{X \in \mathfrak{a} : \|X\| \leq 3\}$) such that the difference $\mu_p - \mu_\infty$ is a signed measure represented by a function of supremum norm $O(p^{-1/2})$. Here to consider μ_∞ as a measure on $\mathfrak{a}_{\text{temp}}^*$ we identify with the our reference torus $V^*/2\pi X^*$ via rescaling by $\log p$.

A.3. The Paley-Wiener theorem. Recall our fixed W -invariant inner product on \mathfrak{a} . The Paley-Wiener theorem asserts that under the transform $k \mapsto \hat{k}$, the preimage of characters of $\mathfrak{a}_{\text{temp}}^*$ supported in $\{X \in \mathfrak{a} : \|X\| \leq R\}$ is contained in functions supported in $K_p \cdot \{X \in \mathfrak{a} : \|X\| \leq R\} \cdot K_p$.

Let us briefly sketch the proof. Let \mathfrak{a}^+ be the closed positive Weyl chamber within \mathfrak{a} . For $\alpha, \beta \in \mathfrak{a}^+$, so that $\nu \mapsto \nu(\beta)$ occurs with a nonzero coefficient in $\widehat{K_p \alpha K_p}$, then, necessarily, $\alpha - \beta$ belongs to the dual cone to \mathfrak{a}^+ .

Now, choose $\alpha \in \mathfrak{a}^+$ so that $k(K_p \alpha K_p) \neq 0$ and $\|\alpha\|$ is maximal subject to that restriction. We claim that $\nu \mapsto \nu(\alpha)$ necessarily occurs in \hat{k} . For, in view of the remarks above, if this were not the case there must exist $K_p \beta K_p$ in the support of k , with $\beta \in \mathfrak{a}^+$ and $\|\alpha\| < \|\beta\|$. This is a contradiction.

A.4. Proof of the amplification lemma. Let $\nu_0 \in \mathfrak{a}^*$ be the parameter of the character Λ of the Hecke algebra specified in the Lemma. We do not, of course, assume that $\nu_0 \in \mathfrak{a}_{\text{temp}}^*$.

Take any $a \in \mathfrak{a}$ with a not in the support of the Fourier transform of μ_∞ , but $\|a\|$ reasonably small. For example, we could take a to be twice the coroot associated to any root of maximal length; then $\|a\| = 4$. Take $R = |W|\|a\|$. Construct the function k_1 with spherical transform

$$(A.3) \quad \hat{k}_1(\nu) = \sum_{|j| \leq |W|} \overline{\sum_{w \in W} w\nu_0(ja)} \sum_{w \in W} w\nu(ja)$$

Note that if $\alpha_1, \dots, \alpha_m$ are any nonzero complex numbers, then by a simple compactness argument.

$$(A.4) \quad \max_{j \neq 0, |j| \leq m} \left| \alpha_1^j + \dots + \alpha_m^j \right| \geq c(m) > 0$$

So $M := \sum_{|j| \leq |W|} \left| \sum_{w \in W} w\nu_0(ja) \right|^2 \gg 1$. We have $\sup_{\nu \in \mathfrak{a}_{\text{temp}}^*} |\hat{k}_1(\nu)| \ll M^{1/2}$. Thus, by (A.2), $\|k_1\|_{L^2}^2 \ll M$; by definition, $\hat{k}_1(\nu_0) = M$; and from the explicit form of Plancherel measure and (A.1), $|k_1(1)| = O(M^{1/2} p^{-1/2})$.

Put $k = k_1 - k_1(1)1_{K_p}$. Then $k(1) = 0$ and – if we suppose that the residue field size, p , is sufficiently large, as we may – $|\hat{k}(\nu_0)| \gg \|k\|_{L^2}$.

By the Paley-Wiener theorem, k is supported in $\bigcup_{|a| \leq R} K_p a K_p$. The number of K_p -double cosets in this set is equal to the number of $a \in \mathfrak{a}$ with $|a| \leq R$, which is, in turn $O(1)$. (a is completely determined by the value of all the roots on it; but

the number of roots is $O(1)$ and each value is an integer $\leq R$). We conclude that there is $|a| \leq R$ so that

$$(A.5) \quad |\widehat{1_{K_p a K_p}}(\nu_0)|^2 \gg \int_{K_p a K_p} dg.$$

On the other hand, it is known ([3, Section 3.5]) that

$$(A.6) \quad \int_{K_p a K_p} dg \asymp \delta(a)^2$$

the notation \asymp meaning that the quotient is bounded above and below, at least for $p \geq p_0(\dim \mathbf{G})$. Since δ is the half-sum of positive roots, and each root has length ≤ 1 , we see that $p \leq \delta(a)^2 \leq p^{\dim(\mathbf{G})R}$. (cf. the end of §A.1.)

We take h_p to be the multiple $1_{K_p a_i K_p}$ by a suitable complex number of absolute value 1, so that $\hat{h}_p(\nu_0)$ is positive. The first three assertions of the lemma follow from remarks already made.

We now turn to establishing the necessary bounds on the denominator of $\rho(g)$. Let $\Phi_\rho \subset X^*$ be the weights of the representation of ρ with reference to \mathbf{A} ; for each $\alpha \in \Phi_\rho$, let $V(\alpha)$ be the weight space. We require two preliminary observations:

Firstly, $\sup_{\alpha \in \Phi_\rho} |\alpha| \leq N$. To verify this we may suppose that the representation ρ is irreducible. Because \mathbf{G} is semisimple, we may choose $w \in W$ so that $\|w\alpha - \alpha\| \geq \|\alpha\|$. Call two elements of X^* *connected* if they differ by an element of Φ . There is a sequence $\alpha = \alpha_0, \alpha_1, \dots, \alpha_r = w\alpha$ where α_i, α_{i+1} are connected and $\alpha_i \in \Phi_\rho$ for all i . So $\dim(V) \geq r + 1 \geq \|w\alpha - \alpha\| \geq \|\alpha\|$.

Secondly, $\mathbb{Z}_p^N = \bigoplus_{\alpha} \mathbb{Z}_p^N \cap V(\alpha)$ so long as the restriction map from characters of X^* to $\text{Hom}(A_p \cap K_p, \mathbb{Z}_p^\times)$ is injective. For, this being the case, the spaces $\mathbb{Z}_p^N \cap V(\alpha)$ are characterized as the “eigenspaces” of a prime-to- p finite group acting on \mathbb{Z}_p^N . This is so, in particular, if $p \geq p_0(N)$.

Combining these remarks, we see at once that the denominator of $\rho(a)$ is $\leq p^{NR}$ when $a \in A_p$ projects to the ball of radius $\leq R$ in \mathfrak{a} . The desired bound on denominators follow. \square

REFERENCES

1. Jean Bourgain and Elon Lindenstrauss, *Entropy of quantum limits*, Comm. Math. Phys. **233** (2003), no. 1, 153–171. MR 1957735 (2004c:11076)
2. Shimon Brooks and Elon Lindenstrauss, *Joint quasimodes, positive entropy, and quantum unique ergodicity*, Invent. Math. **198** (2014), no. 1, 219–259. MR 3260861
3. P. Cartier, *Representations of p -adic groups: a survey*, Automorphic forms, representations and L -functions (Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, American Mathematical Society, Providence, R.I., 1979, pp. 111–155. MR 81e:22029
4. Yves Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys. **102** (1985), no. 3, 497–502. MR 87d:58145
5. Manfred Einsiedler and Anatole Katok, *Invariant measures on G/Γ for split simple Lie groups G* , Comm. Pure Appl. Math. **56** (2003), no. 8, 1184–1221, Dedicated to the memory of Jürgen K. Moser. MR 2004e:37042
6. Manfred Einsiedler, Anatole Katok, and Elon Lindenstrauss, *Invariant measures and the set of exceptions to littlewood’s conjecture*, Ann. of Math. (2) **164** (2006), no. 2, 513–560. MR MR2247967 (2007j:22032)
7. Manfred Einsiedler and Elon Lindenstrauss, *On measures invariant under tori on quotients of semi-simple groups*, To appear in the Annals of Mathematics; preprint available at <http://www.math.ethz.ch/~einsiedl/maxsplit.pdf>.

8. Manfred Einsiedler, Elon Lindenstrauss, Philippe Michel, and Akshay Venkatesh, *Distribution of periodic torus orbits and Duke's theorem for cubic fields*, Ann. of Math. (2) **173** (2011), no. 2, 815–885. MR 2776363 (2012h:37006)
9. Elon Lindenstrauss, *On quantum unique ergodicity for $\Gamma \backslash \mathbb{H} \times \mathbb{H}$* , Internat. Math. Res. Notices **2001** (2001), no. 17, 913–933. MR 1859345 (2002k:11076)
10. ———, *Adelic dynamics and arithmetic quantum unique ergodicity*, Current developments in mathematics, 2004, Int. Press, Somerville, MA, 2006, pp. 111–139. MR 2459293 (2010i:58031)
11. ———, *Invariant measures and arithmetic quantum unique ergodicity*, Ann. of Math. (2) **163** (2006), no. 1, 165–219. MR MR2195133
12. Elon Lindenstrauss and Barak Weiss, *On sets invariant under the action of the diagonal group*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1481–1500. MR MR1855843 (2002j:22009)
13. I. G. Macdonald, *Spherical functions on a group of p -adic type*, Ramanujan Institute, Centre for Advanced Study in Mathematics, University of Madras, Madras, 1971, Publications of the Ramanujan Institute, No. 2. MR 0435301 (55 #8261)
14. Simon Marshall, *Sup norms of maass forms on semisimple groups*, preprint available at [arXiv:math.NT/1405.7033](https://arxiv.org/abs/math/1405.7033), 2014.
15. Djordje Milićević, *Large values of eigenfunctions on arithmetic hyperbolic surfaces*, Duke Math. J. **155** (2010), no. 2, 365–401. MR 2736169 (2012e:11088)
16. ———, *Large values of eigenfunctions on arithmetic hyperbolic 3-manifolds*, Geom. Funct. Anal. **21** (2011), no. 6, 1375–1418. MR 2860192
17. Zeév Rudnick and Peter Sarnak, *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. **161** (1994), no. 1, 195–213. MR 95m:11052
18. Lior Silberman, *Arithmetic quantum chaos on locally symmetric spaces*, Ph.D. thesis, Princeton University, 2005, available at <http://www.math.ubc.ca/~lior/work/>.
19. ———, *Quantum unique ergodicity on locally symmetric spaces: the degenerate lift*, preprint available at [arXiv:math.RT/1104.0074](https://arxiv.org/abs/math.RT/1104.0074), 2011.
20. Lior Silberman and Akshay Venkatesh, *Quantum unique ergodicity for locally symmetric spaces*, Geom. Funct. Anal. **17** (2007), no. 3, 960–998, [arXiv:math.RT/407413](https://arxiv.org/abs/math.RT/407413).
21. Alexander I. Šnirel'man, *Ergodic properties of eigenfunctions*, Uspekhi Mat. Nauk **29** (1974), no. 6(180), 181–182. MR 53 #6648
22. Kannan Soundararajan, *Quantum unique ergodicity for $SL_2(\mathbb{Z}) \backslash \mathbb{H}$* , Ann. of Math. (2) **172** (2010), no. 2, 1529–1538. MR 2680500
23. J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69. MR MR546588 (80h:20064)
24. Scott A. Wolpert, *Semiclassical limits for the hyperbolic plane*, Duke Math. J. **108** (2001), no. 3, 449–509. MR 2003b:11051
25. Asif Zaman, *Escape of mass on hilbert modular varieties*, Master's thesis, The University of British Columbia, 2012, M.Sc. thesis, available at <http://www.math.toronto.edu/asif/Research.html>.
26. Steven Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. J. **55** (1987), no. 4, 919–941. MR 89d:58129
27. ———, *The averaging method and ergodic theory for pseudo-differential operators on compact hyperbolic surfaces*, J. Funct. Anal. **82** (1989), no. 1, 38–68. MR 91e:58194

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