Journal of Pure and Applied Algebra 214 (2010) 1399–1406

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Unstable motivic homotopy categories in Nisnevich and cdh-topologies

Vladimir Voevodsky

School of Mathematics, Institute for Advanced Study, Princeton NJ, USA

ARTICLE INFO

Article history:
Received 6 January 2009
Received in revised form 5 October 2009
Available online 1 December 2009
Communicated by C.A. Weibel

MSC: 14F42

ABSTRACT

The motivic homotopy categories can be defined with respect to different topologies and different underlying categories of schemes. For a number of reasons (mainly because of the Gluing Theorem) the motivic homotopy category of smooth schemes with respect to the Nisnevich topology plays a distinguished role but in some cases it is desirable to be able to work with all schemes instead of the smooth ones. In this paper we prove that, under the resolution of singularities assumption, the unstable motivic homotopy category of all schemes over a field with respect to the cdh-topology is almost equivalent to the unstable motivic homotopy category of smooth schemes over the same field with respect to the Nisnevich topology.

1. Introduction

One can do motivic homotopy theory in the context of different motivic homotopy categories. One can vary the topology on the category of schemes used to define the homotopy category or one can vary the category of schemes itself considering only schemes satisfying certain conditions. The category obtained by taking smooth schemes and the Nisnevich topology seems to play a distinguished role in the theory because of the Gluing Theorem (see [7]) and some other, less significant, nice properties. On the other hand, in the parts of the motivic homotopy theory dealing with the motivic cohomology it is often desirable to be able to work with all schemes instead of just the smooth ones. For example, the motivic Eilenberg–MacLane spaces are naturally representable (in characteristic zero) by singular schemes built out of symmetric products of projective spaces but we do not know of any explicit way to represent these spaces by simplicial smooth schemes.

The goal of this paper is to show that, under the resolution of singularities assumption, the pointed motivic homotopy category of smooth schemes over a field with respect to the Nisnevich topology is almost equivalent to the pointed motivic homotopy category of all schemes over the same field with respect to the cdh-topology. More precisely, we show that the inverse image functor

$$\mathbf{L}\pi_\bullet^* : H_\bullet((Sm/k)_{Nis}, \mathbf{A}^1) \to H_\bullet((Sch/k)_{cdh}, \mathbf{A}^1)$$

from the former category to the later one is a localization and if f is a morphism such that $\mathbf{L}\pi_\bullet^*(f)$ is an isomorphism then the first simplicial suspension of f is an isomorphism. This implies in particular that the corresponding s-stable and T-stable motivic homotopy categories are equivalent.

The present paper is a continuation of [9] and it uses the formalism developed there. In the first section we define the standard cd-structures on the category of Noetherian schemes and prove that they are complete, regular and bounded. In the next section we prove some simple results about the homotopy categories of sites with interval with completely decomposable topologies. Our results also imply that the motivic homotopy categories defined with respect to the standard topologies are homotopy categories of almost finitely generated closed model structures (see [6]). In the last section we apply these results to prove the comparison theorem.

Everywhere below a scheme means a Noetherian scheme.

E-mail address: vladimir@ias.edu.

0022-4049/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2009.11.005

2. The standard cd-structures on categories of schemes

Let us consider the following two cd-structures on the category of Noetherian schemes.

Upper cd-structure or Nisnevich cd-structure where a square of the form

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{e} & X \end{array} \qquad (1)$$

is distinguished if it is a pull-back square such that p is etale, e is an open embedding and $p^{-1}(X-e(A)) \to X-e(A)$ is an isomorphism. Here $X - e(A)$ is considered with the reduced scheme structure.

Lower cd-structure or proper cdh-structure where a square of the form (1) is distinguished if it is a pull-back square such that p is proper, e is a closed embedding and $p^{-1}(X - e(A)) \to X - e(A)$ is an isomorphism.

Remark 2.1. These cd-structures owe their names to the fact that the behavior of the functors of inverse image f^* and $f^!$, which have upper indexes, with respect to etale morphisms is very similar to the behavior of the functors of direct image f_* and $f_!$, which have lower indexes, with respect to proper morphisms.

The topology associated with the upper cd-structure is called the upper cd-topology. We will show below (see Proposition 2.17) that it coincides with the Nisnevich topology. In particular, an etale morphism $f : X \to Y$ is an upper covering if and only if for any y in Y the fiber $p^{-1}(y)$ contains a k_y-rational point. The topology associated with the lower cd-structure is called the lower cd-topology or proper cdh-topology. By Proposition 2.18 a proper morphism of schemes $p : X \to Y$ is a lower cd-covering if and only if for any point y in Y the fiber $p^{-1}(y)$ contains a k_y-rational point.

The intersection of the upper and lower cd-structures is equivalent to the *additive cd-structure* where a square is distinguished if it is of the form

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \coprod Y \end{array} \qquad (2)$$

A presheaf F is a sheaf in the topology associated with the additive cd-structure if and only if

$$F(X \coprod Y) = F(X) \times F(Y)$$

and $F(\emptyset) = pt$.

The union of the upper and lower cd-structures is the *combined cd-structure*. A square is distinguished in it if it is an upper distinguished or a lower distinguished square. Proposition 2.17 and the definition given in [8, Section 4.1] imply that the associated topology is the cdh-topology.

If we consider only squares where both e and p are monomorphisms the upper and lower cd-structures become:

Plain upper cd-structure or Zariski cd-structure where a square of the form (1) is distinguished if both p and e are open embeddings and $X = p(Y) \cup e(A)$. The associated topology is the Zariski topology.

Plain lower cd-structure where a square of the form (1) is distinguished if both p and e are closed embeddings and $X = p(Y) \cup e(A)$. The associated topology is the closed analog of the Zariski topology.

Any combination of the additive, upper, lower, plain upper and plain lower cd-structures is called a standard cd-structure. There are nine standard cd-structures: the five generating ones, the combined cd-structure and the combinations of the plain upper with plain lower, plain upper with lower and plain lower with upper cd-structures. They form the following lattice where arrows indicate inclusions

$$\begin{array}{ccccc} additive & \longrightarrow & p.upper & \longrightarrow & upper \\ \downarrow & & \downarrow & & \downarrow \\ p.lower & \longrightarrow & p & \longrightarrow & p.lower + upper \\ \downarrow & & \downarrow & & \downarrow \\ lower & \longrightarrow & lower + p.upper & \longrightarrow & cdh \end{array}$$

The topology associated with the combination of the lower and the plain upper cd-structures $lower + p.upper$ is considered in [2]. The goal of this section is to prove the following theorem.

Theorem 2.2. *All the standard cd-structures are complete, regular and bounded on the category Sch/S of schemes of finite type over a Noetherian scheme S of finite dimension. In addition the upper and plain upper cd-structures are complete, regular and bounded on the category Sm/S of smooth schemes (of finite type) over a Noetherian scheme S of finite dimension.*

For the category of all schemes the theorem follows from statements 2.3, 2.13 and 2.14 which are proved below. The same arguments apply for Sm/S and the upper and plain upper cd-structures.

Lemma 2.3. *The standard cd-structures are complete on the category of schemes or schemes of finite type over a base. In addition the upper and plain upper cd-structures are complete on the category of smooth schemes over a base and the lower and plain lower cd-structures are complete on the category of proper schemes over a base.*

Proof. It follows immediately from [9, Lemma 2.5]. □

Let us show now that the standard cd-structures considered on the category of schemes of finite dimension are bounded. A sequence of points $x_0, \dots, x_d$ of a topological space X is called an increasing sequence (of length d) if $x_i \neq x_{i+1}$ and $x_i \in cl(x_{i+1})$ where $cl(x_{i+1})$ is the closure of the point x_{i+1} in X. For a scheme X define $D_d(X)$ as the class of open embeddings $j : U \to X$ such that for any $z \in X - U$ there exists an increasing sequence $z = x_0, x_1, \dots, x_d$ of length d. The density structure defined by the classes D_d is called the *standard density structure* on the category of schemes. It is locally of finite dimension on the category of schemes of finite dimension and the dimension of a scheme with respect to it is the dimension of the corresponding topological space.

Lemma 2.4. *If $U, V \in D_d(X)$ then $U \cap V \in D_d(X)$.*

Lemma 2.5. *Let $U \in D_d(X)$ and V be an open subscheme of X. Then $U \cap V \in D_d(V)$.*

Proof. Let x be a point of V outside of $U \cap V$. Considered as a point of X it has an increasing sequence $x = x_0, \dots, x_d$ with $x_i \in X$. But since $x_0 \in V$ we have $x_i \in V$ because $x_0 \in cl(x_i)$ and V is open. □

Lemma 2.6. *Let x_0, x_1, x_2 be an increasing sequence on a scheme X and Z be a closed subset of X such that x_2 lies outside Z. Then there exists an increasing sequence x_0, x'_1, x_2 such that x'_1 lies outside Z.*

Proof. Replacing X by the local scheme of x_0 in the closure of x_2 we may assume that any point of X contains x_0 in its closure and in turn lies in the closure of x_2. It remains to show that the complement to Z contains at least one point which is not equal to x_2. If it were false we would have $x_2 = X - Z$ i.e. x_2 would be a locally closed point. This contradicts our assumption since by [4, 5.1.10(ii)] a locally closed point on a locally Noetherian scheme has dimension ≤ 1. □

Lemma 2.7. *Let X be a scheme, U a dense open subset of X and $x_0, \dots, x_d$ any increasing sequence in X. Then there exists an increasing sequence $x_0, x'_1, \dots, x'_d$ such that $x'_i \in U$ for all $i \geq 1$.*

Proof. We may assume that $d > 0$. If x_d is contained in U set $x'_d = x_d$. Otherwise let x'_d be a point of U such that $x_{d-1} \in cl(x_d)$ which exists since U is dense. Since x_d is not in U, x_{d-1} is not in U and thus $x'_d \neq x_{d-1}$ and $x_0, x_1, \dots, x'_d$ is again an increasing sequence. Assume by induction that we constructed $x'_{i+1}, \dots, x'_d \in U$ such that $x_0, \dots, x_i, x'_{i+1}, \dots, x'_d$ is an increasing sequence. By Lemma 2.6 for any increasing sequence y_0, y_1, y_2 and a closed subset Z which does not contain y_2 there exists an increasing sequence y_0, y'_1, y_2 such that Z does not contain y_1. Applying this result to the sequence x_{i-1}, x_i, x'_{i+1} and $Z = X - U$ we construct x'_i. □

Lemma 2.8. *Let X be a scheme and Y a constructible subset in X. Then any point y' of the closure $cl(Y)$ of Y in X belongs to the closure of a point y of Y.*

Proof. Since Y is constructible it is of the form $Y = \cup_{i=1}^n Y_i$ where each Y_i is open in a closed subset of X (see e.g. [5, Prop. 2.3.3]). It is clearly sufficient to prove our statement for each Y_i. As a topological space Y_i corresponds to a Noetherian scheme. Thus there exists finitely many points y'_i in Y such that any point of Y is in the closure of one of the y_i's. If a point y' in $cl(Y)$ has an open neighborhood U which does not contain any of the points y_i then U does not contain any point of Y which contradicts the assumption that $y \in cl(Y)$. Thus y belongs to the closure of $\{y_i\}$ which coincides with the union of closures of points y_i since there is finitely many of them. □

Lemma 2.9. *Let $f : X \to Y$ be a morphism of finite type of Noetherian schemes and assume that there exists an open subset U in Y such that $f^{-1}(U)$ is dense in X and $f^{-1}(U) \to U$ has fibers of dimension zero. Then for any $d \geq 0$ and $V \in D_d(X)$ there exists $W \in D_d(Y)$ such that $f^{-1}(W) \subset V$.*

Proof. We may clearly assume that $d > 0$. Let $Z = X - V$. We have to show that $Y - cl(f(Z)) \in D_d(Y)$ i.e. that for any y in $cl(f(Z))$ there exists an increasing sequence $y = y_0, \dots, y_d$ in Y. Since f is of finite type $f(Z)$ is constructible and in particular any point of $cl(f(Z))$ is in the closure of a point in $f(Z)$ by Lemma 2.8. Thus we may assume that y belongs to $f(Z)$ i.e. $y = f(x)$ where x is in Z. By Lemma 2.7 we can find an increasing sequence $x = x_0, x_1, \dots, x_d$ for x such that for $i > 0$ we have $x_i \in f^{-1}(U)$. Then $y = f(x_0), \dots, f(x_d)$ is an increasing sequence i.e. $f(x_i) \neq f(x_{i+1})$. Indeed for $i > 0$ it follows from the fact that the fibers of f over U are of dimension zero. For $i = 0$ we have two cases. If $f(x_0) \in U$ then the same argument as for $i > 0$ applies. If $f(x_0)$ is not in U then $f(x_0) \neq f(x_1)$ since $f(x_1) \in U$. □

Proposition 2.10. *The upper cd-structure and the plain upper cd-structures on the category of Noetherian schemes of finite dimension are bounded with respect to the standard density structure.*

Proof. We will only consider the upper cd-structure. The plain case is similar. Let us show that any upper distinguished square is reducing with respect to the standard density structure (see [9, Definition 2.21]). Let our square be of the form

$$\begin{array}{ccc} W & \xrightarrow{j_V} & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array} \tag{3}$$

and let $W_0 \in D_{d-1}(W)$, $U_0 \in D_d(U)$, $V_0 \in D_d(V)$. Applying Lemma 2.9 to the morphism $j \coprod p$ we can find $X_0 \in D_d(X)$ such that $j(U_0) \cup p(V_0) \subset X_0$. Replacing X with X_0 and applying Lemma 2.5 we may assume that $U_0 = U$ and $V_0 = V$. Let $Z = W - W_0$, $C = X - U$ and set $X' = X - (C \cap cl(pj_V(Z)))$. Let us show that the square

$$\begin{array}{ccc} W_0 & \longrightarrow & j_V(W_0) \\ \downarrow & & \downarrow \\ U & \longrightarrow & X' \end{array} \tag{4}$$

is upper distinguished. It is clearly a pull-back square, the right vertical arrow is etale and the lower horizontal one is an open embedding. It is also obvious that $p^{-1}(X' - U) \cap j_V(W_0) = (X' - U)$. To finish the proof it remains to show that $X' \in D_d(X)$. Let x be a point of X outside of X' i.e. a point of $C \cap cl(pj_V(Z))$. Since $pj_V(Z) \cap C = \emptyset$ there exists $x' = pj_V(y) \in pj_V(Z)$ such that $x \in cl(x')$ and $x' \neq x$. Let $y = y_0, \ldots, y_{d-1}$ be an increasing sequence for y in W which exists since $W_0 \in D_{d-1}(W)$. The morphism $q = pj_V$ has fibers of dimension zero and therefore $q(y_0), \ldots, q(y_{d-1})$ is an increasing sequence for x'. Thus we get an increasing sequence $x, q(y_0), \ldots, q(y_{d-1})$ for x of length d. □

Proposition 2.11. *The lower cd-structure and the plain lower cd-structures on the category of Noetherian schemes of finite dimension are bounded with respect to the standard density structure.*

Proof. We will only consider the case of the lower cd-structure. The plain case is similar. Consider a lower distinguished square

$$Q = \left(\begin{array}{ccc} B & \xrightarrow{i_Y} & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{i} & X \end{array}\right). \tag{5}$$

If we replace Y by the scheme-theoretic closure of the open subscheme $p^{-1}(X-A)$ we get another lower distinguished square which is a refinement of the original one. This square satisfies the condition of Lemma 2.12 and therefore it is reducing. □

Lemma 2.12. *A lower distinguished square of the form (5) such that the subset $p^{-1}(X - A)$ is dense in Y is reducing with respect to the lower cd-structure.*

Proof. Let $Y_0 \in D_d(Y)$, $A_0 \in D_d(A)$, $B_0 \in D_{d-1}(B)$. Applying Lemma 2.9 to p and $U = X - A$ we conclude that there exists $X_0 \in D_d(X)$ such that $p(Y_0) \subset X_0$. Applying the same lemma to i we find an open subset $X_1 \in D_d(X)$ such that $i(A_0) \subset X_1$. Then by Lemma 2.4 $X_1 \cap X_0 \in D_d(X)$ and replacing X by $X_1 \cap X_0$ and using Lemma 2.5 we may assume that $A_0 = A$ and $Y_0 = Y$. Let $X' = X - pi_Y(B - B_0)$. To finish the proof it is enough to check that $X' \in D_d(X)$ and define Q' as the pull-back of Q to X'. According to Lemma 2.9 applied again to p and $U = X - A$ it is enough to check that $Y - i_Y(B - B_0) \in D_d(Y)$. Since $B_0 \in D_{d-1}(B)$ and i_Y is a closed embedding it is enough to check that $Y - i_B(B)$ is dense in Y. This follows from our assumption since $Y - i_B(B) = p^{-1}(X - A)$. □

Since all generating cd-structures on the category of Noetherian schemes are bounded by the *same* density structure any combination of such structures is also bounded by this density structure. We get the following result.

Proposition 2.13. *The standard cd-structures on the category of Noetherian schemes of finite dimension are bounded.*

Finally let us show that all the standard cd-structures are regular. It is clearly sufficient to consider the "generating" cd-structures. Then any combination of them will also be regular.

Lemma 2.14. *The additive, upper, plain upper, lower and plain lower cd-structures are regular.*

Proof. The additive case is obvious. Let us show that the upper, palin upper, lower and plain lower cd-structures satisfy the conditions of [9, Lemma 2.11]. The first two conditions are obvious. Consider the third condition in the upper case. The square

$$d(Q) = \begin{pmatrix} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B \times_A B & \longrightarrow & Y \times_X Y \end{pmatrix} \tag{6}$$

is a pull-back square. Since p is etale, and in particular unramified, the diagonal $Y \to Y \times_X Y$ is an open embedding. The morphism $B \times_A B \to Y \times_X Y$ is an open embedding because e is an open embedding. The condition that $p^{-1}(X - e(A)) \to X - e(A)$ is a universal homeomorphism implies that for a pair of geometric points y_1, y_2 of Y such that $p(y_1) = p(y_2) \in X - e(A)$ one has $y_1 = y_2$. Therefore,

$$Y \times_X Y = (B \times_A B) \cup Y$$

i.e. (6) is a (plain) upper distinguished square.

Consider the third condition in the lower case. The square (6) is a pull-back square. Since p is proper, and in particular separated, the diagonal $Y \to Y \times_X Y$ is a closed embedding. The morphism $B \times_A B \to Y \times_X Y$ is a closed embedding because e is a closed embedding. The condition that $p^{-1}(X - e(A)) \to X - e(A)$ is a universal homeomorphism implies that for a pair of geometric points y_1, y_2 of Y such that $p(y_1) = p(y_2) \in X - e(A)$ one has $y_1 = y_2$. Therefore,

$$Y \times_X Y = (B \times_A B) \cup Y$$

i.e. (6) is a (plain) lower distinguished square. □

Definition 2.15. Let $f : \tilde{X} \to X$ be a morphism of schemes. A splitting sequence for f is a sequence of closed embeddings

$$\emptyset = Z_{n+1} \to Z_n \to \cdots \to Z_1 \to Z_0 = X$$

such that for any $i = 0, \ldots, n$ the projection

$$(Z_i - Z_{i+1}) \times_X \tilde{X} \to (Z_i - Z_{i+1})$$

has a section.

Lemma 2.16. *A morphism of finite type of Noetherian schemes $f : \tilde{X} \to X$ has a splitting sequence if and only if for any point x of X there exists a point $\tilde{x}$ of $\tilde{X}$ such that $f(\tilde{x}) = x$ and the corresponding morphism of the residue fields is an isomorphism.*

Proof. The "only if" part is obvious. The "if" part follows easily by the Noetherian induction (cf. [7, Lemma 3.1.5, p. 97]). □

Proposition 2.17. *An etale morphism $f : \tilde{X} \to X$ is a covering in the upper cd-topology if and only if for any point x of X there exists a point $\tilde{x}$ of $\tilde{X}$ such that $f(\tilde{x}) = x$ and the corresponding morphism of the residue fields is an isomorphism.*

Proof. Since the upper cd-structure is complete any upper cd-covering has a refinement which is a simple covering which immediately implies the "only if" part of the proposition. To prove the "if" part we have to show, in view of Lemma 2.16, that any etale morphism $f : \tilde{X} \to X$ which has a splitting sequence $Z_n \to \cdots \to Z_0 = X$ is an upper cd-covering. We will construct an upper distinguished square of the form (1) based on X such that the pull-back of f to Y has a section and the pull-back of f to A has a splitting sequence of length less than n. The result then follows by induction on n. We take $A = X - Z_n$. To define Y consider the section s of $f_n : \tilde{X} \times_X Z_n \to Z_n$ which exists by definition of a splitting sequence. Since f is etale and in particular unramified the image of s is an open subscheme. Let W be its complement. The morphism $\tilde{X} \times_X Z_n \to \tilde{X}$ is a closed embedding thus the image of W is closed in $\tilde{X}$. We take $Y = \tilde{X} - W$. One verifies immediately that the pull-back square defined by $A \to X$ and $Y \to X$ is upper distinguished. The pull-back of f to Y has a section and the pull-back of f to A has a splitting sequence of length $n - 1$. This finishes the proof of the proposition. □

Proposition 2.17 implies that the topology associated with the upper cd-structure on the category of Noetherian schemes is the Nisnevich topology.

Proposition 2.18. *A proper morphism $f : \tilde{X} \to X$ is a covering in the lower cd-topology if and only if for any point x of x there exists a point $\tilde{x}$ of $\tilde{X}$ such that $f(\tilde{x}) = x$ and the corresponding morphism of the residue fields is an isomorphism.*

Proof. Since the lower cd-structure is complete any lower cd-covering has a refinement which is a simple covering which immediately implies the "only if" part of the proposition. To prove the "if" part we have to show, in view of Lemma 2.16, that any proper morphism $f : \tilde{X} \to X$ which has a splitting sequence $Z_n \to \cdots \to Z_0 = X$ is a lower cd-covering. We will construct a lower distinguished square of the form (1) based on X such that the pull-back of f to Y has a section and the pull-back of f to A has a splitting sequence of length less than n. The result then follows by induction on n. We take $A = Z_1$. To define Y consider the section s of $f_n : \tilde{X} \times_X (X - Z_1) \to (X - Z_1)$ which exists by definition of a splitting sequence. Since f is proper and in particular separated, the image of s is a closed subscheme. Let W be its complement. The morphism $\tilde{X} \times_X (X - Z_1) \to \tilde{X}$ is an open embedding thus the image of W is open in $\tilde{X}$. We take $Y = \tilde{X} - W$. One verifies immediately that the pull-back square defined by $A \to X$ and $Y \to X$ is lower distinguished. The pull-back of f to Y has a section and the pull-back of f to A has a splitting sequence of length $n - 1$. This finishes the proof of the proposition. □

3. Motivic homotopy categories

Recall that in [7] we defined for any site T with an interval I a category $H(T, I)$ which we called the homotopy category of (T, I). Applying this definition to a category of schemes with some standard topology and taking I to be the affine line one obtains different motivic homotopy categories. Among these homotopy categories the one denoted in [7] by $H(S)$ and corresponding to the category of smooth schemes over S with the Nisnevich or upper cd-topology seems to play a distinguished role. In this section we prove a number of results which provide a new description for the motivic homotopy categories in the standard topologies and in particular for the category $H(S)$. We start with some results applicable to all sites with interval with good enough completely decomposable topologies.

Let C be a category with a complete regular bounded cd-structure P (see [9]) and an interval I (see [7, Section 2.3, p. 85]). Assume in addition that C has a final object and that for any X in C the product $X \times I$ exists. We can form the homotopy category of (C, P, I) in two ways. First, we may define a new cd-structure (P, I) whose distinguished squares are the distinguished squares of C and squares of the form

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & X \end{array}$$

where X runs through all objects of C and consider the homotopy category of this cd-structure i.e. the localization of $\Delta^{op}PreShv(C)$ with respect to the class $cl_l(G_{(P,I)})$ of $G_{(P,I)}$-local equivalences (see [9, around Lemma 3.7]). For reasons of notation compatibility with [10] we denote it by $H(C^{\amalg_{<\infty}}, G_{(P,I)})$.

On the other hand we may consider the homotopy category $H(C_{t_P}, I)$ of the site with interval (C_{t_P}, I) as defined in [7]. We are going to show that if P is complete, regular and bounded these two constructions agree.

Proposition 3.1. *Let (C, I) be a category with an interval which has a final object and such that the products $X \times I$ exist. Let P be a complete, regular and bounded cd-structure on C. Then the categories $H(C^{\amalg_{<\infty}}, G_{(P,I)})$ and $H(C_{t_P}, I)$ are naturally equivalent.*

Proof. Both categories are defined as localizations of $\Delta^{op}PreShv(C)$ and we only have to prove that the two localizing classes coincide. Both localizing classes are defined as "left orthogonals" to the corresponding classes of local objects. Therefore it is sufficient to prove that the class of $G_{(P,I)}$-local objects coincides, up to projective equivalences, with the class of objects whose associated sheaf is I-local in the sense of [7, Def. 2.1]. This follows easily from [9, Prop. 3.8]. □

We will also need the pointed analog of Proposition 3.1. Let $PreShv_\bullet$ be the category of pointed presheaves of sets on C. This category can be identified with the category of radditive functors on the category $(C^{\amalg_{<\infty}})_+$ where $C^{\amalg_{<\infty}}$ is the category with finite coproducts freely generated by C and, for a category D with a final object and finite coproducts, D_+ is the full subcategory of the category of pointed objects in D which consists of objects pointed by a disjoint point. Therefore, in order to keep the notation compatibility with [10] we should write $H((C^{\amalg_{<\infty}})_+, E_+)$ for the category obtained by the localization of $\Delta^{op}PreShv_\bullet$ by E_+-local equivalences where E is a set of morphisms in $\Delta^{op}PreShv(C)$. By [10, Cor. 4.21(3)] we know that a morphism in $\Delta^{op}PreShv_\bullet$ is a E_+-local equivalence if and only if it is mapped to a E-local equivalence by the functor which forgets the distinguished point. Therefore, Proposition 3.1 has the following corollary.

Corollary 3.2. *Let (C, I) be a category with an interval which has a final object and such that the products $X \times I$ exist. Let P be a complete, regular and bounded cd-structure on C. Then the categories $H((C^{\amalg_{<\infty}})_+, (G_{(P,I)})_+)$ and $H_\bullet(C_{t_P}, I)$ are naturally equivalent.*

Specializing to the case of the motivic homotopy categories and using Theorem 2.2 we get the following results.

Proposition 3.3. *Let P be a standard cd-structure on the category Sch/S. Then one has*

$$H((Sch/S)^{\amalg_{<\infty}}, G_{(P,\mathbf{A}^1_S)}) = H((Sch/S)_{t_P}, \mathbf{A}^1_S)$$

$$H(((Sch/S)^{\amalg_{<\infty}})_+, (G_{(P,\mathbf{A}^1_S)})_+) = H_\bullet((Sch/S)_{t_P}, \mathbf{A}^1_S).$$

Proposition 3.4. *Let P be a standard cd-structure which is contained in the upper cd-structure. Then one has*

$$H((Sm/S)^{\amalg_{<\infty}}, G_{(P,\mathbf{A}^1_S)}) = H((Sm/S)_{t_P}, \mathbf{A}^1_S)$$

$$H(((Sm/S)^{\amalg_{<\infty}})_+, (G_{(P,\mathbf{A}^1_S)})_+) = H_\bullet((Sm/S)_{t_P}, \mathbf{A}^1_S).$$

These identifications allow one to consider the categories on the right hand side of the equalities in Propositions 3.3, 3.4 as particular cases of the homotopy categories considered in [10] and to apply the results of this paper to these categories.

4. The comparison theorem

Let k be a field. We have an obvious continuous map of sites with intervals

$$\pi : (Sch/k)_{cdh} \to (Sm/k)_{Nis}$$

which is reasonable by the results of the first section and [9, Prop. 3.9]. Let

$$\mathbf{L}\pi_\bullet^* : H_\bullet((Sm/k)_{Nis}, \mathbf{A}^1) \to H_\bullet((Sch/k)_{cdh}, \mathbf{A}^1) \tag{7}$$

be the corresponding inverse image functor on the pointed homotopy categories. For a morphism f in the pointed homotopy category we denote by

$$\Sigma_s^1(f) = f \wedge Id_{S_s^1}$$

the first simplicial suspension of f. Let us recall the following definition given in [1].

Definition 4.1. A field k is said to admit resolution of singularities if the following two conditions hold:

1. for any reduced scheme of finite type X over k there exists a proper morphism $f : \tilde{X} \to X$ such that $\tilde{X}$ is smooth and f has a section over a dense open subset of X
2. for any smooth scheme X over k and a proper surjective morphism $Y \to X$ which has a section over a dense open subset of X there exists a sequence of blow-ups with smooth centers $X_n \to X_{n-1} \to \cdots \to X_0 = X$ and a morphism $X_n \to Y$ over X.

Note that any field satisfying the conditions of Definition 4.1 is perfect. For a functor Φ let $iso(\Phi)$ be the class of morphisms which are mapped to isomorphisms by Φ.

Theorem 4.2. *Let k be a field which admits resolution of singularities. Then the functor $\mathbf{L}\pi_\bullet^*$ is a localization and for any f in $iso(\mathbf{L}\pi_\bullet^*)$ the morphism $\Sigma_s^1(f)$ is an isomorphism.*

Proof. Let us start with the following construction. The proof of the theorem is completed right before Corollary 4.8.

Define the smooth blow-up cd-structure on the category Sm/k of smooth schemes over k as the collection of pull-back squares of the form (1) such that e is a closed embedding and p is the blow-up with the center in $e(A)$.

Lemma 4.3. *Let k be a field which admits resolution of singularities. Then the smooth blow-up cd-structure on the category of smooth schemes over k is complete.*

Proof. To show that a cd-structure is complete it is sufficient to show that for any distinguished square of the form (1) and any morphism $f : X' \to X$ the sieve $f^*(e, p)$ contains the sieve generated by a simple covering (see [9, Lemma 2.4]). Let us prove it by induction on $dim(X')$. If $dim(X') = 0$ the sieve $f^*(e, p)$ contains an isomorphism. Assume that the statement is proved for $dim(X') < d$ and let X' be of dimension d. The map $X' \times_X (A \coprod Y) \to X'$ is proper and has a section over a dense open subset of X'. Thus by the resolution of singularities assumption we have a sequence of blow-ups with smooth centers $X'_n \stackrel{p_{n-1}}{\to} X'_{n-1} \stackrel{p_{n-2}}{\to} \cdots \stackrel{p_0}{\to} X'_0 = X'$ such that the pull-back of (e, p) to X'_n contains an isomorphism and in particular a sieve generated by a simple covering. Assume by induction that the pull-back of (e, p) to X'_i contains a sieve generated by a simple covering $\{r_j : U_j \to X'_i\}$ and let us show that the same is true for X'_{i-1}. Let $e_{i-1} : Z'_{i-1} \to X'_{i-1}$ be the center of the blow-up $X'_i \to X'_{i-1}$. The restriction of (e, p) to Z'_{i-1} contains a sieve generated by a simple covering $\{s_l : V_l \to Z'_{i-1}\}$ since $dim(Z'_{i-1}) < d$. Thus the restriction of (e, p) to X'_{i-1} contains the sieve generated by $\{p_{i-1}r_j, e_{i-1}s_l\}$ which is a simple covering by definition. □

Lemma 4.4. *The smooth blow-up cd-structure on the category of smooth schemes over any field is bounded with respect to the standard density structure.*

Proof. The same arguments as in the proof of Lemma 2.12 show that any distinguished square of the smooth blow-up cd-structure is reducing with respect to the standard density structure. □

Lemma 4.5. *The smooth blow-up cd-structure on the category of smooth schemes over any field is regular.*

Proof. The first two conditions of [9, Definition 2.10] are obviously satisfied. To prove the third one we have to show that for a distinguished square of the form (1) the map of representable sheaves of the form

$$\rho(Y) \coprod \rho(B) \times_{\rho(A)} \rho(B) \to \rho(Y) \times_{\rho(X)} \rho(Y) \tag{8}$$

is surjective. Since any smooth scheme has a covering in our topology by connected smooth schemes it is sufficient to show that the map of presheaves corresponding to (8) is surjective on sections on smooth connected schemes. Let U be a smooth connected scheme and $f, g : U \to Y$ be a pair of morphisms such that $p \circ f = p \circ g$. The scheme $Y \times_X Y$ is the union of two closed subschemes namely the diagonal Y and $B \times_A B$ (see the proof of the lower case in Lemma 2.14). Since U is smooth and connected it is irreducible and therefore the closure of the image of $f \times g$ in $Y \times_X Y$ is irreducible. This implies that the image belongs to either Y or $B \times_A B$ and since U is smooth and in particular reduced the morphism $f \times_X g$ lifts to Y or to $B \times_A B$. □

Consider the topology *scdh* associated with the sum of the smooth blow-up cd-structure and the upper cd-structure on the category of smooth schemes over S.

Lemma 4.6. *The scdh cd-structure is bounded and regular. If k admits resolution of singularities then it is also complete.*

Proof. Since the sum of two cd-structures bounded by the same density structure is bounded, Proposition 2.10 and Lemma 4.4 imply that this cd-structure is bounded by the standard density structure on Sm/k. Since the sum of two regular cd-structures is regular, Lemmas 2.14 and 4.5 imply that it is regular. Since the sum of two complete cd-structures is complete, Lemmas 2.3 and 4.3 imply that if k admits resolution of singularities then this cd-structure is complete. □

Lemma 4.7. *If k admits resolution of singularities the inverse image functor $\pi_1^* : Shv_{scdh}(Sm/k) \to Shv_{cdh}(Sch/k)$ is an equivalence.*

Proof. The resolution of singularities assumption implies that any object of Sch/k has a cdh-covering by objects of Sm/k and that any cdh-covering of an object of Sm/k has a refinement which is a scdh-covering. These two facts together imply that the inverse and the direct image functors define equivalences of the corresponding categories of sheaves (see [3]). □

The continuous map $\pi : (Sch/k)_{cdh} \to (Sm/k)_{Nis}$ factors as a composition $\pi = \pi_2 \circ \pi_1$ where $\pi_1 : (Sch/k)_{cdh} \to (Sm/k)_{scdh}$ and $\pi_2 : (Sm/k)_{scdh} \to (Sm/k)_{Nis}$. Both continuous maps are reasonable by the lemmas above and [9, Prop. 3.9]. Therefore, $\mathbf{L}\pi^* = \mathbf{L}\pi_1^*\mathbf{L}\pi_2^*$ and a similar equality holds for the inverse images on pointed sheaves.

Lemma 4.7 implies that up to an equivalence the functor $\mathbf{L}\pi_\bullet^*$ coincides with $\mathbf{L}\pi_{2,\bullet}^*$. The later functor is the localization corresponding to the increase of the cd-structure on Sm/k from Nis to $scdh$. Therefore, by Corollary 3.2 we have

$$iso(\mathbf{L}\pi_{2,\bullet}^*) = cl_l((G_{\mathbf{A}^1})_+ \amalg (G_{scdh})_+).$$

Applying as was explained at the end of the previous section the results of [10] we get from [10, Cor. 3.52]:

$$cl_l((G_{scdh})_+) = cl_{\bar{\Delta}}((G_{\mathbf{A}^1})_+ \amalg (G_{scdh})_+ \amalg W_{proj})$$

where $cl_{\bar{\Delta}}$ is the $\bar{\Delta}$-closure and W_{proj} is the class of pointed projective equivalences.

For any class E and any X one has $Id_X \wedge cl_{\bar{\Delta}}(E) \subset cl_{\bar{\Delta}}(Id_X \wedge E)$. Therefore,

$$\Sigma_s^1(iso(\mathbf{L}\pi_{2,\bullet}^*)) \subset cl_{\bar{\Delta}}(\Sigma_s^1((G_{\mathbf{A}^1})_+) \amalg \Sigma_s^1((G_{scdh})_+) \amalg \Sigma_s^1(W_{proj})).$$

Since $\Sigma_s^1(W_{proj}) \subset W_{proj}$ and $\Sigma_s^1((G_{\mathbf{A}^1})_+) \subset cl_{\bar{\Delta}}((G_{\mathbf{A}^1})_+)$, it remains to show that $\Sigma_s^1((G_{scdh})_+) \subset cl_l((G_{\mathbf{A}^1})_+ \amalg (G_{Nis})_+ \amalg W_{proj})$. This follows from [7, Remark 3.2.30, p.118]. Theorem 4.2 is proved. □

Corollary 4.8. *Let k be as above and X and Y be pointed simplicial sheaves on $(Sm/k)_{Nis}$ such that Y is $\mathbf{A}^1$-equivalent to the simplicial loop space of an $\mathbf{A}^1$-local object. Then the map*

$$Hom(X, Y) \to Hom(\mathbf{L}\pi^*(X), \mathbf{L}\pi^*(Y)),$$

where the morphisms on the left hand side are in $H((Sm/k)_{Nis}, \mathbf{A}^1)$ and on the right hand side in $H_\bullet((Sch/k)_{cdh}, \mathbf{A}^1)$, is bijective.

Acknowledgements

The work on this paper was supported by the NSF grants DMS-97-29992 and DMS-9901219, Sloan Research Fellowship and Veblen Fund.

This paper was written in 2000 while I was a member of the Institute for Advanced Study in Princeton and, part of the time, an employee of the Clay Mathematics Institute. I am very grateful to both institutions for their support. I would also like to thank Charles Weibel who pointed out a number of places in the previous version of the paper which required corrections.

References

[1] Eric M. Friedlander, Vladimir Voevodsky, Bivariant cycle cohomology, in: Cycles, Transfers, and Motivic Homology Theories, in: Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 138–187.
[2] T.G. Goodwillie, S. Lichtenbaum, A cohomological bound for the *h*-topology, Amer. J. Math. 123 (3) (2001) 425–443.
[3] A. Grothendieck, M. Artin, J.-L. Verdier, Theorie des topos et cohomologie etale des schemas (SGA 4), in: Lecture Notes in Math., vols. 269, 270, 305, Springer, Heidelberg, 1972–1973.
[4] A. Grothendieck, J. Dieudonne, Etude Locale des Schemas et des Morphismes de Schemas (EGA 4), in: Publ. Math. IHES, vols. 20, 24, 28, 32, 1964–1967.
[5] A. Grothendieck, J. Dieudonne, Le Langage des Schemas (EGA 1), Springer, Heidelberg, 1971.
[6] Mark Hovey, Spectra and symmetric spectra in general model categories, J. Pure Appl. Algebra 165 (1) (2001) 63–127.
[7] Fabien Morel, Vladimir Voevodsky, $\mathbf{A}^1$-homotopy theory of schemes, Inst. Hautes Études Sci. Publ. Math. 90 (2001) 45–143. 1999.
[8] Andrei Suslin, Vladimir Voevodsky, Relative cycles and Chow sheaves, in: Cycles, Transfers, and Motivic Homology Theories, in: Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 10–86.
[9] Vladimir Voevodsky, Homotopy theory of simplicial sheaves in completely decomposable topologies, J. Pure Appl. Algebra 214 (8) (2010) 1384–1398.
[10] Vladimir Voevodsky, Simplicial radditive functors, Journal of K-theory (2010) (in press) arXiv:0805.4434.