Triangulated categories of motives over a field.

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Contents

1 Introduction.

In this paper we construct for any perfect field k a triangulated category $DM_-^{eff}(k)$ which is called the *triangulated category of (effective) motivic complexes* over k (the minus sign indicates that we consider only complexes bounded from the above). This construction provides a natural categorical framework to study different algebraic cycle cohomology theories ([3],[13],[9],[7]) in the same way as the derived category of the etale sheaves provides a categorical framework for the etale cohomology. The first section of the paper may be considered as a long introduction. In §2.1 we give an elementary construction of a triangulated category $DM_{gm}^{eff}(k)$ of *effective geometrical motives* over k which is equivalent to the full triangulated subcategory in $DM_{-}^{eff}(k)$ generated by "motives" of smooth varieties. In §2.2 we give a detailed summary of main results of the paper.

We do not discuss here the relations of our theory to the hypothetical theory of mixed motives or more generally mixed motivic sheaves ([1],[2]) primarily because it would require giving a definition of what the later theory is (or should be) which deserves a separate carefull consideration. We woluld like to mention also that though for rational coefficients the standard motivic conjectures predict that $DM_{gm}^{eff}(k)$ should be equivalent to the derived category of bounded complexes over the abelian category of mixed motives most probably no such description exists for integral or finite coefficients (see 4.3.8).

Most of the proofs in this paper are based on results obtained in [7] and [15]. In particular due to the resolution of singularities restriction in [7] the results of Section 4 are proven at the moment only for fields of characteristic zero. As a general rule the restrictions on the base field which are given at the beginning of each section are assumed throughout this section if the opposite is not explicitly declared.

We would like to mention two other constructions of categories similar to our $DM_{gm}(k)$. One was given by M. Levine in [8]. Another one appeared in [16]. At the end of §4.1 we give a sketch of the proof that with the rational coefficients it is equivalent to the category defined here if the base field has resolution of singularities.

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2 Geometrical motives.

2.1 *The triangulated category of geometrical motives.*

Let k be a field. We denote by Sm/k the category of smooth schemes over k.

For a pair X, Y of smooth schemes over k denote by $c(X, Y)$ the free abelian group generated by integral closed subschemes W in $X \times Y$ which are finite over X and surjective over a connected component of X. An element of $c(X,Y)$ is called a *finite correspondence*[1] from X to Y.

Let now X_1, X_2, X_3 be a triple of smooth schemes over k, $\phi \in c(X_1, X_2)$ be a finite correspondence from X_1 to X_2 and $\psi \in c(X_2, X_3)$ be a finite correspondence from X_2 to X_3. Consider the product $X_1 \times X_2 \times X_3$ and let $pr_i : X_1 \times X_2 \times X_3 \to X_i$ be the corresponding projections. One can verify easily that the cycles $(pr_1 \times pr_2)^*(\phi)$ and $(pr_2 \times pr_3)^*(\psi)$ on $X_1 \times X_2 \times X_3$ are in general position. Let $\psi * \phi$ be their intersection. We set $\psi \circ \phi = (pr_1 \times pr_3)_*(\psi * \phi)$. Note that the push-forward is well defined since ϕ (resp. ψ) is finite over X_1 (resp. X_2).

For any composable triple of finite correspondences α, β, γ one has

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

and therefore one can define a category $SmCor(k)$ such that objects of $SmCor(k)$ are smooth schemes of finite type over k, morphisms are finite correspondences and compositions of morphisms are compositions of correspondences defined above. We will denote the object of $SmCor(k)$ which corresponds to a smooth scheme X by $[X]$.

For any morphism $f : X \to Y$ its graph Γ_f is a finite correspondence from X to Y. It gives us a functor $[-] : Sm/k \to SmCor(k)$.

One can easily see that the category $SmCor(k)$ is additive and one has $[X \coprod Y] = [X] \oplus [Y]$.

Consider the homotopy category $\mathcal{H}^b(SmCor(k))$ of bounded complexes over $SmCor(k)$. We are going to define the triangulated category of effective geometrical motives over k as a localization of $\mathcal{H}^b(SmCor(k))$. Let T be the class of complexes of the following two forms:

1. For any smooth scheme X over k the complex

$$[X \times \mathbf{A}^1] \stackrel{[pr_1]}{\to} [X]$$

belongs to T.

[1] Note that a correspondence belongs to $c(X,Y)$ if and only if it has a well defined pull-back with respect to any morphism $X' \to X$ and a well defined push-forward with respect to any morphism $Y \to Y'$.

2. For any smooth scheme X over k and an open covering $X = U \cup V$ of X the complex

$$[U \cap V] \stackrel{[j_U]\oplus[j_V]}{\to} [U] \oplus [V] \stackrel{[i_U]\oplus(-[i_V])}{\to} [X]$$

belongs to T (here j_U, j_V, i_U, i_V are the obvious open embeddings).

Denote by $\bar{T}$ the minimal thick subcategory of $\mathcal{H}^b(SmCor(k))$ which contains T. It would be most natural for our purposes to define the category of geometrical motives as the localization of $\mathcal{H}^b(SmCor(k))$ with respect to $\bar{T}$. Unfortunately this definition makes it difficult to formulate results relating our theory to more classical motivic theories (see Proposition 2.1.4 and discussion of 1-motices in Section 3.4). The problem is that in the classical approach one usually replaces additive categories of geometrical nature by their pseudo-abelian (or Karoubian) envelopes, i.e. one formally adds kernels and cokernels of projectors. Note that this operation takes triangulated categories into triangulated categories and tensor categories into tensor categories. We follow this tradition in the definition below, but we would like to mention again that in our case the only reason to do so is to make comparison statements which involve the "classical motives" to look more elegant.

Definition 2.1.1 *Let k be a field. The triangulated category $DM_{gm}^{eff}(k)$ of effective geometrical motives over k is the pseudo-abelian envelope of the localization of the homotopy category $\mathcal{H}^b(SmCor(k))$ of bounded complexes over $SmCor(k)$ with respect to the thick subcategory $\bar{T}$. We denote the obvious functor $Sm/k \to DM_{gm}^{eff}(k)$ by M_{gm}.*

Note that one has the following simple but useful result.

Lemma 2.1.2 *Let X be a smooth scheme over k and $X = U \cup V$ be a Zariski open covering of X. Then there is a canonical distinguished triangle in DM_{gm}^{eff} of the form*

$$M_{gm}(U \cap V) \to M_{gm}(U) \oplus M_{gm}(V) \to M_{gm}(X) \to M_{gm}(U \cap V)[1].$$

For a pair of smooth schemes X, Y over k we set

$$[X] \otimes [Y] = [X \times Y].$$

For any smooth schemes X_1, Y_1, X_2, Y_2 the external product of cycles defines a homomorphism:

$$c(X_1, Y_1) \otimes c(X_2, Y_2) \to c(X_1 \times X_2, Y_1 \times Y_2)$$

which gives us a definition of tensor product of morphisms in $SmCor(k)$. Together with the obvious commutativity and associativity isomorphisms it gives us a tensor category structure on $SmCor(k)$. This structure defines in the usual way a tensor triangulated category structure on $\mathcal{H}^b(SmCor(k))$ which can be descended to the category $DM_{gm}^{eff}(k)$ by the universal property of localization. We proved the following simple result.

Proposition 2.1.3 *The category $DM_{gm}^{eff}(k)$ has a tensor triangulated category structure such that for any pair X, Y of smooth schemes over k there is a canonical isomorphism $M_{gm}(X \times Y) \cong M_{gm}(X) \otimes M_{gm}(Y)$.*

Note that the unit object of our tensor structure is $M_{gm}(Spec(k))$. We will denote it by $\mathbf{Z}$. For any smooth scheme X over k the morphism $X \to Spec(k)$ gives us a morphism in $DM_{gm}^{eff}(k)$ of the form $M_{gm}(X) \to \mathbf{Z}$. There is a canonical distinguished triangle $\tilde{M_{gm}}(X) \to M_{gm}(X) \to \mathbf{Z} \to \tilde{M_{gm}}(X)[1]$ where $\tilde{M_{gm}}(X)$ is the reduced motive of X represented in $\mathcal{H}^b(SmCor(k))$ by the complex $[X] \to [Spec(k)]$.

We define the Tate object $\mathbf{Z}(1)$ of $DM_{gm}^{eff}(k)$ as $\tilde{M_{gm}}(\mathbf{P}^1)[-2]$. We further define $\mathbf{Z}(n)$ to be the n-th tensor power of $\mathbf{Z}(1)$. For any object A of $DM_{gm}^{eff}(k)$ we denote by $A(n)$ the object $A \otimes \mathbf{Z}(n)$.

Finally we define the triangulated category $DM_{gm}(k)$ of geometrical motives over k as the category obtained from $DM_{gm}^{eff}(k)$ by inverting $\mathbf{Z}(1)$. More precisely, objects of $DM_{gm}(k)$ are pairs of the form (A, n) where A is an object of $DM_{gm}^{eff}(k)$ and $n \in \mathbf{Z}$ and morphisms are defined by the following formula

$$Hom_{DM_{gm}}((A,n),(B,m)) = \lim_{k \geq -n,-m} Hom_{DM_{gm}^{eff}}(A(k+n), B(k+m)).$$

The category $DM_{gm}(k)$ with the obvious shift functor and class of distinguished triangles is clearly a triangulated category. The situation with the tensor structure on DM_{gm} is more subtle. In general it is not possible to get a tensor structure on the category obtained from a tensor additive category by inverting an object Q but it is possible when the permutation involution on $Q \otimes Q$ is the identity morphism. We will show below (2.1.5) that it is true in our case. For a field k which admits resolution of singularities we will also show that the canonical functor from $DM_{gm}^{eff}(k)$ to $DM_{gm}(k)$ is a full embedding, i.e. that the Tate object is quasi-invertible in $DM_{gm}^{eff}(k)$ (4.3.1).

To prove that the permutation involution for the Tate object equals identity we need first to establish a connection between our $DM_{gm}^{eff}(k)$ and Chow motives.

Consider the category $\mathcal{C}$ whose objects are smooth projective schemes over k and morphisms are given by the formula

$$Hom_{\mathcal{C}_0}(X,Y) = \oplus_{X_i} A_{dim(X_i)}(X_i \times Y)$$

where X_i are the connected components of X and $A_d(-)$ is the group of cycles of dimension d modulo rational equivalence. Its pseudo-abelian envelope (i.e. the category obtained from $\mathcal{C}$ by formal addition of cokernels of projectors) is called the category of effective Chow motives over k. Denote this category by $Chow^{eff}(k)$ and let

$$Chow : SmProj/k \to Chow^{eff}(k)$$

be the corresponding functor on the category of smooth projective varieties over k.

Proposition 2.1.4 *There exists a functor $Chow^{eff}(k) \to DM_{gm}^{eff}(k)$ such that the following diagram commutes:*

$$\begin{array}{ccccc} & SmProj/k & \to & Sm/k & \\ Chow & \downarrow & & \downarrow & M_{gm} \\ & Chow^{eff}(k) & \to & DM_{gm}^{eff}(k). & \end{array}$$

Proof: It is clearly sufficient to show that for smooth projective varieties X, Y over k there is a canonical homomorphism

$$A_{dim(X)}(X \times Y) \to Hom_{DM_{gm}^{eff}}(M_{gm}(X), M_{gm}(Y)).$$

Denote by $h_0(X,Y)$ the cokernel of the homomorphism

$$c(X \times \mathbf{A}^1, Y) \to c(X,Y)$$

given by the difference of restrictions to $X \times \{0\}$ and $X \times \{1\}$. One can easily see that the obvious homomorphism

$$c(X,Y) \to Hom_{DM_{gm}^{eff}}(M_{gm}(X), M_{gm}(Y))$$

factors through $h_0(X,Y)$. On the other hand by definition of rational equivalence we have a canonical homomorphism

$$h_0(X,Y) \to A_{dim(X)}(X \times Y)$$

which is an isomorphism by [7, Th. 7.1].

Corollary 2.1.5 *The permutation involution on* $\mathbf{Z}(1) \otimes \mathbf{Z}(1)$ *is identity in* DM_{gm}^{eff}.

Proof: It follows from Proposition 2.1.4 and the corresponding well known fact for Chow motives.

Remark: It follows from Corollary 4.2.6 that if k admits resolution of singularities then the functor constructed in Proposition 2.1.4 is a full embedding and any distinguished triangle in $DM_{gm}^{eff}(k)$ with all three vertices being of the form $M_{gm}(X)$ for smooth projective varieties X splits.

2.2 *Summary of main results.*

We will give in this section a summary of main results of the paper. Their proofs are based on the construction given in Section 3. Essentially our main technical tool is an embedding of the category DM_{gm}^{eff} to the derived category of sheaves with some additional structure (transfers) which allow us to apply all the standard machinery of sheaves and their cohomology to our category of motives.

Motives of singular varieties. For a field k which admits resolution of singularities we construct in Section 4.1 an extension of the functor $M_{gm} : Sm/k \to DM_{gm}^{eff}$ to a functor $M_{gm} : Sch/k \to DM_{gm}^{eff}$ from the category of all schemes of finite type over k. This extended functor has the following main properties.

Kunnet formula. For schemes of finite type X, Y over k one has a canonical isomorphism $M_{gm}(X \times Y) = M_{gm}(X) \otimes M_{gm}(Y)$ (4.1.7).

Homotopy invariance For a scheme of finite type X over k the morphism $M_{gm}(X \times \mathbf{A}^1) \to M_{gm}(X)$ is an isomorphism (4.1.8).

Mayer-Vietoris axiom. For a scheme X of finite type over k and an open covering $X = U \cup V$ of X one has a canonical distinguished triangle of the form

$$M_{gm}(U \cap V) \to M_{gm}(U) \oplus M_{gm}(V) \to M_{gm}(X) \to M_{gm}(U \cap V)[1]$$

(4.1.1).

Blow-up distinguished triangle. For a scheme X of finite type over k and a closed subscheme Z in X denote by $p_Z : X_Z \to X$ the blow-up of Z in X. Then there is a canonical distinguished triangle of the form

$$M_{gm}(p_Z^{-1}(Z)) \to M_{gm}(X_Z) \oplus M_{gm}(Z) \to M_{gm}(X) \to M_{gm}(p_Z^{-1}(Z))[1]$$

(4.1.3).

Projective bundle theorem. Let X be a scheme of finite type over k and $\mathcal{E}$ be a vector bundle on X. Denote by $p : \mathbf{P}(\mathcal{E}) \to X$ the projective bundle over X associated with $\mathcal{E}$. Then one has a canonical isomorphism:

$$M_{gm}(\mathbf{P}(\mathcal{E})) = \oplus_{n=0}^{dim\mathcal{E}-1} M_{gm}(X)(n)[2n]$$

(4.1.11).

It should be mentioned that our functor $M_{gm}(-)$ provides a way to extend motivic homology and cohomology type theories to not necessarily smooth varieties. In the case of *homology* this extension is essentially the only one possible. The situation with cohomology is different. For instance the Picard group for a smooth scheme X is canonically isomorphic to the motivic cohomology group $Hom_{DM_{gm}^{eff}}(M_{gm}(X), \mathbf{Z}(1)[2])$. We do not have any "motivic" description for the Picard groups of arbitrary varieties though, since the functor $X \mapsto Pic(X)$ considered on the category of all schemes is not homotopy invariant and does not have the descent property for general blow-ups.

We hope that there is another more subtle approach to "motives" of singular varieties which makes use of some version of "reciprocity functors" introduced by Bruno Kahn instead of homotopy invariant functors considered in this paper which gives "right" answers for all schemes.

Motives with compact support. For any field k which admits resolution of singularities we construct a functor M^c_{gm} from the category of schemes of finite type over k and proper morphisms to the category DM^{eff}_{gm} which has the following properties:

1. For a proper scheme X over k one has a canonical isomorphism
$$M^c_{gm}(X) = M_{gm}(X)$$

2. For a scheme X of finite type over k and a closed subscheme Z in X one has a canonical distinguished triangle
$$M^c_{gm}(Z) \to M^c_{gm}(X) \to M^c_{gm}(X - Z) \to M^c_{gm}(Z)[1]$$
(4.1.5).

3. For a flat equidimensional morphism $f : X \to Y$ of schemes of finite type over k there is a canonical morphism
$$M^c_{gm}(Y)(n)[2n] \to M^c_{gm}(X)$$
where $n = dim(X/Y)$ (4.2.4).

4. For any scheme of finite type X over k one has a canonical isomorphism $M^c_{gm}(X \times \mathbf{A}^1) = M^c_{gm}(X)(1)[2]$ (4.1.8).

Blow-ups of smooth varieties and Gysin distinguished triangles. Let k be a perfect field, X be a smooth scheme over k and Z be a smooth closed subscheme in X everywhere of codimension c. Then one has:

Motives of blow-ups. There is a canonical isomorphism
$$M_{gm}(X_Z) = M_{gm}(X) \oplus (\oplus_{n=1}^{c-1} M_{gm}(Z)(n)[2n])$$
(3.5.3).

Gysin distinguished triangle. There is a canonical distinguished triangle
$$M_{gm}(X - Z) \to M_{gm}(X) \to M_{gm}(Z)(c)[2c] \to M_{gm}(X - Z)[1]$$
(3.5.4).

Quasi-invertibility of the Tate object. Let k be a field which admits resolution of singularities. Then for any objects A, B of $DM_{gm}^{eff}(k)$ the obvious morphism

$$Hom(A, B) \to Hom(A(1), B(1))$$

is an isomorphism. In particular the functor

$$DM_{gm}^{eff} \to DM_{gm}$$

is a full embedding (4.3.1).

Duality. For any field k which admits resolution of singularities the category $DM_{gm}(k)$ is a "rigid tensor triangulated category". More precisely one has (4.3.7):

1. For any pair of objects A, B in DM_{gm} there exists the internal Hom-object $\underline{Hom}_{DM}(A, B)$. We set A^* to be $\underline{Hom}_{DM}(A, \mathbf{Z})$.

2. For any object A in $DM_{gm}(k)$ the canonical morphism $A \to (A^*)^*$ is an isomorphism.

3. For any pair of objects A, B in DM_{gm} there are canonical isomorphisms

$$\underline{Hom}_{DM}(A, B) = A^* \otimes B$$

$$(A \otimes B)^* = A^* \otimes B^*$$

For any smooth equidimensional scheme X of dimension n over k there is a canonical isomorphism (4.3.2)

$$M_{gm}(X)^* = M_{gm}^c(X)(-n)[-2n].$$

Let X be a smooth equi-dimensional scheme of dimension n over k and Z be a closed subscheme of X. Applying duality to the localization sequence for M_{gm}^c we get the following generalized Gysin distinguished triangle

$$M_{gm}(X - Z) \to M_{gm}(X) \to M_{gm}^c(Z)^*(n)[2n] \to M_{gm}(X - Z)[1]$$

Relations to the algebraic cycle homology theories. Let again k be a field which admits resolution of singularities and X be a scheme of finite type over k.

Higher Chow groups. If X is quasi-projective and equidimensional of dimension n the groups $Hom(\mathbf{Z}(i)[j], M^c_{gm}(X))$ (i.e. Borel-Moore homology in our theory) are canonically isomorphic to the higher Chow groups $CH^{n-i}(X, j-2i)$ (Corollary 4.2.9, see also [3], [12], [7]).

Suslin homology. For any X of finite type over a perfect field k the groups (i.e. homology in our theory) $Hom(\mathbf{Z}[j], M_{gm}(X))$ are isomorphic to the Suslin homology groups $h_j(X)$ (3.2.7, see also [9],[13]).

Motivic cohomology. The groups $Hom(M_{gm}(X), \mathbf{Z}(i)[j])$ (i.e. cohomology in our theory) are isomorphic to motivic cohomology groups $H^j_M(X, \mathbf{Z}(i))$ introduced in [7]. In particular one has the desired relations with algebraic K-theory (loc. cit.).

Bivariant cycle cohomology. For any X, Y of finite type over k any $i \geq 0$ and any j the group $Hom(M_{gm}(X)(i)[j], M^c_{gm}(Y))$ is isomorphic to the bivariant cycle cohomology group $A_{i,j-2i}(X, Y)$ (see [7]).

Relations to the Chow motives. For any pair of smooth projective varieties X, Y over a field which admits resolution of singularities the group $Hom_{DM_{gm}(k)}(M_{gm}(X), M_{gm}(Y))$ is canonically isomorphic to the group of cycles of dimension $dim(X)$ on $X \times Y$ modulo rational equivalence (4.2.6(1)). In particular the full additive subcategory in DM_{gm} which is closed under direct summands and generated by objects of the form $M_{gm}(X)(n)[2n]$ for smooth projective X over k and $n \in \mathbf{Z}$ is canonically equivalent as a tensor additive category to the category of Chow motives over k. Moreover any distinguished triangle with all three vertices being in this subcategory splits by 4.2.6(2).

Motivic complexes and the homotopy t-structure. We construct for any perfect field k an embedding of the category $DM^{eff}_{gm}(k)$ to a bigger tensor triangulated category $DM^{eff}_{-}(k)$ of motivic complexes over k. The image of $DM^{eff}_{gm}(k)$ is "dense" in $DM^{eff}_{-}(k)$ in the sense that the smallest triangulated subcategory in $DM_-^{eff}(k)$ which is closed with respect to direct sums amd contains the image of $DM_{gm}^{eff}(k)$ coincides with $DM_-^{eff}(k)$. Almost by definition the category $DM_-^{eff}(k)$ has a (non degenerate) t-structure whose heart is the abelian category $HI(k)$ of homotopy invariant Nisnevich sheaves with transfers on Sm/k. Note that this t-structure is not the desired "motivic t-structure" on $DM_-^{eff}(k)$ whose heart is the abelian category of (effective) mixed motives over k. On the other hand it seems to be important to have at least one non-degenerate t-structure on $DM_-^{eff}(k)$ since one may hope then to construct the "correct" one by the gluing technique.

Motivic complexes in the etale and h- topologies. We construct in Section 3.3 a category $DM_{-,et}^{eff}(k)$ of the etale (effective) motivic complexes. There are canonical functors

$$DM_-^{eff}(k) \to DM_{-,et}^{eff}(k) \to DM_h(k)$$

(where the last category is the one constructed in [16]). We show that the second functor is an equivalence if k admits resolution of singularities and the first one becomes an equivalence after tensoring with $\mathbf{Q}$. We also show that the category $DM_{-,et}^{eff}(k, \mathbf{Z}/n\mathbf{Z})$ of the etale motivic complexes with $\mathbf{Z}/n\mathbf{Z}$ coefficients is equivalent to the derived category of complexes bounded from the above over the abelian category of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules on the small etale site $Spec(k)_{et}$ for n prime to $char(k)$ and $DM_{-,et}^{eff}(k, \mathbf{Z}/p\mathbf{Z}) = 0$ for $p = char(k)$.

3 Motivic complexes.

3.1 *Nisnevich sheaves with transfers and the category $DM_-^{eff}(k)$.*

The definition of the category DM_{gm}^{eff} given above being quite geometrical is very inconvinient to work with. In this section we inroduce another triangulated category - the category of (effective) motivic complexes. We will denote this category by $DM_-^{eff}(k)$.

Definition 3.1.1 *Let k be a field. A presheaf with transfers on Sm/k is an additive contravariant functor from the category $SmCor(k)$ to the category* *of abelian groups. It is called a Nisnevich sheaf with transfers if the corresponding presheaf of abelian groups on Sm/k is a sheaf in the Nisnevich topology.*

We denote by $PreShv(SmCor(k))$ the category of presheaves with transfers on Sm/k and by $Shv_{Nis}(SmCor(k))$ its full subcategory consisting of Nisnevich sheaves with transfers.

For a smooth scheme X over k denote by $L(X)$ the corresponding representable presheaf with transfers, i.e. $L(X)(Y) = c(Y, X)$ for any smooth scheme Y over k. Note that this presheaf is a particular case of Chow presheaves considered in [14]. In the notations of that paper $L(X)$ is the Chow presheaf $c_{equi}(X/Spec(k), 0)$.

The following lemma is straight-forward.

Lemma 3.1.2 *For any smooth scheme X over k the presheaf $L(X)$ is a sheaf in the Nisnevich topology.*

The role of the Nisnevich topology becomes clear in the following proposition:

Proposition 3.1.3 *Let X be a scheme of finite type over k and $\{U_i \to X\}$ be a Nisnevich covering of X. Denote the coproduct $\coprod U_i$ by U and consider the complex of presheaves:*

$$\ldots \to L(U \times_X U) \to L(U) \to L(X) \to 0$$

with the differential given by alternating sums of morphisms induced by the projections. Then it is exact as a complex of Nisnevich sheaves.

Proof: Note first that the presheaves $L(-)$ can be extended in the obvious way to presheaves on the category of smooth schemes over k which are not necessarily of finite type. Since points in the Nisnevich topology are henselian local schemes it is sufficient to verify that for any smooth henselian local scheme S over k the sequence of abelian groups

$$\ldots \to L(U \times_X U)(S) \to L(U)(S) \to L(X)(S) \to 0$$

is exact. For a closed subscheme Z in $X \times S$ which is quasi-finite over S denote by $L(Z/S)$ the free abelian group generated by irreducible components of Z which are finite and surjective over S. Clearly, the groups $L(Z/S)$ are covariantly functorial with respect to morphisms of quasi-finite schemes over S. For a closed subscheme Z in $X \times S$ which is finite over S denote by Z_U the fiber product $Z \times_{X \times S} (U \times S)$. One can easily see that our complex is a filtered inductive limit of complexes K_Z of the form

$$\ldots \to L(Z_U \times_Z Z_U/S) \to L(Z_U/S) \to L(Z/S) \to 0$$

where Z runs through closed subschemes of $X \times S$ which are finite and surjective over S. Since S is henselian any such Z is again henselian and hence the covering $Z_U \to Z$ splits. Let us choose a splitting $s_1 : Z \to Z_U$. We set $s_k : (Z_U)^k_Z \to (Z_U)^{k+1}_Z$ to be the product $s_1 \times_Z Id_{(Z_U)^{k-1}_Z}$. The morphisms s_k induce homomorphisms of abelian groups

$$\sigma_k : L((Z_U)^k_Z/S) \to L((Z_U)^{k+1}_Z/S).$$

One can verify easily that these homomorphisms provide a homotopy of the identity morphism of K_Z to zero.

Remarks:

1. Proposition 3.1.3 is false in the Zariski topology even if the corresponding covering $\{U_i \to X\}$ is a Zariski open covering.

2. The only property of the Nisnevich topology which we used in the proof of Proposition 3.1.3 is that a scheme finite over a "Nisnevich point" is again a "Nisnevich point". In particular exactly the same argument works for the etale topology and moreover for any topology where higher direct images for finite equidimensional morphisms vanish.

Theorem 3.1.4 *The category of Nisnevich sheaves with transfers is abelian and the embedding*

$$Shv_{Nis}(SmCor(k)) \to PreShv(SmCor(k))$$

has a left adjoint functor which is exact.

Proof: Note first that the category $PreShv(SmCor(k))$ is abelian by obvious reasons. To prove that $Shv_{Nis}(SmCor(k))$ is abelian it is clearly sufficient to construct an exact left adjoint functor to the canonical embedding. Its existence is an immediate corollary of Lemma 3.1.6 below.

Lemma 3.1.5 *Let $f : [X] \to [Y]$ be a morphism in $SmCor(k)$ and $p : U \to Y$ be a Nisnevich covering. Then there exists a Nisnevich covering $U' \to X$ and a morphism $f' : [U'] \to [U]$ such that $[p] \circ f' = f \circ [p']$.*

Proof: It follows immediately from the fact that a scheme finite over a henselian local scheme is a disjoint union of henselian local schemes.

Lemma 3.1.6 *Let F be a presheaf with transfers on Sm/k. Denote by F_{Nis} the sheaf in the Nisnevich topology associated with the corresponding presheaf of abelian groups on Sm/k. Then there exists a unique Nisnevich sheaf with transfers such that its underlying presheaf is F_{Nis} and the canonical morphism of presheaves $F \to F_{Nis}$ is a morphism of presheaves with transfers.*

Proof: Let us prove the uniqueness part first. Let F_1 and F_2 be two presheaves with transfers given together with isomorphisms of the corresponding presheaves on Sm/k with F_{Nis}. Consider a morphism $f : [X] \to [Y]$ in $SmCor(k)$. We have to show that for any section ϕ of F_{Nis} on Y one has $F_1(f)(\phi) = F_2(f)(\phi)$. Let $p : U- > Y$ be a Nisnevich covering such that $F_{Nis}(p)(\phi)$ belongs to the image of the homomorphism $F(U)- > F_{Nis}(U)$ (such a covering always exist) and $U' \to X$, $f' : [U'] \to [U]$ be as in Lemma 3.1.5. Then we have

$$F_1([p'])F_1(f)(\phi) = F_2(f')F_2([p])(\phi) = F_2([p'])F_2(f)(\phi) = F_1([p'])F_2(f)(\phi)$$

which implies that $F_1(f)(\phi) = F_2(f)(\phi)$ because p' is a covering and F_1, F_2 are Nisnevich sheaves.

To give F_{Nis} the structure of a presheaf with transfers it is sufficient to construct for any section ϕ of F_{Nis} on a smooth scheme X over k a morphism of presheaves $[\phi] : L(X) \to F_{Nis}$ such that $[\phi]$ takes the tautological section of $L(X)$ over X to ϕ. Let $p : U \to X$ be a Nisnevich covering of X such that $F_{Nis}(p)(\phi)$ corresponds to a section ϕ_U of F over U satisfying the condition $F(pr_1)(\phi_U) = F(pr_2)(\phi_U)$ where $pr_i : U \times_X U \to U$ are the projections. Then ϕ_U defines a morphism of sheaves $[\phi_U] : L(U) \to F_{Nis}$ such that $[\phi_U] \circ L(pr_1) = [\phi_U] \circ L(pr_2)$. Applying Proposition 3.1.3 to our covering we conclude that $[\phi_U]$ can be descended to a morphism $[\phi] : L(X) \to F_{Nis}$.

The proof of the following lemma is standard.

Lemma 3.1.7 *The category $Shv_{Nis}(SmCor(k))$ has sufficiently many injective objects.*

Proposition 3.1.8 *Let X be a smooth scheme over a field k and F be a Nisnevich sheaf with transfers. Then for any $i \in \mathbf{Z}$ there is a canonical isomorphism:*

$$Ext^i_{Shv_{Nis}(SmCor(k))}(L(X), F) = H^i_{Nis}(X, F).$$

Proof: Since the category $Shv_{Nis}(SmCor(k))$ has sufficiently many injective objects by Lemma 3.1.7 and for any Nisnevich sheaf with tranfers G one has $Hom(L(X), G) = G(X)$ we only have to show that for any injective Nisnevich sheaf with transfers I one has $H^n_{Nis}(X, I) = 0$ for $n > 0$. It is sufficient to show that the Chech cohomology groups with coefficients in I vanish for all X (see [10, Prop. III.2.11]). Let $\mathcal{U} = \{U_i \to X\}$ be a Nisnevich covering of X and α be a class in $\check{H}^n_{Nis}(\mathcal{U}/X, I)$. Let us set $U = \coprod U_i$. Then α is given by a section a of I over U_X^{n+1} or equivalently by a morphism $L(U_X^{n+1}) \to I$ in the category of sheaves with transfers. In view of Lemma 3.1.3 the fact that a is a cocycle implies that as a morphism it can be factored through a morphism from $ker(L(U^n) \to L(U^{n-1}))$ to I. Since I is an injective object in $Shv_{Nis}(SmCor(k))$ it implies that a can be factored through U^n, i.e. that $\alpha = 0$.

Consider now the derived category $D^-(Shv_{Nis}(SmCor(k)))$ of complexes of Nisnevich sheaves with transfers bounded from the above. We will use the following generalization of Proposition 3.1.8.

Proposition 3.1.9 *Let X be a smooth scheme over a field k and K be a complex of Nisnevich sheaves with transfers bounded from the above. Then for any $i \in \mathbf{Z}$ there is a canonical isomorphism*

$$Hom_{D^-(Shv_{Nis}(SmCor(k)))}(L(X), K[i]) = \mathbf{H}^i_{Nis}(X, K)$$

where the groups on the right hand side are the hypercohomology of X in the Nisnevich topology with coefficients in the complex of sheaves K .

Proof: By Proposition 3.1.8 and the cohomological dimension theorem for the Nisnevich topology ([11]) we conclude that the sheaf with transfers $L(X)$

has finite Ext-dimension. Therefore it is sufficient to prove our proposition for a bounded complex K. Then morphisms in the derived category can be computed using an injective resolution for K and the statement follows from Proposition 3.1.8.

Remarks:

1. The natural way to prove Proposition 3.1.8 would be to show that if I is an injective Nisnevich sheaf with transfers then I is an injective Nisnevich sheaf. To do so one could construct the free sheaf with transfers functor from the category of Nisnevich sheaves to the category of Nisnevich sheaves with transfers which is left adjoint to the forgetfull functor and then use the same argument as in the standard proof of the fact that an injective sheaf is an injective presheaf. The problem is that while the free sheaf with transfers functor is right exact simple examples show that it is not left exact.

2. Though Proposition 3.1.8 gives us an interpretation of Ext-groups from sheaves $L(X)$ in the category of Nisnevich sheaves with transfers we do not know how to describe the Ext-groups from $L(X)$ in the category of all Nisnevich sheaves.

Definition 3.1.10 *A presheaf with transfers F is called homotopy invariant if for any smooth scheme X over k the projection $X \times \mathbf{A}^1 \to X$ induces isomorphism $F(X) \to F(X \times \mathbf{A}^1)$.*

A Nisnevich sheaf with transfers is called homotopy invariant if it is homotopy invariant as a presheaf with transfers.

The following proposition relates presheaves with transfers to pretheories ([15]). Though the proof given here is fairly long the statement itself is essentially obvious and the only problem is to "unfold" all the definitions involved.

Proposition 3.1.11 *Let k be a field. Then any presheaf with transfers on Sm/k is a pretheory of homological type over k.*

Proof: Let F be a presheaf with transfers on Sm/k, U be a smooth scheme over k and $X \to U$ be a smooth curve over U. Let us remind that we denote in [15] by $c_{equi}(X/U, 0)$ the free abelian group generated by integral closed subschemes of X which are finite over U and surjective over an irreducible component of U. For any morphism of smooth schemes $f : U' \to U$ there is a base change homomorphism

$$cycl(f) : c_{equi}(X/U, 0) \to c_{equi}(X \times_U U'/U', 0)$$

and for any morphism $p : X \to X'$ of smooth curves over U there is a push-forward homomorphism

$$p_* : c_{equi}(X/U, 0) \to c_{equi}(X'/U, 0).$$

To give F a pretheory structure we have to construct a homomorphism

$$\phi_{X/U} : c_{equi}(X/U, 0) \to Hom(F(X), F(U))$$

such that the following two conditions hold ([15, Def. 3.1]):

1. For an element $\mathcal{Z}$ in $c_{equi}(X/U, 0)$ which corresponds to a section $s : U \to X$ of the projection $X \to U$ we have $\phi_{X/U}(\mathcal{Z}) = F(s)$.

2. For any morphism of smooth schemes $f : U' \to U$ an element $\mathcal{Z}$ in $c_{equi}(X/U, 0)$ and an element u in $F(X)$ we have:

   $$F(f)(\phi_{X/U}(\mathcal{Z})(u)) = \phi_{X\times_U U'/U'}(cycl(f)(\mathcal{Z}))(F(f_X)(u))$$

   where f_X is the projection $X \times_U U' \to X$.

For (F, ϕ) to be a pretheory of homological type we require in addition that for any morphism $p : X \to X'$ of smooth curves over U, any element u in $F(X')$ and any element $\mathcal{Z}$ in $c_{equi}(X/U, 0)$ we have

$$\phi_{X'/U}(p_*(\mathcal{Z}))(u) = \phi_{X/U}(\mathcal{Z})(F(p)(u)).$$

Let $g : X \to U$ be a smooth curve over a smooth scheme U and Z be an integral closed subscheme in X which belongs to $c_{equi}(X/U, 0)$. Consider the closed embedding $Id_X \times g : X \to X \times U$. Then the image of Z under this embedding belongs to $c(U, X) = Hom_{SmCor(k)}([U], [X])$.

This construction defines homomorphisms

$$\alpha_{X/U} : c_{equi}(X/U, 0) \to Hom_{SmCor(k)}([U], [X])$$

and since F is a functor on $SmCor(k)$ we can define $\phi_{X/U}$ as the composition of $\alpha_{X/U}$ with the canonical homomorphism

$$Hom_{SmCor(k)}([U],[X]) \to Hom(F(X),F(U)).$$

One can verify easily that homomorphisms $\alpha_{X/U}$ satisfy the following conditions which implies immediately the statement of the proposition.

1. For a section $s : U \to X$ of the projection $X \to U$ we have $\alpha_{X/U}(s) = [s]$.

2. For a morphism of smooth schemes $f : U' \to U$ and an element $\mathcal{Z}$ in $c_{equi}(X/U,0)$ we have

$$\alpha_{X/U}(\mathcal{Z}) \circ [f] = [f_X] \circ \alpha_{X\times_U U'/U'}(cycl(f)(\mathcal{Z}))$$

   where again f_X is the projection $X \times_U U' \to X$.

3. For a morphism $p : X \to X'$ of smooth curves over U and an element $\mathcal{Z}$ in $c_{equi}(X/U,0)$ we have

$$\alpha_{X'/U}(p_*(\mathcal{Z})) = [p] \circ \alpha_{X/U}(\mathcal{Z}).$$

The following proposition summarizes some of the main properties of homotopy invariant presheaves with transfers which follow from the corresponding results about homotopy invariant pretheories proven in [15] by Proposition 3.1.11.

Theorem 3.1.12 *Let F be a homotopy invariant presheaf with transfers on Sm/k. Then the Nisnevich sheaf with transfers F_{Nis} associated with F is homotopy invariant. Moreover as a presheaf on Sm/k it coincides with the Zariski sheaf F_{Zar} associated with F. If in addition the field k is perfect one has:*

1. *The presheaves $H^i_{Nis}(-,F_{Nis})$ have canonical structures of homotopy invariant presheaves with transfers.*

2. *For any smooth scheme over k one has*

$$H^i_{Zar}(X,F_{Zar}) = H^i_{Nis}(X,F_{Nis}).$$

Theorem 3.1.12 implies immediately the following result.

Proposition 3.1.13 *For any perfect field k the full subcategory $HI(k)$ of the category $Shv_{Nis}(SmCor(k))$ which consists of homotopy invariant sheaves is abelian and the inclusion functor $HI(k) \to Shv_{Nis}(SmCor(k))$ is exact.*

We denote by $DM_-^{eff}(k)$ the full subcategory of $D^-(Shv_{Nis}(SmCor(k)))$ which consists of complexes with homotopy invariant cohomology sheaves. Proposition 3.1.13 implies that $DM_-^{eff}(k)$ is a triangulated subcategory. Moreover, the standard t-structure on $D^-(Shv_{Nis}(SmCor(k)))$ induces a t-structure on $DM_-^{eff}(k)$ whose heart is equivalent to the category $HI(k)$. We call this t-structure the homotopy t-structure on $DM_-^{eff}(k)$.

The connection between our geometrical category $DM_{gm}^{eff}(k)$ and the category DM_-^{eff} of (effective) motivic complexes will become clear in the next section. We will use the following simple fact about $DM_-^{eff}(k)$.

Lemma 3.1.14 *The category $DM_-^{eff}(k)$ is pseudo-abelian i.e. each projector in $DM_-^{eff}(k)$ has kernel and cokernel.*

3.2 *The embedding theorem.*

We prove in this section our main technical result - the fact that the category $DM_{gm}^{eff}(k)$ admits a natural full embedding as a tensor triangulated category to the category $DM_-^{eff}(k)$. All through this section we assume that k is a perfect field. Since the objects of $DM_-^{eff}(k)$ are essentially some complexes of sheaves on the category Sm/k of smooth schemes over k the existence of this embedding let us to apply the machinery of sheaves and their cohomology to the category $DM_{gm}^{eff}(k)$.

Let us define first a tensor structure on the category of Nisnevich sheaves with transfers. In view of Lemma 3.1.6 it is sufficient to define tensor products for presheaves with transfers. Note that for two presheaf with transfers their tensor product in the category of presheaves does not have transfers and thus a more sophisticated construction is required.

Let F be a presheaf with transfers. Let A_F be the set of pairs of the form $(X, \phi \in F(X))$ for all smooth schemes X over k. Since a section $\phi \in F(X)$ is the same as a morphism of presheaves with transfers $L(X) \to F$ there is a canonical surjection of presheaves

$$\oplus_{(X,\phi)\in A_F} L(X) \to F.$$

Iterating this construction we get a canonical left resolution $\mathcal{L}(F)$ of F which consists of direct sums of presheaves of the form $L(X)$ for smooth schemes X over k. We set

$$L(X) \otimes L(Y) = L(X \times Y)$$

and for two presheaves with transfers F, G:

$$F \otimes G = \underline{H}_0(\mathcal{L}(F) \otimes \mathcal{L}(G)).$$

One can verify easily that this construction indeed provides us with a tensor structure on the category of presheaves with transfers and thus on the category of Nisnevich sheaves with transfers. Moreover using these canonical "free resolutions" we get immediately a definition of tensor product on the derived category $D^-(Shv_{Nis}(SmCor(k)))$.

Remark: Note that the tensor product of two homotopy invariant presheaves with transfers is almost never a homotopy invariant presheaf. To define tensor structure on $DM_-^{eff}(k)$ we need the description of this category given in Proposition 3.2.3 below.

For presheaves with transfers F, G denote by $\underline{Hom}(F, G)$ the presheaf with transfers of the form:

$$\underline{Hom}(F, G)(X) = Hom(F \otimes L(X), G)$$

where the group on the right hand side is the group of morphisms in the category of presheaves with transfers. One can verify easily that for any three presheaves with transfers F, G, H there is a canonical isomorphism

$$Hom(F, \underline{Hom}(G, H)) \to Hom(F \otimes G, H),$$

i.e. $\underline{Hom}(-,-)$ is the internal Hom-object with respect to our tensor product. Note also that if G is a Nisnevich sheaf with transfers then $\underline{Hom}(F, G)$ is a Nisnevich sheaf with transfers for any presheaf with transfers F.

To construct a functor $DM_{gm}^{eff}(k) \to DM_-^{eff}(k)$ as well as to define the tensor structure on DM_-^{eff} we will need an alternative description of DM_-^{eff} as a localization of the derived category $D^-(Shv_{Nis}(SmCor(k)))$.

Let $\Delta^\bullet$ be the standard cosimplicial object in Sm/k. For any presheaf with transfers F on Sm/k let $\underline{C}_*(F)$ be the complex of presheaves on Sm/k

of the form $\underline{C}_n(F)(X) = F(X \times \Delta^n)$ with differentials given by alternated sums of morphisms which correspond to the boundary morphisms of $\Delta^\bullet$ (note that $\underline{C}_n(F) = \underline{Hom}(L(\Delta^n), F)$). This complex is called the singular simplicial complex of F. One can easily see that if F is a presheaf with transfers (resp. a Nisnevich sheaf with transfers) then $\underline{C}_*(F)$ is a complex of presheaves with transfers (resp. Nisnevich sheaves with transfers). We denote the cohomology sheaves $\underline{H}^{-i}(\underline{C}_*(F))$ by $\underline{h}_i^{Nis}(F)$. In the case when F is of the form $L(X)$ for a smooth scheme X over k we will abbreviate the notation $\underline{C}_*(L(X))$ (resp. $\underline{h}_i^{Nis}(L(X))$) to $\underline{C}_*(X)$ (resp. $\underline{h}_i^{Nis}(X)$). The complex $\underline{C}_*(X)$ is called the Suslin complex of X and its homology groups over $Spec(k)$ are called the Suslin homology of X (see [13], [15]).

Lemma 3.2.1 *For any presheaf with transfers F over k the sheaves $\underline{h}_i^{Nis}(F)$ are homotopy invariant.*

Proof: The cohomology presheaves $\underline{h}_i(F)$ of the complex $\underline{C}_*(F)$ are homotopy invariant for any presheaf F (see [15, Prop. 3.6]). The fact that the associated Nisnevich sheaves are homotopy invariant follows from Theorem 3.1.12.

We say that two morphisms of presheaves with transfers $f_0, f_1 : F \to G$ are homotopic if there is a morphism $h : F \otimes L(\mathbf{A}^1) \to G$ such that

$$h \circ (Id_F \otimes L(i_0)) = f_0$$

$$h \circ (Id_F \otimes L(i_1)) = f_1$$

where $i_0, i_1 : Spec(k) \to \mathbf{A}^1$ are the points 0 and 1 respectively. A morphism of presheaves with transfers $f : F \to G$ is a (strict) homotopy equivalence if there is a morphism $g : G \to F$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity morphisms of G and F respectively. One can verify easily that the composition of two homotopy equivalences is a homotopy equivalence.

Consider homomorphisms

$$\eta_n : F \to \underline{C}_n(F)$$

which take a section of F on a smooth scheme X to its preimage on $X \times \Delta^n$ under the projection $X \times \Delta^n \to X$. We will need the following lemma.

Lemma 3.2.2 *The morphisms η_n are strict homotopy equivalences.*

Proof: Since Δ^n is (noncanonically) isomorphic to $\mathbf{A}^n$ we have for any $n > 0$:

$$\underline{C}_n(F) = \underline{C}_1(\underline{C}_{n-1}(F))$$

and therefore it is sufficient to show that η_1 is a homotopy equivalence. Let $\alpha : \underline{C}_1(F) \to F$ be the morphism which sends a section of F on $X \times \mathbf{A}^1$ to its restriction to $X \times \{0\}$. Then $\alpha \circ \eta_1 = Id$ and it remains to show that there is a morphism $h : \underline{C}_1(F) \otimes L(\mathbf{A}^1) \to \underline{C}_1(F)$ such that

$$h \circ (Id \otimes L(i_1)) = Id$$

$$h \circ (Id \otimes L(i_0)) = \eta_1 \circ \alpha.$$

We set h to be the morphism adjoint to the morphism

$$\underline{C}_1(F) \to \underline{Hom}(L(\mathbf{A}^1), \underline{C}_1(F)) = \underline{C}_2(F)$$

which sends a section of F on $X \times \mathbf{A}^1$ to its preimage on $X \times \mathbf{A}^2$ under the morphism $\mathbf{A}^2 \to \mathbf{A}^1$ given by the multiplication of functions.

Lemma 3.2.1 implies that $\underline{C}_*(-)$ is a functor from the category of Nisnevich sheaves with transfers on Sm/k to $DM_-^{eff}(k)$. The following proposition shows that it can be extended to a functor from the corresponding derived category which provides us with the alternative description of $DM_-^{eff}(k)$ mentioned above.

Proposition 3.2.3 *The functor $\underline{C}_*(-)$ can be extended to a functor*

$$\mathbf{R}C : D^-(Shv_{Nis}(SmCor(k))) \to DM_-^{eff}(k)$$

which is left adjoint to the natural embedding. The functor $\mathbf{R}C$ identifies $DM_-^{eff}(k)$ with localization of $D^-(Shv_{Nis}(SmCor(k)))$ with respect to the localizing subcategory generated by complexes of the form

$$L(X \times \mathbf{A}^1) \stackrel{L(pr_1)}{\to} L(X)$$

for smooth schemes X over k.

Proof: Denote by A the class of objects in $D^-(Shv_{Nis}(SmCor(k)))$ of the form $L(X\times \mathbf{A}^1) \stackrel{L(pr_1)}{\to} L(X)$ for smooth schemes X over k and let $\mathcal{A}$ be the localizing subcategory generated by A, i.e. the minimal triangulated subcategory in $D^-(Shv_{Nis}(SmCor(k)))$ which contains A and is closed under direct sums and direct summands. Consider the localization $D^-(Shv_{Nis}(SmCor(k)))/\mathcal{A}$ of $D^-(Shv_{Nis}(SmCor(k)))$ with respect to the class of morphisms whose cones are in $\mathcal{A}$. Our proposition asserts that the restriction of the canonical projection

$$D^-(Shv_{Nis}(SmCor(k))) \to D^-(Shv_{Nis}(SmCor(k)))/\mathcal{A}$$

to the subcategory $DM_-^{eff}(k)$ is an equivalence and that the composition of this projection with the inverse equivalence coincides on sheaves with transfers with the functor $\underline{C}_*(F)$.

To prove this assertion it is sufficient to show that the following two statements hold:

1. For any sheaf with transfers F on Sm/k the canonical morphism $F\to \underline{C}_*(F)$ is an isomorphism in $D^-(Shv_{Nis}(SmCor(k)))/\mathcal{A}$.

2. For any object T of $DM_-^{eff}(k)$ and any object B of $\mathcal{A}$ one has $Hom(B,T)=0$.

To prove the second statement we may assume (by definition of $\mathcal{A}$) that B is of the form $L(X\times\mathbf{A}^1)\to L(X)$ for a smooth scheme X over k. By Proposition 3.1.9 it is sufficient to show that the projection $X\times\mathbf{A}^1\to X$ induces isomorphisms on the hypercohomology groups

$$\mathbf{H}^*(X,T)\to \mathbf{H}^*(X\times\mathbf{A}^1,T).$$

which follows by the hypercohomology spectral sequence from Theorem 3.1.12.

To prove the first statement we need the following two lemmas.

Lemma 3.2.4 *For any object T of $D^-(Shv_{Nis}(SmCor(k)))$ and an object S of $\mathcal{A}$ the object $T\otimes S$ belongs to $\mathcal{A}$.*

Proof: Since $\mathcal{A}$ is the localizing subcategory generated by A it is sufficient to consider the case of T being of the form $L(X)$ and S being in A. Then it follows from the fact that $L(X)\otimes L(Y) = L(X\times Y)$.

Lemma 3.2.5 *Let $f : F \to G$ be a homotopy equivalence of presheaves with transfers. Then the cone of f belongs to $\mathcal{A}$.*

Proof: We have to show that f becomes an isomorphism after localization with respect to $\mathcal{A}$. It is sufficient to show that a morphism of sheaves with transfers which is homotopic to the identity equals the identity morphism in $D^-(Shv_{Nis}(SmCor(k)))/\mathcal{A}$. By definition of homotopy we have only to show that for any F the morphisms

$$Id \otimes L(i_0) : F \to F \otimes L(\mathbf{A}^1)$$

$$Id \otimes L(i_1) : F \to F \otimes L(\mathbf{A}^1)$$

are equal. Denote by I^1 the kernel of the morphism $L(\mathbf{A}^1) \to L(pt)$. Then the difference $Id \otimes L(i_0) - Id \otimes L(i_1)$ can be factored through $F \otimes I^1$ which is zero in $D^-(Shv_{Nis}(SmCor(k)))/\mathcal{A}$ by Lemma 3.2.4 since I^1 belongs to A.

Let F be a sheaf with transfers. Denote by $\underline{C}_{\geq 1}(F)$ the cokernel of the obvious morphism of complexes of sheaves $F \to \underline{C}_*(F)$. To finish the proof of the proposition we have to show that $\underline{C}_{\geq 1}(F)$ belongs to $\mathcal{A}$. Let $\underline{\tilde{C}}_n(F) = coker(\eta_n)$ where η_n is the morphism defined right before Lemma 3.2.2. Since the differential in $\underline{C}_*(F)$ takes $Im(\eta_n)$ to $Im(\eta_{n-1})$ the sheaves $\underline{\tilde{C}}_n(F)$ form a quotient complex of $\underline{C}_*(F)$ which is clearly quasi-isomorphic to $\underline{C}_{\geq 1}(F)$. Since $\mathcal{A}$ is a localizing subcategory it is sufficient now to note that for each n $\underline{\tilde{C}}_n(F)$ belongs to $\mathcal{A}$ by Lemmas 3.2.5, 3.2.2. Proposition is proven.

We define the tensor structure on $DM_-^{eff}(k)$ as the descent of the tensor structure on $D^-(Shv_{Nis}(SmCor(k)))$ with respect to the projection $\mathbf{R}C$. Note that such a descent exists by the universal property of localization and Lemma 3.2.4.

Remark: Note that the inclusion functor

$$DM_-^{eff}(k) \to D^-(Shv_{Nis}(SmCor(k)))$$

does not preserve tensor structures. In fact many hard conjectures in motivic theory (Beilinson's vanishing conjectutes and Quillen-Lichtenbaum conjectures in the first place) can be formulated as statements about the behavior of the tensor structures on these two categories with respect to the inclusion functor.

The following theorem is the main technical result we use to study the category $DM_{gm}^{eff}(k)$.

Theorem 3.2.6 *Let k be a perfect field. Then there is a commutative diagram of tensor triangulated functors of the form*

$$\begin{array}{ccc} \mathcal{H}^b(SmCor(k)) & \stackrel{L}{\to} & D^-(Shv_{Nis}(SmCor(k))) \\ \downarrow & & \downarrow \mathbf{R}C \\ DM_{gm}^{eff}(k) & \stackrel{i}{\to} & DM_-^{eff}(k) \end{array}$$

such that the following conditions hold:

1. *The functor i is a full embedding with a dense image.*
2. *For any smooth scheme X over k the object $\mathbf{R}C(L(X))$ is canonically ismorphic to the Suslin complex $\underline{C}_*(X)$*

Proof: The only statement which requires a proof is that the functor i exists and is a full embedding. The fact that it is a tensor triangulated functor follows then immediately from the corresponding property of the composition $\mathbf{R}C \circ L$ and the fact that DM_{gm}^{eff} is a localization of $\mathcal{H}^b(SmCor(k))$. In view of Lemma 3.1.14 we may replace $DM_{gm}^{eff}(k)$ by the localization of the category $\mathcal{H}^b(SmCor(k))$ with respect to the thick subcategory generated by objects of the following two types:

1. Complexes of the form $L(X \times \mathbf{A}^1) \to L(X)$ for smooth schemes X over k.
2. Complexes of the form $L(U \cap V) \to L(U) \oplus L(V) \to L(X)$ for Zariski open coverings of the form $X = U \cup V$.

We denote the class of objects of the first type by T_{hom} and the class of objects of the second type by T_{MV}.

To prove the existence of i it is clearly sufficient to show that $\mathbf{R}C$ takes T_{hom} and T_{MV} to zero. The fact that $\mathbf{R}C(T_{hom}) = 0$ follows immediately from the definition of this functor (see Proposition 3.2.3). Note that $\mathbf{R}C(L(Y)) =$

$\underline{C}_*(Y)$ for any smooth scheme Y over k. Thus to prove that $\mathbf{R}C(T_{MV}) = 0$ we have to show that for an open covering $X = U \cup V$ of a smooth scheme X the total complex of the bicomplex

$$0 \to \underline{C}_*(U \cap V) \to \underline{C}_*(U) \oplus \underline{C}_*(V) \to \underline{C}_*(X) \to 0$$

is exact in the Nisnevich topology.

Note that this sequence of complexes of sheaves is left exact and the cokernel of the last arrow is isomorphic to the singular simplicial complex $\underline{C}_*(L(X)/(L(U) + L(V)))$ of the quotient presheaf $L(X)/(L(U) + L(V))$. The sheaf in the Nisnevich topology associated with this presheaf is zero by Proposition 3.1.3 and therefore $\underline{C}_*(L(X)/(L(U)+L(V)))$ is quasi-isomorphic to zero in the Nisnevich topology by [15, Theorem 5.9]. This proves the existence of the functor i.

To prove that i is a full embedding we proceed as follows. Consider the category $D^-(PreShv(SmCor(k)))$. One can verify easily that $D^-(Shv_{Nis}(SmCor(k)))$ is the localization of $D^-(PreShv(SmCor(k)))$ with respect to the localizing subcategory generated by presheaves with transfers F such that $F_{Nis} = 0$. The functor L from our diagram lifts to a functor

$$L_0 : \mathcal{H}^b(SmCor(k)) \to D^-(PreShv(SmCor(k)))$$

which is clearly a full embedding. Moreover the localizing subcategory generated by the image of L_0 coincides with $D^-(PreShv(SmCor(k)))$.

Let $T = T_{hom} \cup T_{MV}$ and let $\mathcal{T}$ be the localizing subcategory in $D^-(PreShv(SmCor(k)))$ generated by T. The general theory of localization of triangulated categories implies now that it is sufficient to show that for any presheaf with transfers F such that $F_{Nis} = 0$ we have $F \in \mathcal{T}$.

Consider the family of functors $H^i : Sm/k \to Ab$ of the form

$$H^i(X) = Hom_{D^-(PreShv(SmCor(k)))/\mathcal{T}}(L(X), F[i]).$$

It clearly suffice to show that $H^i = 0$ for all i. Note that since $\mathcal{T}$ contains T_{MV} our family has the Mayer-Vietoris long exact sequences for Zariski open coverings. Thus by [5, Theorem 1'] we have only to show that the sheaves in the Zariski topology associated with H^i's are zero. By the construction H^i's are presheaves with transfers and since T_{hom} belongs to $\mathcal{T}$ they are homotopy invariant. Thus $(H^i)_{Zar} = (H^i)_{Nis}$ by Theorem 3.1.12(2).

A morphism from $L(X)$ to $F[i]$ in $D^-(PreShv(SmCor(k)))/\mathcal{T}$ can be represented by a diagram of the form

$$\begin{array}{ccccc} L(X) & & & & F[i] \\ & f\searrow & & \swarrow g & \\ & & K & & \end{array}$$

such that the cone of g belongs to $\mathcal{T}$. Since $\underline{C}_*(-)$ is an exact functor from the category of presheaves with transfers to the category of complexes of presheaves with transfers it can be extended to $D^-(PreShv(SmCor(k)))$ and as was established in the proof of Proposition 3.2.3 for any object K of $D^-(PreShv(SmCor(k)))$ the canonical morphism $K \to \underline{C}_*(K)$ is an isomorphism in $D^-(PreShv(SmCor(k)))/\mathcal{T}$. Thus it is sufficient to show that there exists a Nisnevich covering $U \to X$ of X such that the composition:

$$L(U) \to L(X) \xrightarrow{f} K \to \underline{C}_*(K)$$

is zero. It follows from the fact that $(\underline{C}_*(K))_{Nis} = 0$.

The following corollary gives an "explicit" description of morphisms in the category $DM_{gm}^{eff}(k)$ in terms of certain hypercohomology groups. It will be used extensively in further sections to provide "motivic" interpretations for different algebraic cycle homology type theories.

Corollary 3.2.7 *Let k be a perfect field. Then for any smooth schemes X, Y over k and any $j \in \mathbf{Z}$ one has a canonical isomorphism*

$$Hom_{DM_{gm}^{eff}}(M_{gm}(X), M_{gm}(Y)[j]) = \mathbf{H}^j_{Nis}(X, \underline{C}_*(Y)) = \mathbf{H}^j_{Zar}(X, \underline{C}_*(Y)).$$

In particular for any smooth scheme X over k the groups $Hom_{DM_{gm}^{eff}}(\mathbf{Z}[j], M_{gm}(X))$ are isomorphic to the Suslin homology of X (see [13],[9]).

Proof: The first isomorphism follows from Theorem 3.2.6 and Proposition 3.1.9. The second one follows from Theorem 3.1.12 and Lemma 3.2.1.

One of the important advantages of the category $DM_-^{eff}(k)$ is that it has internal Hom-objects for morphisms from objects of DM_{gm}^{eff}.

Proposition 3.2.8 *Let A, B be objects of $DM_-^{eff}(k)$ and assume that A belongs to the image of $DM_{gm}^{eff}(k)$. Then there exists the internal Hom-object $\underline{Hom}_{DM^{eff}}(A, B)$. If $A = \underline{C}_*(X)$ for a smooth scheme X and $p_X : X \to Spec(k)$ is the canonical morphism then*

$$\underline{Hom}_{DM_-^{eff}}(A, B) = \mathbf{R}(p_X)_*((p_X)^*(B)).$$

Proof: We may consider the internal $RHom$-object $\underline{RHom}(A, B)$ in the derived category of unbounded complexes over $Shv_{Nis}(SmCor(k))$. It is clearly sufficient to verify that it belongs to $DM_-^{eff}(k)$. The fact that its cohomology sheaves are homotopy invariant follows from Theorem 3.1.12(1). The fact that $H^i(\underline{RHom}(A, B)) = 0$ for sufficiently large i follows from the lemma below.

Lemma 3.2.9 *Let X be a smooth scheme over a perfect field k and F be a homotopy invariant Nisnevich sheaf with transfers over k. Denote by $p : X \to Spec(k)$ the canonical morphism. Then the sheaves $R^ip_*(p^*(F))$ on Sm/k are zero for $i > dim(X)$.*

Proof: By Theorem 3.1.12 these sheaves are homotopy invariant Nisnevich sheaves with transfers. Therefore, by [15, Cor. 4.19] for any smooth scheme Y over k and any nonempty open subset U of Y the homomorphisms

$$R^ip_*(p^*(F))(Y) \to R^ip_*(p^*(F))(U)$$

are injective and our result follows from the cohomological dimesnion theorem for the Nisnevich cohomology ([11]).

3.3 *Etale sheaves with transfers.*

To avoid unpleasant technical difficulties we assume in this section that k has finite etale cohomological dimension. This section is very sketchy mainly because of Propositions 3.3.2, 3.3.3 below which show that with rational coefficients the etale topology gives the same motivic answers as the Nisnevich topology and with finite coefficients everything degenerates to the usual etale cohomology.

Denote by $Shv_{et}(SmCor(k))$ the category of presheaves with transfers on Sm/k which are etale sheaves. One can easily see that the arguments of

Section 3.1 work for the etale topology as well as for the Nisnevich topology. In particular one has the following result.

Proposition 3.3.1 *For any field k the category $Shv_{et}(SmCor(k))$ is abelian and there exists the associated sheaf functor*

$$Shv_{Nis}(SmCor(k)) \to Shv_{et}(SmCor(k))$$

which is exact. Denote by $D^-(Shv_{et}(SmCor(k)))$ the derived category of complexes bounded from the above over $Shv_{et}(SmCor(k))$. Then for any object A of this category and any smooth scheme X over k one has a canonical isomorphism

$$Hom_{D^-(Shv_{et})}(L(X), A) = \mathbf{H}^0_{et}(X, A).$$

We denote by $DM^{eff}_{-,et}(k)$ the full subcategory of $D^-(Shv_{et}(SmCor(k)))$ which consits of complexes with homotopy invariant cohomology sheaves. Using results of [15] we see immediately that this is a triangulated subcategory, the analog of Proposition 3.2.3 holds in the etale case and the associated etale sheaf functor gives us a functor $DM^{eff}_-(k) \to DM^{eff}_{-,et}(k)$.

Proposition 3.3.2 *The functor*

$$DM^{eff}_-(k) \otimes \mathbf{Q} \to DM^{eff}_{-,et}(k) \otimes \mathbf{Q}$$

is an equivalence of triangulated categories.

Proof: It follows immediately from Proposition 3.3.1 and the comparison Theorem [15, Prop. 5.28].

Denote by $DM^{eff}_{-,et}(k, \mathbf{Z}/n\mathbf{Z})$ the category constructed in the same way as $DM^{eff}_{-,et}(k)$ from the abelian category $Shv_{et}(SmCor(k), \mathbf{Z}/n\mathbf{Z})$ of etale sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules with transfers.

Proposition 3.3.3 *Denote by p the exponential characteristic of the field k. Then one has:*

1. *Let $n \geq 0$ be an integer prime to p. Then the functor*

   $$DM^{eff}_{-,et}(k, \mathbf{Z}/n\mathbf{Z}) \to D^-(Shv(Spec(k)_{et}, \mathbf{Z}/n\mathbf{Z}))$$

   where $Shv(Spec(k)_{et}, \mathbf{Z}/n\mathbf{Z})$ is the abelian category of sheaves of $\mathbf{Z}/n\mathbf{Z}$-modules on the small etale site $Spec(k)_{et}$ which takes a complex of sheaves on Sm/k to its restriction to $Spec(k)_{et}$ is an equivalence of triangulated categories.

2. *For any $n \geq 0$ the category $DM^{eff}_{-,et}(k, \mathbf{Z}/p^n\mathbf{Z})$ is equivalent to the zero category.*

Proof: The first statement follows from the rigidity theorem [15, Th. 5.25]. The second one follows from the fact that $\mathbf{Z}/p\mathbf{Z} = 0$ in $DM^{eff}_{-,et}(k)$ which was proven (in slightly different form) in [16].

3.4 *Motives of varieties of dimension ≤ 1.*

Consider the thick subcategory $d_{\leq n}DM^{eff}_{gm}(k)$ in the category $DM^{eff}_{gm}(k)$ generated by objects of the form $M_{gm}(X)$ for smooth schemes of dimension $\leq n$ over k. Similarly let $d_{\leq n}DM^{eff}_{-}(k)$ be the localizing subcategory in $DM^{eff}_{-}(k)$ generated by objects of the form $\underline{C}_*(X)$ for smooth schemes X of dimension $\leq n$ over k. These categories are called the category of (effective) geometrical n-motives and the category of (effective) n-motivic complexes respectively. One can observe easily that the inclusion functors

$$d_{\leq n}DM^{eff}_{-}(k) \to d_{\leq n+1}DM^{eff}_{-}(k)$$

have right adjoints given on the level of sheaves with transfers by taking the canonical free resolution of the restriction of a sheaf to the category of smooth schemes of dimension $\leq n$. Unfortunately, the corresponding left adjoint functors most probably do not exist for $n \geq 2$. In fact it can be shown that if the standard motivic assumptions hold the existence of such an adjoint for $n = 2$ would imply that the group of 1-cycles modulo algebraic equivalence on a variety of dimension three is either finitely generated or is not countable, which is known to be wrong.

In this section we will describe "explicitly" the category $d_{\leq 0}DM^{eff}_{gm}(k)$ of geometrical 0-motives and give some partial description of the category $d_{\leq 1}DM^{eff}_{gm}(k)$ of geometrical 1-motives. In particular we obtain a description of the motivic cohomology of weight one which will be used in the next section to construct the standard distinguished triangles in $DM^{eff}_{gm}(k)$.

We start with the category $d_{\leq 0}DM^{eff}_{gm}(k)$ of zero motives. As always we assume that k is a perfect field. Choose an algebraic closure $\bar{k}$ of k and let $G_k = Gal(\bar{k}/k)$ be the Galois group of $\bar{k}$ over k. Denote by $Perm(G_k)$ the full additive subcategory of the category of $\mathbf{Z}[G_k]$-modules which consists of permutational representations (i.e. representations which are formal linear envelopes of finite G_k-sets). The tensor structure on the category of $\mathbf{Z}[G_k]$

modules gives us a tensor structure on $Perm(k)$ such that $Perm(k)$ becomes a rigid tensor additive category.

Let $Shv(Perm(G_k))$ be the category of additive contravariant functors from $Perm(k)$ to the category of abelian groups. Clearly the category $Shv(Perm(G_k))$ is abelian and we have a full embedding

$$Perm(k) \to Shv(Perm(G_k))$$

which takes an object to the corresponding representable functor. We denote this functor by $L_G(-)$. Note that objects of the form $L_G(-)$ are projective objects in $Shv(Perm(G_k))$ and therefore the corresponding functor

$$\mathcal{H}^b(Perm(k)) \to D^b(Shv(Perm(k)))$$

is also a full embedding.

Denote by $\mathcal{H}^b(Perm(k))_\oplus$ the pseudo-abelian envelope of $\mathcal{H}^b(Perm(k))$. Note that the embedding

$$\mathcal{H}^b(Perm(k)) \to D^b(Shv(Perm(k)))$$

has a canonical extension to an embedding

$$\mathcal{H}^b(Perm(k))_\oplus \to D^b(Shv(Perm(k)))$$

since the right hand side category is pseudo-abelian.

The following proposition which gives an explicit description of the categories $d_{\leq 0}DM_{gm}^{eff}(k)$ and $d_{\leq 0}DM_-^{eff}(k)$ is an easy corollary of Theorem 3.2.6 and the elementary Galois theory.

Proposition 3.4.1 *There is a commutative diagram of the form*

$$\begin{array}{ccc}
\mathcal{H}^b(Perm(k))_\oplus & \to & D^-(Shv(Perm(k))) \\
\downarrow & & \downarrow \\
d_{\leq 0}DM_{gm}^{eff}(k) & \to & d_{\leq 0}DM_-^{eff}(k)
\end{array}$$

with vertical arrows being equivalences of tensor triangulated categories.

Remarks:

1. Note that while the category $D^-(Shv(Perm(k)))$ (and thus the category $d_{\leq 0}DM_-^{eff}(k)$) has an obvious t-structure it is not clear how to construct any nondegenerate t-structure on the category $\mathcal{H}^b(Perm(k))_\oplus$. In particular, even for zero motives there seems to be no reasonable abelian theory underlying the triangulated theory which we consider.

2. The rational coefficients analog of $\mathcal{H}^b(Perm(k))_\oplus$ is canonically equivalent to the derived category of bounded complexes over the abelian category of G_k-representations over $\mathbf{Q}$. Thus with rational coefficients our theory for zero dimensional varieties gives the usual Artin motives.

The following result which was proven in different forms in [9] and [13] (see also [6]) contains essentially all the information one needs to describe the category $d_{\leq 1}DM_{gm}^{eff}(k)$ of 1-motives.

Theorem 3.4.2 *Let $p : C \to Spec(k)$ be a smooth connected curve over a field k. Denote by $Alb(C)$ the Albanese variety of C and let $\underline{Alb}(C)$ be the sheaf of abelian groups on Sm/k represented by $Alb(X)$. Then one has:*

1. *$\underline{h}_i^{Nis}(C) = 0$ for $i \neq 0, 1$.*

2. *$\underline{h}_1^{Nis}(C) = p_*\mathbf{G}_m$ if C is proper and $\underline{h}_1^{Nis}(C) = 0$ otherwise. In particular $\underline{h}_1^{Nis}(C) = \mathbf{G}_m$ if and only if C is proper and geometrically connected.*

3. *The kernel of the canonical homomorphism $\underline{h}_0^{Nis}(C) \to \mathbf{Z}$ is canonically isomorphic to a subsheaf $\underline{Alb}(C)$.*

Corollary 3.4.3 *Let X be a smooth scheme over k. Then one has:*

$$Hom_{DM_{gm}^{eff}}(M_{gm}(X), \mathbf{Z}(1)[j]) = H_{Zar}^{j-1}(X, \mathbf{G}_m).$$

Proof: It follows from the definition of $\mathbf{Z}(1)$, Corollary 3.2.7 and the fact that

$$\underline{h}_i(\mathbf{P}^1) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{G}_m^* & \text{for } i = 1 \\ 0 & \text{for } i \neq 0, 1. \end{cases}$$

by Theorem 3.4.2.

Unfortunately we are unable to give a reasonable description of the category $d_{\leq 1}DM_{gm}^{eff}(k)$ due to the fact that with the integral coefficients it most probably has no "reasonable" t-structure (see 4.3.8 for a precise statement in the case of the whole category $DM_{gm}^{eff}(k)$). The rational coefficients analog $d_{\leq 1}DM_{gm}^{eff}(k, \mathbf{Q})$ can be easily described in "classical" terms as the derived category of bounded complexes over the abelian category of the Deligne 1-motives over k with rational coefficients. We do not consider this description here mainly because it will not play any role in the rest of the paper.

3.5 *Fundamental distinguished triangles in the category of geometrical motives.*

In this section we will construct several canonical distinguished triangles in $DM_{gm}^{eff}(k)$ which correspond to the standard exact sequences in the cohomology of algebraic varieties. We will also prove the standard decomposition results for motives of projective bundles and blow-ups.

Proposition 3.5.1 *Let X be a smooth scheme over k and $\mathcal{E}$ be a vector bundle over X. Denote by $p : \mathbf{P}(\mathcal{E}) \to X$ the projective bundle over X associated with $\mathcal{E}$. Then one has a canonical isomorphism in $DM_{gm}^{eff}(k)$ of the form:*

$$M_{gm}(\mathbf{P}(\mathcal{E})) = \oplus_{n=0}^{dim\mathcal{E}-1} M_{gm}(X)(n)[2n].$$

Proof: We may assume that $d = dim(\mathcal{E}) > 0$. Let $\mathcal{O}(1)$ be the standard line bundle on $\mathbf{P}(\mathcal{E})$. By Corollary 3.4.3 it defines a morphism $\tau_1 : M_{gm}(X) \to \mathbf{Z}(1)[2]$. For any $n \geq 0$ we set τ_n to be the composition

$$M_{gm}(X) \stackrel{M_{gm}(\Delta)}{\to} M_{gm}(X^n) = M_{gm}(X)^{\otimes n} \stackrel{\tau_1^{\otimes n}}{\to} \mathbf{Z}(n)[2n].$$

Let further σ_n be the composition

$$M_{gm}(X) \stackrel{M_{gm}(\Delta)}{\to} M_{gm}(X) \otimes M_{gm}(X) \stackrel{Id_{M_{gm}(X)} \otimes \tau_n}{\to} M_{gm}(X)(n)[2n].$$

We have a morphism

$$\Sigma = \oplus_{n=0}^{d-1} \sigma_n : M_{gm}(X) \to \oplus_{n=0}^{dim\mathcal{E}-1} M_{gm}(X)(n)[2n].$$

Let us show that it is an isomorphism. Note first that Σ is natural with respect to X. Using the induction on the number of open subsets in a trivializing covering for $\mathcal{E}$ and the distinguished triangles from Lemma 2.1.2 we may assume that $\mathcal{E}$ is trivial. In this case Σ is the tensor product of the corresponding morphism for the trivial vector bundle over $Spec(k)$ and $Id_{M_{gm}(X)}$. It means that we have only to consider the case $X = Spec(k)$. Then our proof goes exactly as the proof of the similar result in [16].

Proposition 3.5.2 *Let X be a smooth scheme over k and $Z \subset X$ be a smooth closed subscheme in X. Denote by $p : X_Z \to X$ the blow-up of Z in X. Then one has a canonical distinguished triangle of the form:*

$$M_{gm}(p^{-1}(Z)) \to M_{gm}(Z) \oplus M_{gm}(X_Z) \to M_{gm}(X) \to M_{gm}(p^{-1}(Z))[1].$$

Proof: Consider the complex $[p^{-1}(Z)] \to [Z] \oplus [X_Z] \to [X]$ in SmCor(k). We have to show that the corresponding object of $DM_{gm}^{eff}(k)$ is zero. Consider the complex Φ of sheaves with transfers of the form

$$L(p^{-1}(Z)) \to L(Z) \oplus L(X_Z) \to L(X).$$

By Theorem 3.2.6 we have only to show that for any homotopy invariant sheaf with transfers F on Sm/k and any $i \in \mathbf{Z}$ we have

$$Hom_{D^-(Shv_{Nis}(SmCor(k)))}(\Phi, F[i]) = 0.$$

Let Φ_0 be the complex of Nisnevich sheaves of the form

$$\mathbf{Z}_{Nis}(p^{-1}(Z)) \to \mathbf{Z}_{Nis}(Z) \oplus \mathbf{Z}_{Nis}(X_Z) \to \mathbf{Z}_{Nis}(X)$$

where $\mathbf{Z}_{Nis}(-)$ denote the freely generated Nisnevich sheaf. By Proposition 3.1.8 the canonical homomorphisms:

$$Hom_{D^-(Shv_{Nis}(SmCor(k)))}(\Phi, F[i]) \to Hom_{D^-(Shv_{Nis}(Sm/k))}(\Phi_0, F[i])$$

are isomorphisms for all $i \in \mathbf{Z}$. The complex Φ_0 is clearly left exact and

$$coker(\mathbf{Z}_{Nis}(Z) \oplus \mathbf{Z}_{Nis}(X_Z) \to \mathbf{Z}_{Nis}(X)) = coker(\mathbf{Z}_{Nis}(X_Z) \stackrel{\mathbf{Z}_{Nis}(p)}{\to} \mathbf{Z}_{Nis}(X))$$

since $p^{-1}(Z) \to Z$ is the projective bundle of a vector bundle and thus $\mathbf{Z}_{Nis}(p^{-1}(Z)) \to \mathbf{Z}_{Nis}(Z)$ is a surjection in the Nisnevich topology. Therefore we have canonical isomorphisms:

$$Hom_{D^-(Shv_{Nis}(Sm/k))}(\Phi_0, F[i]) = Ext^i_{Shv_{Nis}(Sm/k)}(coker(\mathbf{Z}_{Nis}(p)), F).$$

The last groups are zero by [15, Prop. 5.21].

Proposition 3.5.3 *Let X be a smooth scheme over k and Z be a smooth closed subscheme in X everywhere of codimension c. Denote by $p : X_Z \to X$ the blow-up of Z in X. Then there is a canonical isomorphism*

$$M_{gm}(X_Z) = M_{gm}(X) \oplus (\oplus_{n=1}^{c-1} M_{gm}(Z)(n)[2n]).$$

Proof: In view of Propositions 3.5.2 and 3.5.1 we have only to show that the morphism $M_{gm}(X_Z) \to M_{gm}(X)$ has a canonical splitting. Consider the following diagram (morphisms are numbered for convinience):

$$\begin{array}{ccc}
M_{gm}(p^{-1}(Z)) & \stackrel{1}{\to} & M_{gm}(q^{-1}(Z \times \{0\})) \\
3 \downarrow & & \downarrow 4 \\
M_{gm}(Z) \oplus M_{gm}(X_Z) & \stackrel{2}{\to} & M_{gm}(Z \times \{0\}) \oplus M_{gm}((X \times \mathbf{A}^1)_{Z \times \{0\}}) \\
5 \downarrow & & \downarrow 6 \\
M_{gm}(X) & \stackrel{M_{gm}(Id \times \{0\})}{\to} & M_{gm}(X \times \mathbf{A}^1) \\
7 \downarrow & & \downarrow 8 \\
M_{gm}(p^{-1}(Z))[1] & \stackrel{1[1]}{\to} & M_{gm}(q^{-1}(Z \times \{0\}))[1].
\end{array}$$

where $q : (X \times \mathbf{A}^1)_{Z \times \{0\}} \to X \times \mathbf{A}^1$ is the blow-up of $Z \times \{0\}$ in $X \times \mathbf{A}^1$.

Note that the morphism $M_{gm}(Id \times \{0\})$ is an isomorphism equal to the isomorphism $M_{gm}(Id \times \{1\})$. Since the later morphism obviously has a lifting to $M_{gm}(Z \times \{0\}) \oplus M_{gm}((X \times \mathbf{A}^1)_{Z \times \{0\}})$ we conclude that the morphism marked (6) has a canonical splitting. Since the vertical triangles are distinguished by Proposition 4.1.3 it gives us a canonical splitting of the morphism (4).

The morphism $p^{-1}(Z) \to q^{-1}(Z \times \{0\})$ is the canonical embedding of the projective bundle $\mathbf{P}(N(Z))$ over Z (where $N(Z)$ is the normal bundle to Z in X) to the projective bundle $\mathbf{P}(N(Z) \oplus \mathcal{O})$ over Z (where $\mathcal{O}$ is the trivial line bundle on Z). Thus by Proposition 3.5.1 the morphism (1) is a splitting monomorphism with a canonical splitting. Thus the composition of the morphism (2) with the canonical splitting of the morphism (4) with the canonical splitting of the morphism (1) gives us a canonical splitting of the morphism (3). Using the fact that the left vertical triangle is distinguished we conclude that the morphism (5) also has a canonical splitting.

The diagram from the proof of Proposition 3.5.3 has another important application. Let X be a smooth scheme over k and Z be a smooth closed subscheme in X everywhere of codimension c.

Using this diagram we can construct a morphism

$$g_Z : M_{gm}(X) \to M_{gm}(Z)(c)[2c]$$

as follows. Consider two morphisms from $M_{gm}(X)$ to $M_{gm}(Z \times \{0\}) \oplus M_{gm}((X \times \mathbf{A}^1)_{Z\times\{0\}})$. One is the composition of the canonical splitting of the morphism (5) with the morphism (2). Another one is the canonical lifting of the morphism $Id \times \{1\}$. Let f be their difference. Composition of f with the morphism (6) is zero. Thus f has a canonical lifting to a morphism $M_{gm}(X) \to M_{gm}(q^{-1}(Z \times \{0\}))$. To get the morphism g_Z we note that $q^{-1}(Z \times \{0\})$ is the projective bundle associated with a vector bundle of dimension $c+1$ over Z and use Proposition 3.5.1.

The composition of this morphism with the canonical morphism

$$M_{gm}(Z)(c)[2c] \to \mathbf{Z}(c)[2c]$$

which is induced by the projection $Z \to Spec(k)$ is the class of Z in the motivic cohomology group $H^{2c}_M(X, \mathbf{Z}(c))$.

The following proposition shows that g_Z fits into a canonical distinguished triangle which leads to the Gysin exact sequences in cohomology theories.

Proposition 3.5.4 *Let X be a smooth scheme over k and Z be a smooth closed subscheme in X everywhere of codimension c. Then there is a canonical distinguished triangle in $DM^{eff}_{gm}(k)$ of the form*

$$M_{gm}(X - Z) \stackrel{M_{gm}(j)}{\to} M_{gm}(X) \stackrel{g_Z}{\to} M_{gm}(Z)(c)[2c] \to M_{gm}(X - Z)[1]$$

(here j is the open embedding $X - Z \to X$).

Proof: Denote by $M_{gm}(X/(X - Z))$ the object in $DM^{eff}_{gm}(k)$ which corresponds to the complex $[X - Z] \to [X]$. Note that its image in $DM^{eff}_{-}(k)$ is canonically isomorphic to $\underline{C}_*(L(X)/L(X - Z))$ and that there is a distinguished triangle of the form:

$$M_{gm}(X - Z) \to M_{gm}(X) \to M_{gm}(X/(X - Z)) \to M_{gm}(X - Z)[1].$$

To prove the proposition it is sufficient to show that there is an isomorphism $M_{gm}(X/(X - Z)) \to M_{gm}(Z)(c)[2c]$.

To do it we consider again the diagram from the proof of Proposition 3.5.3. Exactly the same arguments as before show that there is a canonical isomorphism

$$M_{gm}(X_Z/(X_Z - p^{-1}(Z))) = M_{gm}(X/(X-Z)) \oplus (\oplus_{n=1}^{c-1} M_{gm}(Z)(n)[2n])$$

which is compatible in the obvious sense with the distinguished triangles for $M_{gm}(X_Z/(X_Z - p^{-1}(Z)))$ and $M_{gm}(X/(X-Z))$. Proceeding as in the construction of the morphism g_Z we see that it can in fact be factored through a canonical morphism

$$\alpha_{(X,Z)} : M_{gm}(X/(X-Z)) \to M_{gm}(Z)(c)[2c].$$

The word "canonical" here means that the following conditions hold:

1. Let $f : X' \to X$ be a smooth morphism. Denote $f^{-1}(Z)$ by Z'. Then the diagram

$$\begin{array}{ccc} M_{gm}(X'/(X'-Z')) & \stackrel{\alpha_{(X',Z')}}{\to} & M_{gm}(Z')(c)[2c] \\ \downarrow & & \downarrow \\ M_{gm}(X/(X-Z)) & \stackrel{\alpha_{(X,Z)}}{\to} & M_{gm}(Z)(c)[2c] \end{array}$$

   commutes.

2. For any smooth scheme Y over k we have

$$\alpha_{(X\times Y, Z\times Y)} = \alpha_{(X,Z)} \otimes Id_{M_{gm}(Y)}.$$

Consider an open covering $X = U \cup V$ of X. Let

$$Z_U = Z \cap U$$

$$Z_V = Z \cap V.$$

One can easily see that there is a canonical distinguished triangle of the form

$$M_{gm}(U\cap V/(U\cap V - Z_U \cap Z_V)) \to M_{gm}(U/(U-Z_U)) \oplus M_{gm}(V/(V-Z_V)) \to$$

$$\to M_{gm}(X/(X-Z)) \to M_{gm}(U\cap V/(U\cap V - Z_U\cap Z_V))[1]$$

and morphisms $\alpha_{-,-}$ map it to the corresponding Mayer-Vietoris distinguished triangle for the open covering $Z = Z_U \cup Z_V$ of Z.

Thus to prove that $\alpha_{(X,Z)}$ is an isomorphism it is sufficient to show that X has an open covering $X = \cup U_i$ such that $\alpha_{(V,Z_V)}$ is an isomorphism for any open subset V which lies in one of the U_i's. In particular we may assume that there exists an etale morphism $f : X \to \mathbf{A}^d$ such that $Z = f^{-1}(\mathbf{A}^{d-c})$. Consider the Cartesian square:

$$\begin{array}{ccc} Y & \to & Z \times \mathbf{A}^c \\ \downarrow & & \downarrow f_{|Z} \times Id \\ X & \stackrel{f}{\to} & \mathbf{A}^d. \end{array}$$

Since f (and therefore $f_{|Z}$) is an etale morphism the diagonal is a connected component of $Z \times_{\mathbf{A}^{d-c}} Z$ and we may consider the open subscheme $X' = Y - (Z \times_{\mathbf{A}^{d-c}} Z - Z)$ of Y. Let Z' be the image of Z in Y. Then pr_1 maps Z' isomorphically to $Z \times \{0\} \subset Z \times \mathbf{A}^c$ and pr_2 maps Z' isomorphically to $Z \subset X$. Moreover

$$pr_1^{-1}(Z \times \{0\}) = Z'$$

and

$$pr_2^{-1}(Z) = Z'.$$

Thus by [15, Prop. 5.18] the obvious morphisms of Nisnevich sheaves

$$\mathbf{Z}_{Nis}(X')/\mathbf{Z}_{Nis}(X' - Z') \to \mathbf{Z}_{Nis}(X)/\mathbf{Z}_{Nis}(X - Z)$$

$$\mathbf{Z}_{Nis}(X')/\mathbf{Z}_{Nis}(X' - Z') \to \mathbf{Z}_{Nis}(Z \times \mathbf{A}^c)/\mathbf{Z}_{Nis}(Z \times \mathbf{A}^c - Z \times \{0\})$$

are isomorphisms. It follows immediately from Propositoon 3.1.8 that the same holds for the sheaves $L(-)/L(-)$ and therefore by Theorem 3.2.6 the morphisms

$$M_{gm}(X'/(X' - Z')) \to M_{gm}(X/(X - Z))$$

$$M_{gm}(X'/(X' - Z')) \to M_{gm}(Z \times \mathbf{A}^c/(Z \times \mathbf{A}^c - Z \times \{0\}))$$

are isomorphisms. Due to the naturality properties of the morphisms $\alpha_{(-,-)}$ stated above it remains to show that $\alpha_{(\mathbf{A}^c,\{0\})}$ or equivalently $\alpha_{(\mathbf{P}^c,pt)}$ is an isomorphism. It follows easily from the construction of $\alpha_{(-,-)}$ (see [16]).

Corollary 3.5.5 *Let k be a field which admits resolution of singularities. Then DM_{gm}^{eff} is generated as a triangulated category by direct summands of objects of the form $M_{gm}(X)$ for smooth projective varieties X over k.*

4 Homology of algebraic cycles and duality.

4.1 *Motives of schemes of finite type.*

Let X be a scheme of finite type over a field k. For any smooth scheme U over k consider the following two abelian groups

1. The group $L(X)(U)$ is the free abelian group generated by closed integral subschemes Z of $X \times U$ such that Z is finite over U and dominant over an irreducible (=connected) component of U,

2. The group $L^c(X)(U)$ is the free abelian group generated by closed integral subschemes Z of $X \times U$ such that Z is quasi-finite over U and dominant over an irreducible (=connected) component of U.

One can define easily Nisnevich sheaves with transfers $L(X), L^c(X)$ on Sm/k such that for any smooth U over k the groups $L(X)(U), L^c(X)(U)$ are the groups described above. In the case when X is smooth over k our notation agrees with the notation $L(X)$ for the presheaf with transfers represented by X. The presheaves $L(X)$ (resp. $L^c(X)$) are covariantly functorial with respect to X (resp. with respect to proper morphisms $X \to X'$) which gives us two functors:

$$L(-) : Sch/k \to PreShv(SmCor(k))$$

$$L^c(-) : Sch^{prop}/k \to PreShv(SmCor(k))$$

(here Sch^{prop}/k is the category of schemes of finite type over k and proper morphisms). Note that the functor $L(-)$ from Sch/k extends the functor $L(-)$ from Sm/k which we considered before.

For a scheme of finite type X over k we write $\underline{C}_*(X)$ (resp. $\underline{C}^c_*(X)$) instead of $\underline{C}_*(L(X))$ (resp. $\underline{C}_*(L^c(X))$). By Lemma 3.2.1 these complexes of Nisnevich sheaves with transfers are objects of the category $DM_-^{eff}(k)$. It provides us with two functors:

$$\underline{C}_*(-) : Sch/k \to DM_-^{eff}(k)$$

$$\underline{C}^c_*(-) : Sch^{prop}/k \to DM_-^{eff}(k).$$

We will show latter in this section that if k admits resolution of singularities they can be factored through the canonical embedding $DM_{gm}^{eff}(k) \to$

$DM_-^{eff}(k)$ and therefore define "motives" and "motives with compact support" for schemes which are not necessarily smooth over k.

Proposition 4.1.1 *Let X be a scheme of finite type over k and $X = U \cup V$ be an open covering of X. Then there is a canonical distinguished triangle in DM_-^{eff} of the form*

$$\underline{C}_*(U \cap V) \to \underline{C}_*(U) \oplus \underline{C}_*(V) \to \underline{C}_*(X) \to \underline{C}_*(U \cap V)[1].$$

Proof: It is sufficient to notice that the sequence of Nisnevich sheaves

$$0 \to L(U \cap V) \to L(U) \oplus L(V) \to L(X) \to 0$$

is exact.

The main technical result which allow us to work effectively with objects of the form $L(X), L^c(X)$ for singular varieties is the theorem below which is a particular case of [7, Th. 5.5(2)].

Theorem 4.1.2 *Let k be a field which admits resolution of singularities and F be a presheaf with transfers on Sm/k such that for any smooth scheme X over k and a section $\phi \in F(X)$ there is a proper birational morphism $p : X' \to X$ with $F(p)(\phi) = 0$. Then the complex $\underline{C}_*(F)$ is quasi-isomorphic to zero.*

The first application of this theorem is the following blow-up distinguished triangle.

Proposition 4.1.3 *Consider a Cartesian square of morphisms of schemes of finite type over k of the form*

$$\begin{array}{ccc} p^{-1}(Z) & \to & X_Z \\ \downarrow & & \downarrow p \\ Z & \to & X \end{array}$$

such that the following conditions hold:

1. *The morphism $p : X_Z \to X$ is proper and the morphism $Z \to X$ is a closed embedding.*

2. *The morphism $p^{-1}(X - Z) \to X$ is an isomorphism.*

Then there is a canonical distinguished triangle in $DM_-^{eff}(k)$ of the form

$$\underline{C}_*(p^{-1}(Z)) \to \underline{C}_*(Z) \oplus \underline{C}_*(X_Z) \to \underline{C}_*(X) \to \underline{C}_*(p^{-1}(Z))[1].$$

Proof: It is sufficient to notice that the sequence of presheaves

$$0 \to L(p^{-1}(Z)) \to L(X_Z) \oplus L(Z) \to L(X)$$

is exact and the quotient presheaf $L(X)/(L(X_Z) \oplus L(Z))$ satisfies the condition of Theorem 4.1.2.

Corollary 4.1.4 *Let k be a field which admits resolution of singularities. Then for any scheme X of finite type over k the object $\underline{C}_*(X)$ belongs to $DM_{gm}^{eff}(k)$.*

The following proposition explains why $\underline{C}_*^c(X)$ is called the motivic complex with compact support.

Proposition 4.1.5 *Let k be a field which admits resolution of singularities, X be a scheme of finite type over k and Z be a closed subscheme of X. Then there is a canonical distinguished triangle of the form*

$$\underline{C}_*^c(Z) \to \underline{C}_*^c(X) \to \underline{C}_*^c(X - Z) \to \underline{C}_*^c(Z)[1].$$

If X is proper than there is a canonical isomorphism $\underline{C}_*^c(X) = \underline{C}_*(X)$.

Proof: The second statement is obvious. To prove the first one it is again sufficient to notice that the sequence of presheaves

$$0 \to L^c(Z) \to L^c(X) \to L^c(X - Z)$$

is exact and the cokernel $L^c(X-Z)/L^c(X)$ satisfies the condition of Theorem 4.1.2.

Corollary 4.1.6 *Let k be a field which admits resolution of singularities. Then for any scheme X of finite type over k the object $\underline{C}_*^c(X)$ belongs to $DM_{gm}^{eff}(k)$.*

Proposition 4.1.7 *Let X, Y be schemes of finite type over k. Then there are canonical isomorphisms:*

$$\underline{C}_*(X \times Y) = \underline{C}_*(X) \otimes \underline{C}_*(Y)$$

$$\underline{C}^c_*(X \times Y) = \underline{C}^c_*(X) \otimes \underline{C}^c_*(Y)$$

Proof: One can construct easily natural morphisms

$$\underline{C}_*(X) \otimes \underline{C}_*(Y) \to \underline{C}_*(X \times Y)$$

$$\underline{C}^c_*(X) \otimes \underline{C}^c_*(Y) \to \underline{C}^c_*(X \times Y).$$

For smooth projective X, Y they are isomorphisms in DM_-^{eff} by the definition of the tensor structure on this category. The general case follows now formally from Propositions 4.1.3 and 4.1.5.

Corollary 4.1.8 *For any scheme of finite type X over k one has canonical isomorphisms:*

$$\underline{C}_*(X \times \mathbf{A}^1) = \underline{C}_*(X)$$

$$\underline{C}^c_*(X \times \mathbf{A}^1) = \underline{C}^c_*(X)(1)[2].$$

In particular we have:

$$\underline{C}^c_*(\mathbf{A}^n) = \mathbf{Z}(n)[2n].$$

Proof: In view of Proposition 4.1.7 it is sufficient to show that

$$\underline{C}_*(\mathbf{A}^1) = \mathbf{Z}$$

$$\underline{C}^c_*(\mathbf{A}^1) = \mathbf{Z}(1)[2].$$

The first fact follows immediately from our definitions. The second follows from the definition of the Tate object and Proposition 4.1.5.

We want to describe now morphisms of the form $Hom(\underline{C}_*(X), \underline{C}_*(F))$ for schemes of finite type X over k and Nisnevich sheaves with transfers F. The first guess that this group is isomorphic to $\mathbf{H}_{Nis}(X, \underline{C}_*(F))$ as it was proved in 3.2.7 for smooth schemes X turns out to be wrong.

To get the correct answer we have to consider the cdh-topology on the category Sch/k of schemes of finite type over k.

Definition 4.1.9 *The cdh-topology on Sch/k is the minimal Grothendieck topology on this category such that the following two types of coverings are cdh-coverings.*

1. *the Nisnevich coverings.*

2. *Coverings of the form $X' \coprod Z \overset{p\coprod i}{\to} X$ such that p is a proper morphism, i is a closed embedding and the morphism $p^{-1}(X - i(Z)) \to X - i(Z)$ is an isomorphism.*

Denote by $\pi : (Sch/k)_{cdh} \to (Sm/k)_{Nis}$ the obvious morphism of sites. Note that the definition of $\underline{C}_*(F)$ given above for presheaves on Sm/k also works for presheaves on Sch/k. The theorem below follows formally from Theorem 4.1.2.

Theorem 4.1.10 *Let X be a scheme of finite type over k and F be a presheaf with transfers on Sm/k. Then for any $i \geq 0$ there are canonical isomorphisms*

$$Hom(\underline{C}_*(X), \underline{C}_*(F)[i]) = \mathbf{H}^i_{cdh}(X, \underline{C}_*(\pi^*(F))) = \mathbf{H}^i_{cdh}(X, \pi^*(\underline{C}_*(F))).$$

In particular if X is smooth one has

$$\mathbf{H}^i_{cdh}(X, \underline{C}_*(\pi^*(F))) = \mathbf{H}^i_{cdh}(X, \pi^*(\underline{C}_*(F))) = \mathbf{H}^i_{Nis}(X, \underline{C}_*(F)).$$

Corollary 4.1.11 *Let k be a field which admits resolution of singularities and X be a scheme of finite type over k. Let further $\mathcal{E}$ be a vector bundle over X. Denote by $p : \mathbf{P}(\mathcal{E}) \to X$ the projective bundle over X associated with $\mathcal{E}$. Then one has a canonical isomorphism in $DM^{eff}_-(k)$ of the form:*

$$\underline{C}_*(\mathbf{P}(\mathcal{E})) = \oplus_{n=0}^{dim\mathcal{E}-1} \underline{C}_*(X)(n)[2n].$$

Proof: Theorem 4.1.10 implies that we can construct a natural morphism

$$\underline{C}_*(\mathbf{P}(\mathcal{E})) \to \oplus_{n=0}^{dim\mathcal{E}-1} \underline{C}_*(X)(n)[2n]$$

in exactly the same way as in Proposition 3.5.1. The fact that it is an isomorphism follows now formally from Proposition 3.5.1, Proposition 4.1.3 and resolution of singularities.

Let us recall that in [16] a triangulated category $DM_h(k)$ was defined as a localization of the derived category $D^-(Shv_h(Sch/k))$ of complexes bounded from the above over the category of sheaves of abelian groups in the h-topology on Sch/k. One can easily see that there is a canonical functor

$$DM_{-,et}^{eff}(k) \to DM_h(k).$$

Using the comparison results of [16] (for sheaves of $\mathbf{Q}$-vector spaces) together with the technique described in this section one can verify easily that this functor is an equivalence after tensoring with $\mathbf{Q}$. Moreover, using the description of the category $DM_{-,et}^{eff}(k, \mathbf{Z}/n\mathbf{Z})$ given in Section 3.3 and comparison results for torsion sheaves from [16] one can also show that it is an equivalence for finite coefficients. Combining these two results we obtain the following theorem.

Theorem 4.1.12 *Let k be a field which admits resolution of singularities. Then the functor*

$$DM_{-,et}^{eff}(k) \to DM_h(k)$$

is an equivalence of triangulated categories. In particular, the categories $DM_-^{eff}(k) \otimes \mathbf{Q}$ and $DM_h(k) \otimes \mathbf{Q}$ are equivalent.

4.2 *Bivariant cycle cohomology.*

Let us recall the definition of the bivariant cycle cohomology given in [7]. For any scheme of finite type X over k and any $r \geq 0$ we denote by $z_{equi}(X, r)$ the presheaf on the category of smooth schemes over k which takes a smooth scheme Y to the free abelian group generated by closed integral subschemes Z of $Y \times X$ which are equidimensional of relative dimension r over Y (note that it means in particular that Z dominates an irreducible component of Y). One can verify easily that $z_{equi}(X, r)$ is a sheaf in the Nisnevich topology and moreover that it has a canonical structure of a presheaf with transfers. The presheaves with transfers $z_{equi}(X, r)$ are covariantly functorial with respect to proper morphisms of X by means of the usual proper push-forward of cycles and contravariantly functorial with an appropriate dimension shift with respect to flat equidimensional morphisms. For $r = 0$ the presheaf $z_{equi}(X, r)$ is isomorphic to the presheaf $L^c(X)$ defined in the previous section.

We will also use below the notation $z_{equi}(Y, X, r)$ introduced in [15] for the presheaf which takes a smooth scheme U to $z_{equi}(X, r)(U \times Y)$.

Definition 4.2.1 *Let X, Y be schemes of finite type over k and $r \geq 0$ be an integer. The bivariant cycle cohomology $A_{r,i}(Y, X)$ of Y with coefficients in r-cycles on X are the hypercohomology $\mathbf{H}^{-i}_{cdh}(Y, \pi^*(\underline{C}_*(z_{equi}(X, r))))$.*

It follows immediately from this definition and Theorem 4.1.10 that we have

$$A_{r,i}(Y, X) = Hom_{DM_-^{eff}}(\underline{C}_*(Y)[i], \underline{C}_*(z_{equi}(X, r))).$$

We will show below (4.2.3) that in fact these groups admit more explicit interpretation in terms of the category DM_-^{eff}.

The following theorem summarizes the most important for us properties of bivariant cycle cohomology proven in [7].

Theorem 4.2.2 *Let k be a field which admits resolution of singularities, X, Y be schemes of finite type over k and $d \geq 0$ be an integer.*

1. *For any $i, r \geq 0$ there are canonical isomorphisms of the form*

$$A_{r,i}(Y \times \mathbf{P}^1, X) = A_{r,i}(Y, X) \oplus A_{r+1,i}(Y, X)$$

   where the projection $A_{r,i}(Y \times \mathbf{P}^1, X) \to A_{r,i}(Y, X)$ (resp. embedding $A_{r,i}(Y, X) \to A_{r,i}(Y \times \mathbf{P}^1, X)$) is given by the embedding of Y to $Y \times \mathbf{P}^1$ by means of a point of $\mathbf{P}^1$ (resp. by the projection $Y \times \mathbf{P}^1 \to Y$) ([7, Th. 8.3(3)]).

2. *For any $i, r \geq 0$ there are canonical isomorphisms of the form*

$$A_{r+1,i}(Y, X \times \mathbf{A}^1) = A_{r,i}(Y, X)$$

   ([7, Th. 8.3(1)]).

3. *If Y is smooth and equidimensional of dimension n there are canonical isomorphisms*

$$A_{r,i}(Y, X) = A_{r+n,i}(Spec(k), Y \times X)$$

   ([7, Th. 8.2]).

4. *For any smooth quasi-projective Y the obvious homomorphisms*

$$\underline{h}_i(\underline{C}_*(z_{equi}(X, r))(Y)) \to A_{r,i}(Y, X)$$

   are isomorphisms ([7, Th. 8.1]).

Proposition 4.2.3 *Let X, Y be schemes of finite type over k. Then for any $r \geq 0$ there are canonical ismorphisms*

$$Hom_{DM_-^{eff}}(\underline{C}_*(Y)(r)[2r+i], \underline{C}_*^c(X)) = A_{r,i}(Y, X).$$

Proof: For $r = 0$ it follows immediately from Definition 4.2.1 and Theorem 4.1.10 since $z_{equi}(X, 0) = L^c(X)$. We proceed by the induction on r. By definition we have $\underline{C}_*(Y)(1)[2] = \underline{C}_*(Y) \otimes \mathbf{Z}(1)[2]$ and $\mathbf{Z}(1)[2]$ is canonically isomorphic to the direct summand of $\underline{C}_*(\mathbf{P}^1)$ which is the kernel of the projector induced by the composition of morphisms $\mathbf{P}^1 \to Spec(k) \to \mathbf{P}^1$. Thus by Proposition 4.1.7 $\underline{C}_*(Y)(r)[2r]$ is canonically isomorphic to the kernel of the corresponding projector on $\underline{C}_*(Y \times \mathbf{P}^1)(r-1)[2r-2]$. By the inductive assumption we have a canonical isomorphism

$$Hom_{DM_-^{eff}}(\underline{C}_*(Y \times \mathbf{P}^1)(r-1)[2r-2+i], \underline{C}_*^c(X)) = A_{r-1,i}(Y \times \mathbf{P}^1, X)$$

and thus by Theorem 4.2.2(1) we have

$$Hom_{DM_-^{eff}}(\underline{C}_*(Y)(r)[2r+i], \underline{C}_*^c(X)) = A_{r,i}(Y, X).$$

Corollary 4.2.4 *Let $f : X \to Y$ be a flat equidimensional morphism of relative dimension n of schemes of finite type over k. Then there is a canonical morphism in DM_-^{eff} of the form:*

$$f^* : \underline{C}_*^c(Y)(n)[2n] \to \underline{C}_*^c(X)$$

and these morphisms satisfy all the standard properties of the contravariant functoriality of algebraic cycles.

Proof: Let $\Gamma_f \subset X \times Y$ be the graph of f. Considered as a cycle on $X \times Y$ it clearly belongs to $z_{equi}(X, n)(Y)$ and our statement follows from Proposition 4.2.3.

Corollary 4.2.5 *Let X be a smooth scheme over k. Denote by $A^i(X)$ the group of cycles of codimension i on X modulo rational equivalence. Then there is a canonical isomorphism*

$$A^i(X) = Hom_{DM_{gm}^{eff}}(M_{gm}(X), \mathbf{Z}(i)[2i]).$$

Proof: It follows immediately from the fact that $\mathbf{Z}(i)[2i] = \underline{C}^c_*(\mathbf{A}^i)$ (Lemma 4.1.8), Proposition 4.2.3 and Theorem 4.2.2(2).

Corollary 4.2.6 *Let X, Y be smooth proper schemes over k. Then one has:*

$$Hom_{DM^{eff}_{gm}}(M_{gm}(X), M_{gm}(Y)) = A_{dim(X)}(X \times Y)$$

$$Hom_{DM^{eff}_{gm}}(M_{gm}(X), M_{gm}(Y)[i]) = 0 \textit{ for } i > 0.$$

Proof: It follows immediately from Proposition 4.2.3 and Theorem 4.2.2(3).

Corollary 4.2.7 *Let X be a smooth scheme over k and Y be any scheme of finite type over k. Then there is a canonical isomorphism in $DM^{eff}_-(k)$ of the form:*

$$\underline{Hom}_{DM^{eff}_-}(\underline{C}_*(X), \underline{C}^c_*(Y)) = \underline{C}_*(z_{equi}(X, Y, 0))$$

where $z_{equi}(X, Y, n)$ is the sheaf of the form

$$z_{equi}(X, Y, r)(-) = z_{equi}(Y, r)(- \times X).$$

Proposition 4.2.3 has the following "global" reformulation.

Proposition 4.2.8 *Let X be a scheme of finite type over a field k which admits resolution of singularities. Then for any $r \geq 0$ there is a canonical isomorphism in $DM^{eff}_-(k)$ of the form:*

$$\underline{C}_*(z_{equi}(X, r)) = \underline{Hom}_{DM^{eff}}(\mathbf{Z}(r)[2r], \underline{C}^c_*(X)).$$

Proof: It is sufficient to show that the isomorphisms of Proposition 4.2.3 are induced by a morphism

$$\underline{Hom}_{DM^{eff}_-}(\mathbf{Z}(r)[2r], \underline{C}^c_*(X)) \to \underline{C}_*(z_{equi}(X, r))$$

in DM^{eff}_-.

Denote by p_r the formal linear combination of endomorphisms of $(\mathbf{P}^1_k)^r$ which gives in DM^{eff}_- the projector from $\underline{C}_*((\mathbf{P}^1_k)^r)$ to $\mathbf{Z}(r)[2r]$ and by p the canonical morphism $(\mathbf{P}^1_k)^r \to Spec(k)$. By Proposition 3.2.8 the object $\underline{Hom}_{DM^{eff}_-}(\mathbf{Z}(r)[2r], \underline{C}^c_*(X))$ is the direct summand of the complex of sheaves

$\mathbf{R}p_*(p^*(\underline{C}_*(z_{equi}(X,0))))$ defined by p_r. Consider the obvious morphism of complexes:

$$\underline{C}_*(z_{equi}(\mathbf{P}^1_k)^r, X, 0) = \underline{C}_*(p_*(p^*(z_{equi}(X,0)))) \to \mathbf{R}p_*(p^*(\underline{C}_*(z_{equi}(X,0)))).$$

By Theorem 4.2.2(4) this morphism is a quasi-isomorphism of complexes of sheaves on Sm/k and thus the same holds for the direct summands of both complexes defined by p_r. It remains to construct a morphism

$$z_{equi}((\mathbf{P}^1_k)^r, X, 0) \to z_{equi}(X, r).$$

Let U be a smooth scheme over k. The group $z_{equi}((\mathbf{P}^1_k)^r, X, 0)(U)$ is by definition the subgroup of cycles on $(\mathbf{P}^1_k)^r \times U \times X$ which consists of cycles equidimensional of relative dimension 0 over $(\mathbf{P}^1_k)^r \times U$. Pushing them forward with respect to the projection $(\mathbf{P}^1_k)^r \times U \times X \to U \times X$ we get the required homomorphism.

The following proposition is essentially a reformulation of a result proven by A. Suslin in [12].

Proposition 4.2.9 *Let X be a quasi-projective equidimensional scheme over k of dimension n. Then for all $i, j \in \mathbf{Z}$ there are canonical isomorphisms:*

$$CH^{n-i}(X, j-2i) = \begin{cases} Hom_{DM^{eff}_-}(\mathbf{Z}(i)[j], \underline{C}^c_*(X)) & for \quad i \geq 0 \\ Hom_{DM^{eff}_-}(\mathbf{Z}, \underline{C}^c_*(X)(-i)[-j]) & for \quad i \leq 0. \end{cases}$$

which commute with the boundary maps in the localization long exact sequences.

Proof: Consider first the case $i \geq 0$. By Proposition 4.2.3 the left hand side group is isomorphic to the group $A_{i,j-2i}(X)$ which is by definition the $(j-2i)$-th homology group of the complex $C_*(z_{equi}(X,i))$. Consider the Bloch's complex $\mathcal{Z}^{n-i}(X,*)$ which computes the higher Chow groups ([3]). The group $C_k(z_{equi}(X,i))$ is the group of cycles on $X \times \Delta^k$ which are equidimensional of relative dimension i over Δ^k while the group $\mathcal{Z}^{n-i}(X,k)$ consists of cycles of codimension $n-i$ on $X \times \Delta^k$ which itersect all faces of Δ^k properly. One can easily see that since $i \geq 0$ we have an inclusion $C_k(z_{equi}(X,i)) \subset \mathcal{Z}^{n-i}(X,k)$. The differencials in both complexes are defined by the intersection with the faces which implies that for any X we have a monomorphism of complexes of abelian groups of the form:

$$\psi : C_*(z_{equi}(X,i)) \to \mathcal{Z}^{n-i}(X,*)$$

which is clearly canonical with respect to both flat (the contravariant functoriality) and proper (the covariant functoriality) morphisms in X.

Let Z be a closed subscheme in X of pure codimension m. To define the boundary homomorphism in the localization long exact sequences forthe bivariant cycle cohomology and the higher Chow groups one considers the exact sequences of complexes:

$$0 \to C_*(z_{equi}(Z,i)) \to C_*(z_{equi}(X,i)) \to C_*(z_{equi}(X-Z,i)) \to coker_1 \to 0$$

$$0 \to \mathcal{Z}^{n-i-m}(Z,*) \to \mathcal{Z}^{n-i}(X,*) \to \mathcal{Z}^{n-i}(X-Z,*) \to coker_2 \to 0$$

and then shows that both cokernels are quasi-isomorphic to zero (see [7] for $coker_1$ and [4] for $coker_2$). Our inclusion ψ gives a morphism of these exact sequences which implies immediately that the corresponding homomorphisms on (co-)homology groups commute with the boundary homomorphisms.

By [12] the monomorphism ψ is a quasi-isomorphism of complexes for any *affine* scheme X. By induction on $dim(X)$ and the five isomorphisms lemma applied to the localization long exact sequences in both theories it implies that ψ is a quasi-isomorphism for all X.

Suppose now that $i < 0$. In this case the right hand side group is isomorphic to $Hom(\mathbf{Z}[j-2i], \underline{C}^c_*(X)(-i)[-2i])$. By Lemma 4.1.8 we have $\underline{C}^c_*(X)(-i)[-2i]) = \underline{C}^c_*(X \times \mathbf{A}^{-i})$ and by Proposition 4.2.3 we conclude that the group in question is isomorphic to the group $A_{0,j-2i}(X \times \mathbf{A}^{-i})$. By the reasoning given above it is canonically isomorphic to the group $CH^{n-i}(X \times \mathbf{A}^{-i}, j-2i)$. Finally applying the homotopy invariance theorem for higher Chow groups we get the assertion of the proposition for $i < 0$.

4.3 *Duality in the triangulated categories of motives.*

One of the most important for us corollaries of the results of the previous section is the following theorem.

Theorem 4.3.1 *For any objects A, B in $DM^{eff}_{gm}(k)$ the natural map*

$$- \otimes Id_{\mathbf{Z}(1)} : Hom(A,B) \to Hom(A(1), B(1))$$

is an ismorphism. Thus the canonical functor

$$DM_{gm}^{eff}(k) \to DM_{gm}(k)$$

is a full embedding.

Proof: By Corollary 3.5.5 we may assume that $A = M_{gm}(X)[i]$ and $B = M_{gm}(Y)$ for some smooth projective varieties X, Y over k and some $i \in \mathbf{Z}$. By Corollary 4.1.8 we have $\underline{C}_*(Y)(1)[2] = \underline{C}_*^c(Y \times \mathbf{A}^1)$ and therefore the right hand side group is isomorphic by Proposition 4.2.3 to the group $A_{1,i}(X, Y \times \mathbf{A}^1)$. By Theorem 4.2.2(2) it is isomorphic to $A_{0,i}(X, Y)$ which is (again by Proposition 4.2.3) isomorphic to $Hom_{DM_-^{eff}}(\underline{C}_*(X), \underline{C}_*(Y))$. We have shown that there is a canonical isomorphism

$$Hom_{DM_{gm}^{eff}}(M_{gm}(X)[i], M_{gm}(Y)) = Hom_{DM_{gm}^{eff}}(M_{gm}(X)(1)[i], M_{gm}(Y)(1)).$$

The fact that it coincides with the morphism induced by the tensor multiplication with $Id_{\mathbf{Z}(1)}$ follows easily from the explicit form of morphisms which we used to construct it.

We will use Theorem 4.3.1 to indentify DM_{gm}^{eff} with its image in DM_{gm}.

Let us show now that the properties of the bivariant cycle cohomology given by Theorem 4.2.2 imply in particular that $DM_{gm}(k)$ is a "rigid tensor triangulated category".

Theorem 4.3.2 *Let X be a smooth proper scheme of dimension n over k. Consider the morphism*

$$M_{gm}(X) \otimes M_{gm}(X) \to \mathbf{Z}(n)[2n]$$

which corresponds to the diagonal $X \to X \times X$ by Corollary 4.2.5. Then the corresponding morphism

$$M_{gm}(X) \to \underline{Hom}_{DM_-^{eff}}(M_{gm}(X), \mathbf{Z}(n)[2n])$$

is an isomorphism.

Proof: We have to show that for any object A of DM_{gm}^{eff} the morphism

$$Hom(A, M_{gm}(X)) \to Hom(A \otimes M_{gm}(X), \mathbf{Z}(n)[2n])$$

given by the diagonal is an isomorphism. Since these morphisms are natural with respect to A we may assume that $A = M_{gm}(U)[i]$ for a smooth scheme U over k. In this case the statement follows from Theorem 4.2.2(3) and Proposition 4.2.3.

Proposition 4.3.3 *Let X be a scheme of dimension n over k. Then for any $m \geq n$ the morphism*

$$\underline{Hom}_{DM_-^{eff}}(\underline{C}_*(X), \mathbf{Z}(n))(m-n) \to \underline{Hom}_{DM_-^{eff}}(\underline{C}_*(X), \mathbf{Z}(m))$$

is an isomorphism.

Proof: Since our morphisms are natural we may assume using resolution of singularities that X is smooth and projective. Then the statement follows in the standard way from Proposition 4.2.3 and Theorem 4.2.2(3).

Corollary 4.3.4 *For any scheme X of dimension n over k and any $m \geq n$ the object $\underline{Hom}_{DM_-^{eff}}(\underline{C}_*(X), \mathbf{Z}(m))$ belongs to DM_{gm}^{eff}.*

Corollary 4.3.5 *For any pair of objects A, B in $DM_{gm}(k)$ there exists the internal Hom-object $\underline{Hom}_{DM_{gm}}(A, B)$ in DM_{gm}.*

Corollary 4.3.6 *Let X be a variety of dimension n. Then for any $m \geq n$ the object $\underline{Hom}_{DM_{gm}}(M_{gm}(X), \mathbf{Z}(m))$ belongs to $DM_{gm}^{eff}(k)$ and its image in $DM_-^{eff}(k)$ is canonically isomorphic to $\underline{Hom}_{DM_-^{eff}}(\underline{C}_*(X), \mathbf{Z}(m))$.*

Remark: Note that even if the internal Hom-object $\underline{Hom}(\underline{C}_*(X), \underline{C}_*(Y))$ in DM_-^{eff} belongs to DM_{gm}^{eff} its image in DM_{gm} does not in general coincide with $\underline{Hom}_{DM_{gm}}(M_{gm}(X), M_{gm}(Y))$. Consider for example $X = \mathbf{P}^1$ and $Y = Spec(k)$. Then

$$\underline{Hom}_{DM_-^{eff}}(\underline{C}_*(X), \underline{C}_*(Y)) = \mathbf{Z}$$

while

$$\underline{Hom}_{DM_{gm}}(M_{gm}(X), M_{gm}(Y)) = \mathbf{Z} \oplus \mathbf{Z}(-1)[-2].$$

For any object A of $DM_{gm}(k)$ we define A^* as the internal Hom-object $\underline{Hom}_{DM_{gm}}(A, \mathbf{Z})$. The following theorem can be stated by saying that $DM_{gm}(k)$ is a rigid tensor triangulated category.

Theorem 4.3.7 *Let k be a field which admits resolution of singularities. Then one has.*

1. *For any object A in $DM_{gm}(k)$ the canonical morphism $A \to (A^*)^*$ is an isomorphism.*

2. *For any pair of objects A, B of $DM_{gm}(k)$ there are canonical isomorphisms:*
$$(A \otimes B)^* = A^* \otimes B^*$$
$$\underline{Hom}(A, B) = A^* \otimes B$$

3. *For a smooth scheme X of pure dimension n over k one has canonical isomorphisms*
$$M_{gm}(X)^* = M^c_{gm}(X)(-n)[-2n]$$
$$M^c_{gm}(X)^* = M_{gm}(X)(-n)[-2n].$$

Proof: To prove statements (1),(2) we may assume by Corollary 3.5.5 that $A = M_{gm}(X)$, $B = M_{gm}(Y)[i]$ for smooth projective varieties X, Y over k and some $i \in \mathbf{Z}$. Then everything follows easily from Theorem 4.3.2.

Let us show that the second statement holds. Since $(-)^{**} = (-)$ it is sufficient by Corollary 4.3.6 to construct a canonical isomorphism

$$\underline{Hom}_{DM^{eff}_-}(\underline{C}_*(X), \mathbf{Z}(n)[2n]) \to \underline{C}^c_*(X).$$

By Corollaries 4.2.7 and 4.1.8 the left hand side object is canonically isomorphic to $\underline{C}_*(z_{equi}(X, \mathbf{A}^n, 0))$. By [7, Th. 7.4] there is a canonical morphism of sheaves with transfers

$$z_{equi}(X, \mathbf{A}^n, 0) \to z_{equi}(X \times \mathbf{A}^n, n)$$

which induces a quasi-isomorphism on the corresponding complexes $\underline{C}_*(-)$. Finally the flat pull-back morphism

$$z_{equi}(X, 0) \to z_{equi}(X \times \mathbf{A}^n, n)$$

induces a quasi-isomorphism of $\underline{C}_*(-)$ complexes by Theorem 4.2.2(2).

Finally we are going to formulate in this section a simple result which shows that integrally $DM^{eff}_{gm}(k)$ does not have a "reasonable" t-structure. More precisely let us say that a t-structure $\tau = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $DM^{eff}_{gm}(k)$ is reasonable if the following conditions hold:

1. τ is compatible with the Tate twist, i.e. an object M belongs to $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$) if and only if $M(1)$ does.

2. For a smooth affine scheme X of dimesnion n one has:

$$H_i^\tau(M_{gm}(X)) = 0 \text{ for } i < 0 \text{ or } i > n$$

$$H_i^\tau(M_{gm}^c(X)) = 0 \text{ for } i < n \text{ or } i > 2n$$

where H_i^τ are the cohomology objects with respect to τ.

Note that in view of Theorem 4.3.7 the last two conditions are dual to each other.

Proposition 4.3.8 *Let k be a field such that there exists a conic X over k with no k-rational points. Then $DM_{gm}^{eff}(k)$ has no reasonable t-structure.*

Proof: One can easily see that our conditions on τ imply that for any smooth plane curve X we have

$$H_i^\tau(M_{gm}(X)) = \begin{cases} 0 & \text{for} \quad i \neq 0,1,2 \\ \mathbf{Z} & \text{for} \quad i = 0 \\ \mathbf{Z}(1) & \text{for} \quad i = 2. \end{cases}$$

and for a smooth hypersurface Y in $\mathbf{P}^3$ we have $H_1^\tau(M_{gm}(Y)) = 0$.

Let now X be a conic over k with no rational points. The diagonal gives an embedding of $X \times X$ in $\mathbf{P}^3$ and since $M_{gm}(X)$ is clearly a direct summand of $M_{gm}(X \times X)$ for any X we conclude that

$$H_i^\tau(M_{gm}(X)) = \begin{cases} 0 & \text{for} \quad i \neq 0,2 \\ \mathbf{Z} & \text{for} \quad i = 0 \\ \mathbf{Z}(1) & \text{for} \quad i = 2. \end{cases}$$

Therefore we have a distinguished triangle of the form

$$\mathbf{Z}(1)[2] \to M_{gm}(X) \to \mathbf{Z} \to \mathbf{Z}(1)[3].$$

By Corollary 3.4.3 we have:

$$Hom_{DM_{gm}^{eff}}(\mathbf{Z}, \mathbf{Z}(1)[3]) = H^2_{Zar}(Spec(k), \mathbf{G}_m) = 0.$$

Thus our triangle splits and we have an isomorphism $M_{gm}(X) = \mathbf{Z} \oplus \mathbf{Z}(1)[2]$. It implies that the canonical morphism

$$Hom_{DM_{gm}^{eff}}(\mathbf{Z}, M_{gm}(X)) \to \mathbf{Z}$$

is surjective which contradicts our assumption on X since the left hand side group is $A_0(X)$ and X has no zero cycles of degree one.

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