Relative cycles and Chow sheaves.

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Contents

1 Introduction.

Let X be a scheme. A cycle on X is a formal linear combination of points of the Zariski topological space of X. A cycle is called an effective cycle if all points appear in it with non negative coefficients. Suppose that X is a projective scheme over a field k of characteristic zero. Then for any projective embedding $i : X \to \mathbf{P}^n$ the classical construction produces a projective variety $C_r(X,i)$ called the Chow variety of effective cycles of dimension r on X such that k-valued points of $C_r(X,i)$ are in natural bijection with effective cycles of dimesnion r on X. Moreover, for any field extension E/k an E-valued point of $C_r(X,i)$ defines a cycle on $X\times_{Spec(k)} Spec(E)$. In particular for any Noetherian scheme S over k and any morphism $\phi: S\rightarrow C_r(X)$ we get a cycle $\mathcal{Z}_\phi$ on $S\times_{Spec(k)} X$ which lies over generic points of S. For any such cycle $\mathcal{Z}_\phi$ and any morphism $f: S'\rightarrow S$ of Noetherian schemes over k the composition $f\circ\phi$ defines a new cycle $cycl(f)(\mathcal{Z})$ on $X\times_{Spec(k)} S'$. Thus existence of Chow verieties implies that for any Noetherian scheme S over k there is a natural class of cycles on $X\times_{Spec(k)} S$ which are contravariantly functorial with respect to all morphisms $S'\rightarrow S$ of Noetherian schemes over k.

Let now $X\rightarrow S$ be any scheme of finite type over a Noetherian scheme S. In this paper we introduce a class of cycles on X which are called *relative cycles* on X over S. Their most important property is the existence of well defined base change homomorphisms for arbitrary morphisms $S'\rightarrow S$ of Noetherian schemes. In the case when $X = X_k\times_{Spec(k)} S$ where X_k is a projective scheme over a field of characteristic zero and S is a semi-normal Noetherian scheme over k the class of effective relative cycles on X over S coincides with the class of cycles of the form $\mathcal{Z}_\phi$ considered above.

Informally speaking a *relative cycle* on X over S is a cycle on X which lies over generic points of S and has a well defined specialization to any fiber of the projection $X\rightarrow S$. We denote the group of relative cycles of relative dimension r on X over S by $Cycl(X/S,r)$. Unfortunately if S is not a scheme of characterstic zero the specializations $\mathcal{Z}_s$ of a relative cycle $\mathcal{Z}$ on X over S to points s of S do not have in general integral coefficients - the characteristic of the residue field of s may appear in denominators. We denote by $z(X/S,r)$ the subgroup in $Cycl(X/S,r)$ which consists of relative cycles $\mathcal{Z}$ such that for any point s of S the specialization $\mathcal{Z}_s$ of $\mathcal{Z}$ to s has integral coefficients. The groups $z(X/S,r)$ are contravariantly functorial with respect to S, i.e. for any morphism of Noetherian schemes $f: S'\rightarrow S$ there is a homomorphism $cycl(f): z(X/S,r)\rightarrow z(X\times_S S'/S',r)$ and for any composable pair of morphisms f,g one has $cycl(f\circ g) = cycl(g)\circ cycl(f)$. It gives us a presheaf of abelian groups on the category of Noetherian schemes over S which we also denote by $z(X/S,r)$. These *Chow presheaves* are the main objects of our study.

The paper is organized as follows. We start in sections 2.1, 2.2 with some elementary properties of equidimensional morphisms and equidimensional closed subschemes. Besides the standard theorems of Chevalley (Theorem 2.1.1, Proposition 2.1.7(3)) our main technical tool here as well as in the rest of the paper is the "platification theorem" (Theorem 2.2.2).

In section 2.3 we prove some basic results about cycles on Noetherian schemes. All material here is well known and is included only for reader's convenience.

In section 3.1 we introduce the groups $Cycl(X/S, r)$ of relative cycles on a scheme X of finite type over a Noetherian scheme S. Relative cycles over general Noetherian schemes demonstrate all kinds of "pathological" behavior. For instance the group $Cycl(X/S, r)$ is not necessarily generated by the corresponding submonoid $Cycl^{eff}(X/S, r)$ of effective relative cycles (see example 3.4.7) and supports of noneffective relative cycles of relative dimension r over S do not have to be equidimensional of relative dimension r over S (example 3.1.9).

We also define in this section different versions of our main object - the groups of relative cycles with proper support and the groups of relative cycles with equidimensional support.

The main theorem 3.3.1 of section 3.3 says that for any morphism $S' \to S$ of Noetherian schemes there is a well defined base change homomorphism $Cycl(X/S, r) \to Cycl(X \times_S S'/S', r) \otimes \mathbf{Q}$. It gives us a construction of presheaves $Cycl(X/S, r)_{\mathbf{Q}}$ on the category of Noetherian schemes over S such that for any such scheme S' the group $Cycl(X/S, r)_{\mathbf{Q}}(S')$ is the group of relative cycles of dimension r on $X \times_S S'$ over S' with *rational* coefficients.

An example of a relative cycle with integral coefficients whose specialization does not have integral coefficients which is due to A.Merkurjev is given in 3.5.10(1).

In order to obtain a good definition of Chow presheaves with integral coefficients we consider a formal condition on a cycle to be *universally* with integral coefficients. It turns out that the corresponding groups $z(X/S, r)$ are subgroups in the groups $Cycl(X/S, r)$ with torsion quotient for all Noetherian schemes S (Proposition 3.3.14). We call the corresponding presheaves of abelian grous $z(X/S, r)$ the *Chow presheaves* of relative cycles on X over S.

We also define in the similar way the presheaves $c(X/S, d)$ which correspond to proper relative cycles and presheaves $z_{equi}(X/S, r)$, $c_{equi}(X/S, r)$ of relative cycles (resp. proper relative cycles) with equidimensional support.

In section 3.4 we consider relative cycles over geometrically unibranch schemes (in particular over normal schemes). It turns out that over such a scheme S the group of relative cycles with equidimensional support is generated by the cycles of integral closed subschemes of X which are equidimensional over S (Theorem 3.4.2) which implies that in this case our definition coincide with the naive one. On the other hand the corresponding statement does not hold in generally for the groups $z(X/S, r)$ as shown in example 3.5.10(2).

One of the important properties of presheaves $z(X/S, r)$ is that for any regular base scheme S the groups $z(X/S, r)$ coincide with the groups $Cycl(X/S, r)$, i.e. in this case any relative cycle has universally integral coefficients. Our base change homomorphisms over a regular base scheme coincide with base change homomorphisms which one can define by means of the Tor-formula (see the end of the section 3.5). We also show in section 3.5 that our base change homomorphisms coincide in the case of finite cycles over normal schemes with base change homomorphisms defined in [16].

In section 3.6 we study functoriality of Chow presheaves $z(X/S, r)$ and $c(X/S, r)$ with respect to X. We show in particular that for any morphism (resp. any proper morphism) $p : X \to Y$ of schemes of finite type over S there is a push-forward homomorphism of presheaves $p_* : c(X/S, r) \to c(Y/S, r)$ (resp. $p_* : z(X/S, r) \to z(Y/S, r)$) and for any flat (resp. flat and proper) equidimensional morphism $f : X \to Y$ of relative dimension n there is a pull-back homomorphism of presheaves $f^* : z(Y/S, r) \to z(X/S, r+n)$ (resp. $f^* : c(Y/S, r) \to c(X/S, r+n)$).

In the next section we define for Chow presheaves the *correspondence homomorphisms* which were considered in context of Chow varieties by Eric M. Friedlander. In particular we show that there is a well defined homomorphism of external product

$$z(X/S, r_1) \otimes z(Y/S, r_2) \to z(X \times_S Y/S, r_1 + r_2)$$

for any schemes of finite type X, Y over S.

In the last section we consider Chow presheaves as sheaves in the h-topologies. In particular we construct in section 4.3 some exact sequences of Chow sheaves which are important for localization-type theorems in algebraic cycle homology and Suslin homology (see [17]). Finally in section 4.4 we study representability of Chow sheaves. One can easily show that except for some trivial cases the Chow presheaves are not representable as presheaves.

To avoid this difficulty we introduce a notion of the h-representability. We construct then the Chow scheme $C_{r,d}$ of cycles of degree d and dimension r on $\mathbf{P}^n$ over $Spec(\mathbf{Z})$ using essentially the classical construction due to Chow (see also [14]) and show that it h-represents the Chow sheaf $z_d^{eff}(\mathbf{P}^n/Spec(\mathbf{Z}), \mathrm{r})$ of effective relative cycles of relative dimension r and degree d on $\mathbf{P}^n$. A rather formal reasoning shows then that for any quasi-projective scheme X over a Noetherian scheme S and any $r \geq 0$ the Chow sheaf $c(X/S, r)$ is h-representable by disjoint union of quasi-projective schemes over S. As an application of this representability result we show that for a quasi-projective scheme X over a Noetherian scheme S the group $z(X/S, r)_h$ of sections of the h-sheaf associated with the Chow presheaf $z(X/S, r)$ can be described using the notion of " continuous algebraic maps" introduced by Eric M. Friedlander and O. Gabber ([3]) which generalizes a similar result obtained in [2] for quasi-projective schemes over a field.

Everywhere in this text a scheme means a separated scheme.

2 Generalities.

2.1 *Universally equidimensional morphisms.*

For a scheme X we denote by $dim(X)$ the dimension of the Zariski topological space of X. By definition $dim(X)$ is either a positive integer of infinity. If $x \in X$ is a point of a locally Noetherian scheme X we denote by $dim_x(X)$ the limit $\lim dim(U)$ taken over the partially ordered set of open neighborhoods of x in X. One can easily see that it is well defined and equals $dim(U)$ for a sufficiently small U (see [8, Ch.0,14.4.1]).

For a morphism $p : X \to S$ denote by $dim(X/S)$ the function on the set of points of X of the form $dim(X/S)(x) = dim_x(p^{-1}(p(x)))$. The most important property of the dimension functions is given by the following well known theorem.

Theorem 2.1.1 (Chevalley) *Let $p : X \to S$ be a morphism of finite type. Then for any $n \geq 0$ the subset $\{x \in X : dim_{X/S}(x) \geq n\}$ is closed in X.*

Proof: See [8, Th. 13.1.3].

Definition 2.1.2 *A morphism of schemes $p : X \to S$ is called an equidimensional morphism of dimension r if the following conditions hold:*

1. *p is a morphism of finite type.*
2. *The function $dim(X/S)$ is constant and equals r.*
3. *Any irreducible component of X dominates an irreducible component of S.*

A morphism of schemes $p : X \to S$ is called universally equidimensional of dimension r if for any morphism $S' \to S$ the projection $X \times_S S' \to S'$ is equidimensional of dimension r.

Finally, we say that $p : X \to S$ is a morphism of dimension $\leq r$ if $dim(X/S)(x) \leq r$ for all points x of X.

One can easily see that in the definition of equidimensional morphism given above one can replace the condition $dim(X/S) = r$ by the condition that for any point y of S the dimension of all irreducible components of the topological space $p^{-1}(p(x))$ equals r.

Proposition 2.1.3 *Let $p : X \to S$ be a morphism of finite type of Noetherian schemes. Then p is equidimensional of dimension r if and only if for any point x of X there is an open neighborhood U in X and a factorization of the morphism $p_U : U \to S$ of the form $U \stackrel{p_0}{\to} \mathbf{A}^r_S \to S$ such that p_0 is a quasi-finite morphism and any irreducible component of U dominates an irreducible component of $\mathbf{A}^r_S$.*

Proof: See [8, 13.3.1(b)].

Definition 2.1.4 *A morphism of schemes $p : X \to S$ is called an open morphism if for any open subset U of X the subset $p(U)$ is open in S. It is called universally open if for any morphism $S' \to S$ the projection $X \times_S S' \to S'$ is an open morphism.*

Definition 2.1.5 *Let S be a Noetherian scheme. It is called unibranch (resp. geometrically unibranch) if for any point s of S the scheme $Spec(\mathcal{O}^h_{s,S})$ where $\mathcal{O}^h_{s,S}$ is the henselization of the local ring of s in S (resp. the scheme $Spec(\mathcal{O}^{sh}_{s,S})$ where $\mathcal{O}^{sh}_{s,S}$ is the strict henselization of the local ring of s in S) is irreducible.*

Remark: Our definition is consistent with the one given in [8, 6.15.1] in view of [8, 18.8.15].

Proposition 2.1.6 *Let S be a Noetherian geometrically unibranch scheme and $f : S' \to S$ be a proper birational morphism. Then for any point s of S the fiber S'_s of f over s is geometrically connected.*

Proof: It follows from [7, 4.3.5] and [8, 18.8.15(c)].

Proposition 2.1.7 *Let $p : X \to S$ be a morphism of finite type of Noetherian schemes. Then the following implications hold:*

1. *If p is a universally equidimensional morphism then p is universally open.*

2. *If $dim(X/S) = r$ and p is open (resp. universally open) then p is equidimensional (resp. universally equidimensional) of dimension r.*

3. *If S is geometrically unibranch and p is equidimensional then p is universally equidimensional (and hence universally open).*

Proof: (1) It follows from [8, 14.4.8.1 and 14.4.4].
(2) Obvious.
(3) It is known that any equidimensional morphism over a geometrically unibranch scheme is universally open (see [8, 14.4.4]). It implies immediately that for any morphism $S' \to S$ the projection $p' : X \times_S S' \to S'$ satisfies the condition (3) of Defenition 2.1.2. Since the conditions (1) and (2) are obviously stable under a base change the morphism p' is equidimensional.

Remarks:

1. Note that an equidimensional morphism does not have to be open.

2. One can easily see that the inverse statement to the third part of this proposition holds. Namely a Noetherian scheme X is geometrically unibranch if any equidimensional morphism over X is universally equidimensional.

3. Let $p : X \to S$ be a morphism of finite type such that $S = Spec(k)$ where k is a field. Proposition 2.1.7(3) implies that p is universally equidimensional of dimension r if and only if all irreducible components of X have dimension r.

Proposition 2.1.8 *Let $p : X \to S$ be a flat morphism of finite type. Then p is universally equidimensional of dimension r if and only if for any generic point $y : Spec(K) \to S$ of S the projection $X \times_S Spec(K) \to Spec(K)$ is equidimensional of dimension r.*

Proof: Since any flat morphism of finite type is universally open ([11, I.2.12]) it is sufficient to verify that under the conditions of the proposition we have $dim(X/S) = r$. It follows immediately from [8, 12.1.1(iv)] and Theorem 2.1.1.

Proposition 2.1.9 *Let $p : X \to S$ be an equidimensional morphism of relative dimension r such that X is irreducible. Suppose that p admits a decomposition of the form $X \stackrel{p_0}{\to} W \stackrel{p_1}{\to} S$ such that p_0 is surjective and proper and p_1 has at least one fiber of dimension r. Then p_1 is equidimensional of dimension r and p_0 is finite in the generic point of W.*

Proof: It follows easily from Theorem 2.1.1.

Lemma 2.1.10 *Let $p : X \to S$ be a morphism such that any irreducible component of X dominates an irreducible component of S and $i : X_0 \to X$ be a closed embedding which is an isomorphism over the generic points of S. Then i is defined by a nilpotent sheaf of ideals. In particular, p is a universally equidimensional morphism of dimension r if and only if $p_0 : X_0 \to S$ is a universally equidimensional morphism of dimension r.*

Lemma 2.1.11 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $Z \subset X$ be a closed subscheme universally equidimensional of relative dimension r over S and $S' \to S$ be a blow-up of S. Let $\tilde{Z}$ be the proper transform of Z in $X \times_S S'$. Then $\tilde{Z}$ is a closed subscheme of $Z \times_S S'$ defined by a nilpotent sheaf of ideals.*

Proof: Since $Z \times_S S'$ is equidimensional over S' and hence its generic points lie over generic points of S' our statement follows from Lemma 2.1.10.

2.2 *Universally equidimensional closed subschemes.*

Let $p : X \to S$ be a morphism of finite type. We denote by $Z_i(X/S)$ (resp. by $Z_{\leq i}(X/S)$) the set of closed reduced subschemes Z in X such that the morphism $p_{|Z} : Z \to S$ is universally equidimensional of relative dimension i (resp. of dimension $\leq i$).

For any morphism $f : S' \to S$ and any reduced closed subscheme Z of X we denote by $f^{-1}(Z)$ the maximal reduced subscheme of the closed subscheme $Z' = Z \times_S S'$ of $X' = X \times_S S'$. One can easily see that if Z is an element of $Z_{\leq i}(X/S)$ (resp. an element of $Z_i(X/S)$) then $f^{-1}(Z)$ is an element of $Z_{\leq i}(X'/S')$ (resp. an element of $Z_i(X'/S')$).

Lemma 2.2.1 *Let $p : X \to S$ be a morphism of schemes and Z_1, Z_2 be a pair of elements in $Z_i(X/S)$ (resp. in $Z_{\leq i}(X/S)$). Then $Z_1 \cup Z_2$ is an element of $Z_i(X/S)$ (resp. of $Z_{\leq i}(X/S)$).*

Proof: Obvious.

Theorem 2.2.2 *Let $p : S' \to S$ be a morphism of Noetherian schemes and U be an open subscheme in S such that p is flat over U. Then there exists* *a closed subscheme Z in S such that $U \cap Z = \emptyset$ and the proper transform of S' with respect to the blow-up $S_Z \to S$ of S with center in Z is flat over S_Z.*

Proof: See [13, 5.2].

Theorem 2.2.3 *Let $p : X \to S$ be a morphism of finite type, U be an open subscheme of X, Z be an element of $Z_i(U/S)$ and V be an open subscheme of S such that the closure $\bar{Z}$ of Z in X is flat over V. Then there exists a blow-up $f : S' \to S$ of S such that the closure of $f^{-1}(Z)$ in $X' = X \times_S S'$ belongs to $Z_i(X'/S')$ and the morphism $f^{-1}(V) \to V$ is an isomorphism.*

Proof: By Theorem 2.2.2 there is a blow-up $f : S' \to S$ such that the proper transform $\tilde{Z}$ of $\bar{Z}$ with respect to f is flat over S'. One can easily see now that the closure $\bar{Z}'$ of $Z' = Z \times_S S'$ in $X' = X \times_S S'$ is a closed subscheme in $\tilde{Z}$ and the corresponding closed embedding is an isomorphism over the generic points of S'. Therefore, $\bar{Z}'$ is universally equidimensional over S' by Lemma 2.1.10.

Definition 2.2.4 *Let $f : S' \to S$ be a morphism of Noetherian schemes. We say that f is an* abstract blow-up *of S if the morphism f is proper, any irreducible component of S' dominates an irreducible component of S and there exists a dense open subscheme U in S such that the morphism $(f^{-1}(U)_{red}) \to U_{red}$ is an isomorphism (here X_{red} denote the maximal reduced subscheme of X).*

Note that any abstract blow-up in the sense of Defenition 2.2.4 is surjective.

The following lemma lists some trivial properties of abstract blow-ups.

Lemma 2.2.5

1. *A composition of abstract blow-ups is an abstract blow-up.*
2. *If S is a Noetherian scheme and $S' \to S$, $S'' \to S$ is a pair of abstract blow-ups of S then the morphism $S' \times_S S'' \to S$ is an abstract blow-up of S.*

Corollary 2.2.6 *Let $p : X \to S$ be a morphism of finite type and U be an open subscheme of X, Z be an element of $Z_i(U/S)$. Then there exists an abstract blow-up $f : S' \to S$ of S such that the closure of $f^{-1}(Z)$ in $X' = X \times_S S'$ belongs to $Z_i(X'/S')$*

Proof: It is sufficient to note that we may replace S by the disjoint union of its reduced irreducible components and that in the case of integral S there is an open nonempty subset V in S such that the closure $\bar{Z}$ is flat over V.

Corollary 2.2.7 *Let $p : X \to S$ be a universally eqidimensional quasi-projective morphism of dimension r. Then there is an abstract blow-up $S' \to S$, a universally equidimensional projective morphism $\bar{X}' \to S'$ of dimension r and an open embedding $i : X \times_S S' \to \bar{X}'$ over S'.*

Proof: Obvious.

Theorem 2.2.8 *Let $p : X \to S$ be a universally equidimensional morphism and Z be an element in $Z_{\leq i}(X/S)$. Then there exists an abstract blow-up $f : S' \to S$ of S and an element W in $Z_i(X \times_S S'/S')$ such that $f^{-1}(Z)$ is contained in W.*

Proof: We start with the following lemma.

Lemma 2.2.9 *Let S be a local Noetherian scheme and $p : X \to S$ be an affine flat equidimensional morphism of dimension r. Let further Z be a closed subscheme of the closed fiber of X over S which does not contain any of the generic points of this fiber.*

Then there exists a closed subscheme W of X which is flat and equidimensional of dimension $r - 1$ over S such that Z lies in W.

Proof: There exists a finite set $\{x_1, \ldots, x_k\}$ of closed points of X such that the following conditions hold:

1. $\{x_1, \ldots, x_k\} \cap Z = \emptyset$.
2. Any irreducible component of X contains at least one of the points $x_1, \ldots, x_k$.
3. Any irreducible component of the closed fiber X_s of X contains al least one of the points $x_1, \ldots, x_k$.

Since X is affine there is a regular function f on X such that $f = 0$ on Z and $f = 1$ on the set $\{x_1, \ldots, x_k\}$. The third property of this set implies that the divisor $W = (f)$ of f is flat over S (see [11, I.2.5]) and the second one that this divisor is equidimensional of dimension $r-1$ over generic points of S. By Proposition 2.1.8 we conclude that W is equidimensional of dimension $r-1$ over S. Lemma is proven.

To prove our theorem it is obviously sufficient to show that if Z is an integral closed subscheme of X of dimension $\leq i$ over S where $i < r$ then there exist an abstract blow-up $S' \to S$ and an element $W \in Z_{r-1}(X'/S')$ such that Z' is contained in W.

Assume first that X is affine over S. By Theorem 2.2.2 it is sufficient to consider the case of a flat morphism $p : X \to S$. Let z be the generic point of Z and $s = p(z)$. Consider the fiber X_s of X over s. Since Z_s is of codimension at least one in X_s by Lemma 2.2.9 there is an open subscheme U of S which contains s and a closed subscheme W_U in $p^{-1}(U)$ which is flat and equidimensional of dimension $r-1$ over U and which contains $Z \cap p^{-1}(U)$. By Theorem 2.2.3 there is an abstract blow-up $f : S' \to S$ such that $f^{-1}(U) \to U$ is an isomorphism and the closure $\bar{W}'$ of $W' = W_U \times_S S'$ in $X' = X \times_S S'$ belongs to $Z_{r-1}(X'/S')$.

In particular the closure of the pre-image in X' of the generic point of Z is contained in $\bar{W}'$. Let Q be the complement to U in S and Z' be the intesection $Z \cap p^{-1}(Q)$. By our construction we have $dim(Z') < dim(Z)$. Using induction on $dim(Z)$ we may assume that there is an abstract blow-up $S'' \to S$ such that the $Z'' = Z \times_S S''$ is contained in an element W_1 of $Z_{d-1}(X''/S'')$. Considering an abstract blow-up $S''' \to S$ which dominates both S' and S'' (Lemma 2.2.5) we conclude that the statement of our theorem is correct for affine morphisms $p : X \to S$.

Let now $p : X \to S$ be an arbitrary universally equidimensional morphism of dimension r. Then there is a finite open covering $X = \cup U_i$ such that the morphisms $p_i : U_i \to S$ are affine. Since our statement holds for affine morphisms there is an abstract blow-up $S' \to S$ and a family of elements $W_i \in Z_{d-1}(U_i'/S')$ such that $Z' \cap U_i'$ is contained in W_i. By Theorem 2.2.3 we may choose S' such that the closure $\bar{W}_i$ of each W_i in X' is universally equidimensional of dimension $r-1$. It implies the result we need since one obviously has $Z' \subset \cap \bar{W}_i$. Theorem is proven.

2.3 *Cycles on Noetherian schemes.*

Let X be a Noetherian scheme. We denote by $Cycl(X)$ (resp. by $Cycl^{eff}(X)$) the free abelian group (resp. the free abelian monoid) generated by points of the Zariski topological space of X.

For any element $\mathcal{Z}$ of $Cycl(X)$ we denote by $supp(\mathcal{Z})$ the closure of the set of points on X which appear in $\mathcal{Z}$ with nonzero coefficients. We consider $supp(\mathcal{Z})$ as a reduced closed subscheme of X.

Let Z be a closed subscheme of X and ζ_i, $i = 1, \ldots, k$ be the generic points of the irreducible components of Z. We define an element $Cycl_X(Z)$ of the abelian monoid $Cycl^{eff}(X)$ as the formal linear combination of the form

$$cycl_X(Z) = \sum_{i=1}^{k} m_i \zeta_i$$

where $m_i = length(\mathcal{O}_{Z,\zeta_i})$. Each number m_i is a positive integer which is called the multiplicity of Z in the point ζ_i.

This construction gives us a map from the set of closed subschemes of X to the abelian monoid $Cycl^{eff}(X)$ which can be canonically extended to a homomorphism from the free abelian monoid generated by this set to $Cycl^{eff}(X)$. We denote this homomorphism by $cycl_X$.

Let $p : X \to S$ be a flat morphism of Noetherian schemes and let $\mathcal{Z} = \sum n_i z_i$ be a cycle on S. Denote by Z_i the closure of the point z_i which we consider as a closed integral subscheme in S and set $p^*(\mathcal{Z}) = \sum n_i cycl_X(Z_i \times_S X)$. In this way we get a homomorphism (flat pull-back) $p^* : Cycl(S) \to Cycl(X)$. The following lemma is straightforward (cf. [4, Lemma 1.7.1]).

Lemma 2.3.1 *1. If Z is any closed subscheme of S then $p^*(cycl_S(Z)) = cycl_X(Z \times_S X)$.*

2. $supp(p^*(\mathcal{Z})) = (p^{-1}(supp(\mathcal{Z})))_{red}$. In particular the homomorphism $p^* : Cycl(S) \to Cycl(X)$ is injective provided that p is surjective.

Let $X \to Spec(k)$ be a scheme of finite type over a field k and let L/k be any field extension. The corresponding morphism $p : X_L \to X$ is flat and hence defines a homomorphism $p^* : Cycl(X) \to Cycl(X_L)$. The image of a cycle $\mathcal{Z} \in Cycl(X)$ under this homomorphism will be usually denoted by $\mathcal{Z} \otimes_k L$ of $\mathcal{Z}_L$.

Lemma 2.3.2 *Let $X \to Spec(k)$ be a scheme of finite type over a field k and let k'/k be a finite normal field extension with the Galois group G. If $\mathcal{Z}' \in Cycl(X_{k'})^G$ then there is a unique cycle $\mathcal{Z} \in Cycl(X)$ such that $[k' : k]_{insep}\mathcal{Z}' = \mathcal{Z}_{k'}$.*

Proof: The uniqueness of $\mathcal{Z}$ follows immediately from Lemma 2.3.1(2). To prove the existence note that the group $Cycl(X_{k'})^G$ is generated by cycles of the form $\mathcal{Z}' = \sum_{\tau \in G/H} \tau(z')$ where z' is a point of $X_{k'}$ and $H = Stab_G(z')$. Let z be the image of z' in X and let Z be the closure of z which we consider as a closed integral subscheme of X. The points $\tau(z')$ are precisely the generic points of the scheme $Z' = Z \times_{Spec(k)} Spec(k')$ and the multiplicities with which they appear in the cycle $\mathcal{Z}_{k'}$ are all equal to the length of the local Artinian ring $\mathcal{O}_{Z',z'}$. The elementary Galois theory shows that this length is a factor of $[k' : k]_{insep}$. Thus the cycle $\mathcal{Z} = [k' : k]_{insep} z / length(\mathcal{O}_{Z',z'})$ has the required property.

Corollary 2.3.3 *In the assumptions and notations of the previous lemma denote by p the exponential characteristic of the field k. Then the homomorphism*

$$Cycl(X)[1/p] \to (Cycl(X_{k'})[1/p])^G$$

is an isomorphism.

Let $X \to Spec(k)$ be a scheme of finite type over a field k. Then we have a direct sum decomposition $Cycl(X) = \coprod Cycl(X, r)$ where $Cycl(X, r)$ is a subgroup of $Cycl(X)$ generated by points of dimension r. Furthermore one sees easily that for a field extension k'/k the homomorphism

$$Cycl(X) \to Cycl(X_{k'})$$

preserves this decomposition.

Let S be a Noetherian scheme and $p : X \to S$ be a proper morphism of finite type. For any cycle $\mathcal{Z} = \sum n_i z_i \in Cycl(X)$ set

$$p_*(\mathcal{Z}) = \sum n_i m_i p(z_i)$$

where m_i is the degree of the field extension $k_{z_i}/k_{p(z_i)}$ if this extension is finite and zero otherwise. The proof of the following statement is similar to that of Proposition 1.7 of [4] and we omit it.

Proposition 2.3.4 *Consider a pull-back square of morphisms of finite type of Noetherian schemes*

$$\begin{array}{ccc} \tilde{X} & \stackrel{f_X}{\to} & X \\ \tilde{p}\downarrow & & \downarrow p \\ \tilde{S} & \stackrel{f}{\to} & S \end{array}$$

in which f is flat and p is proper. Then for any cycle $\mathcal{Z} \in Cycl(X)$ we have

$$f^*(p_*(\mathcal{Z})) = \tilde{p}_*(f_X^*(\mathcal{Z})).$$

The following lemma is straighforward.

Lemma 2.3.5 *Let $f : X \to Y$ be a finite flat morphism of connected Noetherian schemes. Denote by $deg(f)$ the degree of f. Then $f_*f^* = deg(f)Id_{Cycl(Y)}$.*

3 Relative cycles.

3.1 *Relative cycles.*

Definition 3.1.1 *Let S be a Noetherian scheme, k be a field and $x : Spec(k) \to S$ be a k-point of S. A fat point over x is a triple (x_0, x_1, R), where R is a discrete valuation ring and $x_0 : Spec(k) \to Spec(R)$, $x_1 : Spec(R) \to S$ are morphisms such that*

1. $x = x_1 \circ x_0$
2. *The image of x_0 is the closed point of $Spec(R)$.*
3. *x_1 takes the generic point of $Spec(R)$ to a generic point of S.*

Usually we will abbreviate the notation (x_0, x_1, R) to (x_0, x_1).

Lemma 3.1.2 *Let S be a Noetherian scheme, $X \to S$ be a scheme over S and Z be a closed subscheme in X. Let further R be a discrete valuation ring and $f : Spec(R) \to S$ be a morphism. Then there exists a unique closed subscheme $\phi_f(Z)$ in $Z \times_S Spec(R)$ such that:*

1. *The closed embedding $\phi_f(Z) \to Z \times_S Spec(R)$ is an isomorphism over the generic point of $Spec(R)$.*
2. *$\phi_f(Z)$ is flat over $Spec(R)$.*

Proof: See [8, 2.8.5].

Let S be a Noetheiran scheme, $X \to S$ be a scheme of finite type over S and Z be a closed subscheme of X. For any fat point (x_0, x_1) over a k-point x of S we denote by $(x_0, x_1)^*(Z/S)$ the cycle on $X \times_{Spec(k)} S$ associated with the closed subscheme $\phi_{x_1}(Z) \times_{Spec(R)} Spec(k)$.

If $\mathcal{Z} = \sum m_i z_i$ is any cycle on X we denote by $(x_0, x_1)^*(\mathcal{Z})$ the cycle $\sum m_i (x_0, x_1)^*(Z_i)$ where Z_i is the closure of the point z_i (considered as a reduced closed subscheme of X).

Definition 3.1.3 *Let S be a Noetherian scheme and $X \to S$ be a scheme of finite type over S. A relative cycle on X over S is a cycle $\mathcal{Z} = \sum m_i z_i$ on X satisfying the following requirements:*

1. *The points z_i lie over generic points of S.*

2. *For any field k, k-point x of S and a pair of fat points (x_0, x_1), (y_0, y_1) of S over x one has:*

$$(x_0, x_1)^*(\mathcal{Z}) = (y_0, y_1)^*(\mathcal{Z}).$$

We say that $\mathcal{Z} = \sum n_i z_i$ is a relative cycle of dimension r if each point z_i has dimension r in its fiber over S. We denote the corresponding abelian groups by $Cycl(X/S, r)$.

We say that $\mathcal{Z}$ is an equidimensional relative cycle of dimension r if $supp(\mathcal{Z})$ is equidimensional of dimesnion r over S. We denote the corresponding abelian groups by $Cycl_{equi}(X/S, r)$.

We say that $\mathcal{Z}$ is a proper relative cycle if $supp(\mathcal{Z})$ is proper over S. We denote the corresponding abelian groups by $PropCycl(X/S, r)$ and $PropCycl_{equi}(X/S, r)$.

We will also use the notations $Cycl^{eff}(X/S, r)$, $PropCycl^{eff}(X/S, r)$ etc. for the corresponding abelian monoids of effective relative cycles.

The following lemma gives us means to construct fat points.

Lemma 3.1.4 *Let S be a Noetherian scheme, η be a generic point of S and s be a point in the closure of η. Let further L be an extension of finite type of the field of functions on S in η. Then there is a discrete valuation ring R and a morphism $f : Spec(R) \to S$ such that the following conditions hold:*

1. *f maps the generic point of $Spec(R)$ to η and the field of functions of R is isomorphic to L over k_η,*

2. *f maps the closed point of $Spec(R)$ to s.*

Proof: See [6, 7.1.7].

Let S be a Noetherian scheme, $X \to S$ be a scheme of finite type over S and $\mathcal{Z} = \sum n_i z_i$ be a cycle on X such that the points z_i lie over generic points of S and are of dimension r in the corresponding fibers. Let Z_i denote the closure of z_i considered as a closed integral subscheme of X. It is clear from the above definition that $\mathcal{Z} \in Cycl(X/S, r)$ if and only if $\mathcal{Z} \in Cycl(X_{red}/S_{red}, r)$. Furthemore the schemes Z_i are flat over generic points of S_{red} and according to Theorem 2.2.2 one can find a blow-up $S' \to S_{red}$ such that the proper transforms $\tilde{Z}_i$ of Z_i's are flat over S'. Now we can formulate the following usefull criterion.

Proposition 3.1.5 *Under the above assumptions the following conditions are equivalent:*

1. $\mathcal{Z} \in Cycl(X/S, r)$.
2. *If* $x : Spec(k) \to S$ *is any geometric point of* S *and* $x'_1, x'_2 : Spec(k) \to S'$ *is a pair of its liftings to* S' *then the cycles* $\mathcal{W}_1$, $\mathcal{W}_2$ *on* $X_x = X \times_S Spec(k)$ *given by the formulae*

$$\mathcal{W}_1 = \sum_{i=1}^{k} n_i cycl_{X_s}(\tilde{Z}_i \times_{x'_1} Spec(k))$$

$$\mathcal{W}_2 = \sum_{i=1}^{k} n_i cycl_{X_s}(\tilde{Z}_i \times_{x'_2} Spec(k))$$

coincide.

Proof: **(1=>2)** the geometric points x, x'_1, x'_2 give us set-theoretical points $s \in S$, $s'_1, s'_2 \in S'$ such that s'_1, s'_2 lie over s. We may assume that s (and hence also s'_1, s'_2) is not generic. Using Lemma 3.1.4 we construct discrete valuation rings R'_i and morphisms $Spec(R'_i) \to S'$ which map the closed point of $Spec(R'_i)$ to s'_i and the generic point of $Spec(R'_i)$ to a generic point of S'.

Denote the residue fields of R'_i by k'_i. One checks easily that the scheme $(Spec(k'_1) \times_S Spec(k'_2)) \times_{S' \times_S S'} Spec(k)$ is not empty. Choosing any geometric L-point of this scheme for a field L we get a commutative diagram

$$\begin{array}{ccccccc} & & Spec(k'_1) & \to & Spec(R'_1) & \to & S' & & \\ & \nearrow & & & & \overset{x'_1}{\nearrow} & & \searrow & \\ Spec(L) & & \to & & Spec(k) & & \xrightarrow{x} & & S \\ & \searrow & & & & \underset{x'_2}{\searrow} & & \nearrow & \\ & & Spec(k'_2) & \to & Spec(R'_2) & \to & S' & & \end{array}$$

Thus we get a geometric point $Spec(L) \to S$ and two fat points $Spec(L) \to Spec(R'_i) \to S$ over it. The inverse images of the cycle $\mathcal{Z}$ with respect to these fat points are equal to $\sum n_i cycl[(\tilde{Z}_i \times_{x'_1} Spec(k)) \times_{Spec(k)} Spec(L)]$ and $\sum n_i cycl[(\tilde{Z}_i \times_{x'_2} Spec(k)) \times_{Spec(k)} Spec(L)]$ respectively. These cycles coincide according to the condition $\mathcal{Z} \in Cycl(X/S, r)$. Lemma 2.3.1 shows now that $\mathcal{W}_1 = \mathcal{W}_2$.

(2=>1) Let $x : Spec(k) \to S$ be a geometric point of S and let (x_0, x_1), (y_0, y_1) be a pair of fat points over x. According to the valuative criterion of properness (see [6, 7.3]) these fat points have canonical liftings to fat points $(x'_0, x'_1), (y'_0, y'_1)$ of S'. This gives us two geometric points

$$x' = x'_1 \circ x'_0 : Spec(k) \to S'$$

$$y' = y'_1 \circ y'_0 : Spec(k) \to S'$$

of S' over x. Our statement follows now from obvious equalities:

$$(x_0, x_1)^*(\mathcal{Z}) = (x_0, x'_1)^*(\sum n_i \tilde{Z}_i) = \sum n_i cycl(\tilde{Z}_i \times_{x'} Spec(k))$$

$$(y_0, y_1)^*(\mathcal{Z}) = (y_0, y'_1)^*(\sum n_i \tilde{Z}_i) = \sum n_i cycl(\tilde{Z}_i \times_{y'} Spec(k)).$$

Corollary 3.1.6 *Let k be a field and $X \to Spec(k)$ be a scheme of finite type over k. Then the group $Cycl(X/Spec(k), r)$ is the free abelian group generated by points of dimension r on X, i.e one has*

$$Cycl(X/Spec(k), r) = Cycl_{equi}(X/Spec(k), r) = Cycl(X, r).$$

Proposition 3.1.7 *Let S be a Noetherian scheme, $X \to S$ be a scheme of finite type over S and $\mathcal{Z} = \sum_{i=1}^{k} n_i z_i$ be an effective cycle on X which belongs to $Cycl(X/S, r)$ for some $r \geq 0$. Denote by Z_i the closure of the point z_i in X which we consider as an integral closed subscheme in X. Then Z_i is equidimensional of dimension r over S.*

Proof: According to the Chevalley theorem 2.1.1 all components of all fibers of the projection $Z_i \to S$ are of dimension $\geq r$. Assume that there exists a point s of S such that the fiber $(Z_0)_s$ of Z_0 over s has a component of dimension $> r$. Let η be the generic point of this component.

By Theorem 2.2.2 there is a blow-up $f : S' \to S_{red}$ of S_{red} such that the proper transforms $\tilde{Z}_i$ of the subschemes Z_i with respect to f are flat (and hence equidimensional of dimension r over S'). The morphism $\tilde{Z}_0 \to Z_0$ is proper and dominant and hence surjective. Let τ be any point of $\tilde{Z}_0$ over η and let s'_1 be its image in S'. On the other hand let s'_2 be any closed point of the fiber $(S')_s$. Choosing an algebraically closed field k which contains a composite of the fields $k_{s'_1}$ and $k_{s'_2}$ over k_s we get two geometric points $x'_1, x'_2 : Spec(k) \to S'$ over the same geometric point $x : Spec(k) \to S$. Consider the following cycles on $X_k = X \times_S Spec(k)$:

$$\mathcal{W}_1 = \sum n_i cycl_{X_k}(\tilde{Z}_i \times_{s'_1} Spec(k))$$

$$\mathcal{W}_2 = \sum n_i cycl_{X_k}(\tilde{Z}_i \times_{s'_2} Spec(k)).$$

In view of Proposition 3.1.5 it is sufficient to show that these cycles are different. We will do so by showing that the images of $supp(\mathcal{W}_1)$ and $supp(\mathcal{W}_2)$ in X_s are different. The image of $supp(\mathcal{W}_2)$ coincides with the image of $\cup(\tilde{Z}_i)_{s'_2} \subset X_{k_{s'_2}}$ in X_s and hence is of dimension $\leq r$ (since the morphism $X_{s'_2} \to X_s$ is finite). On the other hand the image of $supp(\mathcal{W}_1)$ contains η and hence is of dimension $> r$.

Corollary 3.1.8 *Let S be a Noetherian scheme and $X \to S$ be a scheme of finite type over S. Then one has:*

$$Cycl^{eff}_{equi}(X/S, r) = Cycl^{eff}(X/S, r)$$

$$PropCycl^{eff}_{equi}(X/S, r) = PropCycl^{eff}(X/S, r).$$

Example 3.1.9 Proposition 3.1.7 fails for cycles which are not effective. Consider the scheme $X = \mathbf{P}^1_k \times \mathbf{A}^2$ over $S = \mathbf{A}^2_k$ where k is a field. Consider the following two rational functions on S:

$$f(x, y) = y/x$$

$$g(x, y) = y/(x + y^2).$$

Let $\Gamma_f, \Gamma_g \in X$ be the graphs of these functions and let $\mathcal{Z}$ be the cycle on X of the form $\Gamma_f - \Gamma_g$. Obviously $supp(\mathcal{Z})$ is not equidimensional over S. We claim that $\mathcal{Z}$ nevertheless belongs to $Cycl(X/S, 0)$.

Let $S' \to S$ be the blow-up of the point $(0, 0)$ in $\mathbf{A}^2$. Denote by f', g' the rational functions on S' which correspond to f and g. One can easily see that f', g' are in fact regular on S' and moreover if $S'_0 \subset S'$ denotes the exceptional divisor of S' we have $f'_{S'_0} = g'_{S'_0}$. Since the proper transforms of the closed subschemes Γ_f, Γ_g are the graphs of f' and g' in $S' \times_S X = S' \times \mathbf{P}^1_k$ Proposition 3.1.5 shows that $\mathcal{Z}$ indeed belongs to $Cycl(X/S, 0)$.

3.2 *Cycles associated with flat subschemes.*

Let $p : X \to S$ be a morphism of finite type of Noetherian schemes. We denote by $Hilb(X/S, r)$ (resp. by $PropHilb(X/S, r)$) the set of closed subschemes Z of $X \times_S S$ which are flat (resp. flat and proper) and equidimensional of dimension r over S.

Let $\mathbf{N}(Hilb(X/S, r))$, $\mathbf{N}(PropHilb(X/S, r))$ (resp. $\mathbf{Z}(Hilb(X/S, r))$, $\mathbf{Z}(PropHilb(X/S, r)))$ be the corresponding freely generated abelian monoids (resp. abelian groups).

The assigment $S'/S \to \mathbf{N}(Hilb(X \times_S S'/S', r))$ etc. defines a presheaf of abelian monoids (groups) on the category of Noetherian schemes over S. If $\mathcal{Z} = \sum n_i Z_i$ is an element of $\mathbf{Z}(\mathrm{Hilb(X/S, r)})$ and S' is a Noetherian scheme over S we denote by $\mathcal{Z} \times_S S'$ the corresponding element $\sum n_i(Z_i \times_S S')$ of $\mathbf{Z}(\mathrm{Hilb}(\mathrm{X} \times_{\mathrm{S}} \mathrm{S}'/\mathrm{S}', \mathrm{r}))$.

Lemma 3.2.1 *Let $p : X \to S$ be a finite flat morphism of Noetherian schemes of constant degree and S'/S be any Noetherian scheme over S. Let further τ' be a generic point of S' and $\eta'_1, \ldots, \eta'_k$ be all points of $X' = X \times_S S'$ lying over τ'. Then one has*

$$\sum_{j=1}^{k} length(\mathcal{O}_{X',\eta'_j})[k_{\eta'_j} : k_{\tau'}] = deg(p) length(\mathcal{O}_{S',\tau'})$$

Proof: The module $\coprod \mathcal{O}_{X',\eta'_j} = [(p')_*(\mathcal{O}_{X'})]_{\tau'}$ is a free $\mathcal{O}_{S',\tau'}$-module of rank $deg(p)$. Computing its length over $\mathcal{O}_{S',\tau'}$ in two different ways we get the desired equality.

Proposition 3.2.2 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and $S' \to S$ be any Noetherian scheme over S. Let further $\mathcal{Z} = \sum n_i Z_i$ be an element of $\mathbf{Z}(\mathrm{Hilb(X/S, r)})$. If $cycl_X(\mathcal{Z}) = 0$ then $cycl_{X \times_S S'}(\mathcal{Z} \times_S S') = 0$.*

Proof: Replacing X by $\cup Z_i$ we may assume that X is equidimensional of relative dimension r over S. Generic points of $Z_i \times_S S'$ lie over generic points of S' and are generic in their fibers. Let $\eta' \in X' = X \times_S S'$ be any of these generic points. Computing the multiplicity of η' in $cycl(\mathcal{Z} \times_S S')$ we may replace X by any open neighborhood of the point $\eta = pr_1(\eta') \in X$.

Lemma 3.2.3 *Let $p : X \to S$ be a flat equidimensional morphism of dimension r and x be a generic point of a fiber of p. Then for any decomposition of p of the form $X \stackrel{p_0}{\to} \mathbf{A}^r_S \to S$ such that p_0 is an equidimensional quasi-finite morphism there exists an open neighborhood U of x in X such that $(p_0)_{|U}$ is a flat quasi-finite morphism.*

Proof: It follows immediately from the fact that p is flat and [5, Ex. IV, Cor. 5.9,p.99].

Using Lemma 3.2.3 and Proposition 2.1.3 and replacing S by $\mathbf{A}^r_S$ we see that it is sufficient to treat the case $r = 0$. Furthemore replacing S' by $Spec(\mathcal{O}_{S',\tau'})$ and S by $Spec(\mathcal{O}_{s,\tau})$ (where $\tau = p(\eta)$ and $\tau' = p'(\eta')$) we may assume that S and S' are local schemes, S' is Artinian and $f : S' \to S$ takes the closed point of S' to the closed point of S.

Let S^{sh} (resp. $(S')^{sh}$) denote the strict henselization of the local scheme S (resp. S') in the closed point. Lemma 2.3.1 shows that $cycl(\mathcal{Z} \times_S S^{sh}) = 0$ and moreover that it is sufficient to check that

$$cycl((\mathcal{Z} \times_S S') \times_{S'} (S')^{sh}) = cycl(\mathcal{Z} \times_S (S')^{sh}) = 0.$$

Since there exists a morphism $(S')^{sh} \to S^{sh}$ over S we see that we may replace S and S' by S^{sh} and $(S')^{sh}$ and assume the schemes S and S' to be strictly henselian.

Since η lies over the closed point of S and S is henselian we conclude that $Spec(\mathcal{O}_{X,\eta})$ is an open neighborhood of η in X and is finite over S (see [11, I.4.2(c)]). Thus replacing X by $Spec(\mathcal{O}_{X,\eta})$ we may additionally assume that X is local and finite over S. In these assumptions η is the only point over τ (and hence η' is the only point over τ') and the schemes Z_i are finite and flat over S of constant degree $deg(Z_i/S)$. Lemma 3.2.1 shows now that the multiplicity of η' in $cycl_{X'}(\mathcal{Z} \times_S S')$ is equal to

$$\sum_i n_i length(\mathcal{O}_{Z'_i,\eta'}) = (\sum_i n_i deg(Z_i/S)) length(\mathcal{O}_{S',\tau'})/[k_{\eta'} : k_{\tau'}].$$

We only have to show that $\sum n_i deg(Z_i/S) = 0$. To do so let τ^0 be a generic point of S and let $\eta^0_1, \dots, \eta^0_k$ be all points of X over τ^0. The multiplicity of η^0_j in $cycl(\mathcal{Z})$ is equal to $\sum n_i length(\mathcal{O}_{Z_i,\eta^0_j})$. Using once again Lemma 3.2.1 we get

$$0 = \sum_j [k_{\eta^0_j} : k_{\tau^0}] \sum_i n_i length(\mathcal{O}_{Z_i,\eta^0_j}) = (\sum_i n_i deg(Z_i/S)) length(\mathcal{O}_{S,\tau^0}).$$

Proposition is proven.

Corollary 3.2.4 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, Z be an element of $Hilb(X/S, r)$ and (x_0, x_1, R) be a fat point over a k-point $x : Spec(k) \to S$ of S. Then*

$$(x_0, x_1)^*(cycl_X(Z)) = cycl_{X\times_S Spec(k)}(Z \times_S Spec(k)).$$

Proof: Let Z_i be the irreducible components of Z, z_i be their generic points and n_i be their multiplicities such that $cycl_X(Z) = \sum n_i z_i$. One checks easily that

$$cycl_{X\times_S Spec(R)}(Z \times_S Spec(R)) = \sum n_i cycl_{X\times_S Spec(R)}(\phi_{x_1}(Z_i)).$$

Proposition 3.2.2 shows now that

$$(x_0, x_1)^*(cycl_X(Z)) = \sum n_i cycl_{X\times_S Spec(k)}(\phi_{x_1}(Z_i) \times_{Spec(R)} Spec(k)).$$

Corollary 3.2.5 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the image of the cycle map* $\mathbf{Z}(\mathrm{Hilb(X/S, r)}) \to \mathrm{Cycl(X)}$ *lies in $Cycl_{equi}(X/S, r)$.*

Corollary 3.2.6 *Let R be a discrete valuation ring and $X \to Spec(R)$ be a scheme of finite type over $Spec(R)$. Then a cycle $\mathcal{Z} = \sum n_i Z_i$ belongs to $Cycl(X/Spec(R), r)$ if and only if the points z_i belong to the generic fiber of X over $Spec(R)$ and are of dimension r in this fiber.*

3.3 *Chow presheaves.*

Theorem 3.3.1 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $\mathcal{Z}$ be an element of the group $Cycl(X/S, r)$ and $f : T \to S$ be a Noetherian scheme over S.*

Then there is a unique element $\mathcal{Z}_T$ in $Cycl(X \times_S T/T, r) \otimes_{\mathbf{Z}} \mathbf{Q}$ such that for any commutative diagram of the form

$$\begin{array}{ccccc} & \overset{y_0}{\nearrow} & Spec(A) & \overset{y_1}{\to} & T \\ Spec(k) & & & & \downarrow f \\ & \underset{x_0}{\searrow} & Spec(R) & \overset{x_1}{\to} & S \end{array}$$

where (x_0, x_1) and (y_0, y_1) are fat k-points of S and T respectively one has:

$$(y_0, y_1)^*(\mathcal{Z}_T) = (x_0, x_1)^*(\mathcal{Z}).$$

Proof: The uniqueness of $\mathcal{Z}_T$ follows easily form 3.1.4 and 2.3.1(2).

In the proof of existence we start with the special case $T = Spec(k_s)$ where s is a set-theoretic point of S. In this case we have the following slightly more precise statement.

Lemma 3.3.2 *Denote by p the exponential characteristic of the field k_s. Then there exists a unique cycle $\mathcal{Z}_s$ in $Cycl(X_s, r)[1/p]$ such that for any field extension k/k_s and any fat point (x_0, x_1) over the k-point $Spec(k) \to Spec(k_s) \to S$ one has $(x_0, x_1)^*(\mathcal{Z}) = \mathcal{Z}_s \otimes_{k_s} k$.*

Proof: Denote by Z_i the closure of z_i which we consider as an integral closed subscheme in X and choose a blow-up $S' \to S_{red}$ such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S'. If k/k_s is a field extension such that the k-point $Spec(k) \to Spec(k_s) \to S$ admits a lifting to S' we get a cycle $\mathcal{Z}_k = \sum n_i cycl(\tilde{Z}_i \times_{S'} Spec(k)) \in Cycl(X \times_S Spec(k), r)$ which is independent of the choice of the lifting according to Proposition 3.1.5. Moreover if L/k is a field extension then the cycle $\mathcal{Z}_L$ is also defined and we have

$$\mathcal{Z}_L = \mathcal{Z}_k \otimes_k L$$

.

The morphism $S' \to S$ being a surjective morphism of finite type we can find a finite normal extension k_0/k_s such that the point $Spec(k_0) \to Spec(k_s) \to S$ admits a lifting to S'. The formula above shows that the cycle $\mathcal{Z}_{k_0}$ is $Gal(k_0/k_s)$-invariant and hence descends to a cycle $\mathcal{Z}_s \in Cycl(X_s, r)[1/p]$ by Lemma 2.3.2.

Let now k be any extension of k_s such that the point $Spec(k) \to Spec(k_s) \to S$ admits a lifting to S' and let L be a composite of k and k_0 over k_s. Then

$$\mathcal{Z}_k \otimes_k L = \mathcal{Z}_L = \mathcal{Z}_{k_0} \otimes_{k_0} L = \mathcal{Z}_s \otimes_{k_s} L = (\mathcal{Z}_s \otimes_{k_s} k) \otimes_k L$$

and hence $\mathcal{Z}_k = \mathcal{Z}_s \otimes_{k_s} k$.

Finally let k/k_s be a field extension and (x_0, x_1, R) be a fat point over a k-point $Spec(k) \to Spec(k_s) \to S$. The morphism $x_1 : Spec(R) \to S$ has a canonical lifting to S' (according to the valuative criterion of properness).

This gives us a lifting to S' of our k-point $Spec(k) \to S$ and it follows immediately from the construction of $\mathcal{Z}_s$ that one has $(x_0, x_1)^*(\mathcal{Z}) = \mathcal{Z}_k = \mathcal{Z}_s \otimes_{k_s} k$. Lemma is proven.

In the course of the proof of Lemma 3.3.2 we have established that the cycle $\mathcal{Z}_s$ has the following somewhat more general property.

Lemma 3.3.3 *Let $S' \to S$ be a blow-up such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S' and let k/k_s be a field extension such that the k-point $Spec(k) \to Spec(k_s) \to S$ admits a lifting to S'. Then $\mathcal{Z}_s \otimes_{k_s} k = \sum n_i cycl(\tilde{Z}_i \times_{S'} Spec(k))$.*

Let $\tau_1, \ldots, \tau_n$ be the generic points of T and $\sigma_1, \ldots, \sigma_n$ be their images in S. Consider the cycles

$$\mathcal{Z}_{\sigma_j} \otimes_{k_{\sigma_j}} k_{\tau_j} = \sum_l n_{jl} z_{jl} \in Cycl(X \times_S Spec(k_{\tau_j}), r) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Here n_{jl} are rational numbers and z_{jl} are points of $X \times_S T$ lying over τ_j and having dimension r in their fibers. Set $\mathcal{Z}_T = \sum_{j,l} n_{jl} z_{jl}$. We are going to show that $\mathcal{Z}_T$ belongs to $Cycl(X \times_S T/T, r) \otimes_{\mathbf{Z}} \mathbf{Q}$ and has the desired property. To do so we need the following lemma.

Lemma 3.3.4 *Let $f : S' \to S$ be a proper surjective morphism of finite type of Noetherian schemes, A be a discrete valuation ring and $Spec(A) \to S$ be a morphism of schemes. Then there exists a commutative diagram of the form:*

$$\begin{array}{ccc} Spec(A') & \to & S' \\ g \downarrow & & \downarrow f \\ Spec(A) & \to & S \end{array}$$

In which A' is a discrete valuation ring and the morphism g is surjective.

Proof: Let K be the quotient field of A. Since f is a surjective morphism of finite type there exists a finite extension K' of K such that the K'-point $Spec(K') \to Spec(K) \to S$ admits a lifting to S'. Let ν be the discrete valuation of K which corresponds to A. It can be extended to a discrete valuation ν' of K' and we take A' to be the corresponding discrete valuation ring. The valuative criterion of properness ([6, 7.3]) shows that the morphism $Spec(A') \to Spec(A) \to S$ admits a lifting to S'.

Consider a commutative diagram of the form

$$\begin{array}{ccccc} & \overset{y_0}{\nearrow} & Spec(A) & \overset{y_1}{\to} & T \\ Spec(k) & & & & \downarrow f \\ & \overset{x_0}{\searrow} & Spec(R) & \overset{x_1}{\to} & S \end{array}$$

in which (x_0, x_1) (resp. (y_0, y_1)) is a fat k-point of S (resp. of T). Let as before $S' \to S_{red}$ denote a blow-up of S_{red} such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S'. Lemma 3.3.4 shows that there is a discrete valuation ring A' and a surjective morphism $Spec(A') \to Spec(A)$ such that the morphism $Spec(A') \to S$ admits a lifting to S'. Denote the residue field of A (resp. of A') by k_A (resp. by $k_{A'}$) and let k' be a composite of k and $k_{A'}$ over k_A so that we have the following commutative diagram:

$$\begin{array}{ccccc} Spec(k') & \to & Spec(A') & \to & S' \\ \downarrow & & \downarrow & & \downarrow \\ Spec(k) & \overset{y_0}{\to} & Spec(A) & \overset{y_1}{\to} T \to & S \end{array}$$

Assume that y_1 maps the generic point of $Spec(A)$ to τ_1 and consider the following two elements in $\mathbf{Q}(\mathrm{Hilb}(X \times_S \mathrm{Spec}(A')/\mathrm{Spec}(A'), r))$:

$$\mathcal{W} = \sum_i n_i(\tilde{Z}_i \times_{S'} Spec(A'))$$

$$\mathcal{W}_1 = \sum_{j,l} n_{jl}(\phi_{y_1}(Z_{jl}) \times_{Spec(A)} Spec(A')) = \sum_l n_{1l}(\phi_{y_1}(Z_{1l}) \times_{Spec(A)} Spec(A'))$$

where Z_{jl} denotes the closure of the points z_{jl} considered as a closed integral subscheme of $X \times_S T$.

Lemma 3.3.5

$$cycl(\mathcal{W}) = cycl(\mathcal{W}_1)$$

Proof: Let K (resp. K') denote the quotient field of A (resp. of A'). Since the map

$$Cycl(X \times_S Spec(A')/Spec(A'), r) \otimes \mathbf{Q} \to \mathrm{Cycl}(X \times_S \mathrm{Spec}(K')/\mathrm{Spec}(K'), r) \otimes \mathbf{Q}$$

is clearly injective we may replace A' by K' everywhere. Furthermore

$$cycl(\mathcal{W} \times_{Spec(A')} Spec(K)) = \sum n_i cycl(\tilde{Z}_i \times_{S'} Spec(K'))$$

and according to Lemma 3.3.3 this cycle is equal to $\mathcal{Z}_{\sigma_1} \otimes_{k_{\sigma_1}} K'$.

On the other hand we have:

$$cycl(\mathcal{W}_1 \times_{Spec(A')} Spec(K')) =$$

$$= \sum_l n_{1l} cycl([\phi_{y_1}(Z_{1l}) \times_{Spec(A)} Spec(K)] \times_{Spec(K)} Spec(K')) =$$

$$= \sum_l n_{1l} cycl((Z_{1l} \times_T Spec(K)) \times_{Spec(K)} Spec(K')) =$$

$$= [\sum_l n_{1l} cycl(Z_{1l} \times_T Spec(k_{\tau_1}))] \otimes_{k_{\tau_1}} K' =$$

$$= (\sum_l n_{1l} z_{1l}) \otimes_{k_{\tau_1}} K' = (\mathcal{Z}_{\sigma_1} \otimes_{k_{\sigma_1}} k_{\tau_1}) \otimes_{k_{\tau_1}} K'$$

$$= \mathcal{Z}_{\sigma_1} \otimes_{k_{\sigma_1}} K'.$$

Lemma is proven.

Proposition 3.2.2 implies now that

$$cycl(\mathcal{W} \times_{Spec(A')} Spec(k')) = cycl(\mathcal{W}_1 \times_{Spec(A')} Spec(k'))$$

i.e.

$$\sum n_i cycl(\tilde{Z}_i \times_{S'} Spec(k')) = \sum_l n_{1l} cycl(\phi_{y_1}(Z_{1l}) \times_{Spec(A')} Spec(k')) =$$

$$= (y_0, y_1)^*(\mathcal{Z}_T) \otimes_k k'.$$

On the other hand the cycle $(x_0, x_1)^*(\mathcal{Z}) \otimes_k k'$ is equal to $\sum n_i cycl(\tilde{Z}_i \times_{S'} Spec(k'))$ where this time the morphism $Spec(k') \to S'$ is a lifting of the same point $Spec(k') \to Spec(k) \to Spec(R) \to S$ obtained using the canonical lifting of the morphism $Spec(R) \to S$. Proposition 3.1.5 shows that $(x_0, x_1)^*(\mathcal{Z}) \otimes_k k' = (y_0, y_1)^*(\mathcal{Z}_T) \otimes_k k'$ and hence $(x_0, x_1)^*(\mathcal{Z}) = (y_0, y_1)^*(\mathcal{Z}_T)$. Theorem 3.3.1 is proven.

Remark: In general cycles of the form $cycl(f)(\mathcal{Z})$ do not have integral coefficients. See example 3.5.10(1) below.

Lemmas 3.3.6 and 3.3.8 below describe the behavior of supports of cycles with respect to the base change homomorphisms.

Lemma 3.3.6 *In the notations and assumptions of Theorem 3.3.1 we have*

$$supp(\mathcal{Z}_T) \subset (supp(\mathcal{Z}))_T = supp(\mathcal{Z}) \times_S T.$$

Proof: Since $supp(\mathcal{Z}_T) = \cup supp(\mathcal{Z}_{k_{\tau_j}})$ where τ_j are the generic points of T it is sufficient to consider the case $T = Spec(k)$ where k is a field. According to Lemma 3.1.4 there exists an extension k'/k and a fat point (x_0, x_1, R) over the k'-point $Spec(k') \to Spec(k) \to S$. The defining property of the cycle $\mathcal{Z}_k$ shows that

$$supp(\mathcal{Z}_k) \times_{Spec(k)} Spec(k') = supp(\mathcal{Z}_{k'}) = supp((x_0, x_1)^*(\mathcal{Z})) \subset$$

$$\subset \cup_i \phi_{x_1}(Z_i) \times_{Spec(R)} Spec(k') \subset \cup_i (Z_i \times_S Spec(R)) \times_{Spec(R)} Spec(k') =$$

$$= supp(\mathcal{Z}) \times_S Spec(k') = (supp(\mathcal{Z}) \times_S Spec(k)) \times_{Spec(k)} Spec(k').$$

Since the morphism $X_{k'} \to X_k$ is surjective the above inclusion implies the desired one $supp(\mathcal{Z}_k) \subset supp(\mathcal{Z}) \times_S Spec(k)$.

Lemma 3.3.7 *Consider a pull-back square of morphisms of finite type of Noetherian schemes of the form*

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ S' & \stackrel{f}{\to} & S \end{array}$$

and assume that the morphism f is universally open and any generic point of X lies over a generic point of S. Then any generic point of X' lies over a generic point of S'.

Proof: Any generic point of X' obviously lies over a generic point of S. Replacing S by this point we may assume that $S = Spec(k)$ where k is a field. Then the morphism $X \to S$ is universally open (being flat) and hence the morphism $X' \to S'$ is open, which implies that it takes generic points to generic points.

Lemma 3.3.8 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $\mathcal{Z} = \sum n_i z_i$ be an element of $Cycl(X/S, r) \otimes \mathbf{Q}$, $f : S' \to S$ be a Noetherian scheme over S and $\mathcal{Z}' = cycl(f)(\mathcal{Z})$ be the corresponding element of $Cycl(X \times_S S'/S', r) \otimes \mathbf{Q}$.*

1. *If f is a universally open morphism then $supp(\mathcal{Z}') = (supp(\mathcal{Z}) \times_S S')_{red}$.*

2. *If f is dominant then $Supp(\mathcal{Z})$ is the closure of $(X \times_S f)(Supp(\mathcal{Z}'))$.*

Proof: 1. The inclusion

$$supp(\mathcal{Z}') \subset (supp(\mathcal{Z}) \times_S S')_{red}$$

follows from Lemma 3.3.6. Lemma 3.3.7 implies immediately that generic points of $supp(\mathcal{Z}) \times_S S'$ lie over generic points of S'. We may assume therefore that $S' = Spec(k)$ and the image of S' in S is a generic point η of S. Then k is an extension of k_η and according to Lemma 2.3.1(2) we have

$$Supp(\mathcal{Z}') = Supp(\mathcal{Z}_\eta \otimes_{k_\eta} k) = (Supp\mathcal{Z}_\eta \times_{Spec(k_\eta)} Spec(k))_{red}.$$

Now it suffices to note that

$$\mathcal{Z}_\eta = \sum_{z_i/\eta} n_i z_i$$

(the sum being taken over those points z_i which lie over η) and hence $Supp(\mathcal{Z}_\eta) = Supp(\mathcal{Z}) \times_S Spec(k_\eta)$. 3. It suffices to show that $z_i \in (X \times_S f)(Supp(\mathcal{Z}'))$. Denote by η_i the image of z_i in S and let η_i' be a generic point of S' over η_i. Using Lemma 3.3.6 and the part (1) of the present lemma we get

$$Supp(\mathcal{Z} \times_{Spec(k_{\eta_i})} Spec(k_{\eta_i'}))_{red} = Supp(\mathcal{Z}_{k_{\eta_i'}}) =$$

$$= Supp(\mathcal{Z}'_{k_{\eta_i'}}) \subset Supp(\mathcal{Z}')$$

It suffices to note now that the morphism

$$Supp(\mathcal{Z}) \times_{Spec(k_{\eta_i})} Spec(k_{\eta_i'}) \to Supp(\mathcal{Z})$$

is surjective.

Let $f : T \to S$ be a morphism of Noetherian schemes and $X \to S$ be a scheme of finite type over S. We denote by

$$cycl(f) : Cycl(X/S, r) \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathrm{Cycl(X \times_S T/T, r)} \otimes_{\mathbf{Z}} \mathbf{Q}$$

the homomorphism $\mathcal{Z} \to \mathcal{Z}_T$ constructed in Theorem 3.3.1. Lemma 3.3.6 shows that this homomorphism takes $Cycl_{equi}(X/S,r)\otimes_{\mathbf{Z}}\mathbf{Q}$ to $Cycl_{equi}(X\times_S T/T,r)\otimes_{\mathbf{Z}}\mathbf{Q}$, $PropCycl(X/S,r)\otimes_{\mathbf{Z}}\mathbf{Q}$ to $PropCycl(X\times_S T/T,r)\otimes_{\mathbf{Z}}\mathbf{Q}$ etc. We use the same notation $cycl(f)$ for the homomorphisms induced on the corresponding groups and monoids.

It follows immediately from definitions and Lemma 3.1.4 that

$$cycl(g\circ f) = cycl(f)\circ cycl(g)$$

for any composable pair of morphisms of Noetherian schemes.

This shows that for any scheme of finite type $X \to S$ over a Noetherian scheme S there is a presheaf of $\mathbf{Q}$-vector spaces $Cycl(X/S,r)_{\mathbf{Q}}$ on the category of Noetherian schemes over S such that for any Noetherian scheme S' over S one has $Cycl(X/S,r)_{\mathbf{Q}}(S') = Cycl(X\times_S S'/S',r)\otimes\mathbf{Q}$.

Similarly we have presheaves of $\mathbf{Q}$-vector spaces $PropCycl(X/S,r)_{\mathbf{Q}}$, $Cycl_{equi}(X/S,r)_{\mathbf{Q}}$, $PropCycl_{equi}(X/S,r)_{\mathbf{Q}}$ and presheaves of uniquely divisible abelian monoids $Cycl^{eff}(X/S,r)_{\mathbf{Q}^+}$, $PropCycl^{eff}(X/S,r)_{\mathbf{Q}^+}$.

For any Noetherian scheme S' over S we have a canonical lattice $Cycl(X\times_S S',r)$ in the $\mathbf{Q}$-vector space $Cycl(X/S,r)_{\mathbf{Q}}(S')$. Unfortunately these lattices do not form in general a subpresheaf in $Cycl(X/S,r)_{\mathbf{Q}}$ (see example 3.5.10(1)).

Lemma 3.3.9 *Let S be a Noetherian scheme, $X \to S$ be a scheme of finite type over S and $\mathcal{Z}$ be an element of $Cycl(X/S,r)$. Then the following conditions are equivalent:*

1. *For any Noetherian scheme T over S the cycle $\mathcal{Z}_T$ belongs to $Cycl(X\times_S T/T,r)$.*

2. *For any point $s\in S$ the cycle $\mathcal{Z}_s$ belongs to $Cycl(X_s,r)$.*

3. *For any point $s\in S$ there exists a separable field extension k/k_s such that the cycle $\mathcal{Z}_k = Z_s\otimes_{k_s} k$ belongs to $Cycl(X\times_S Spec(k),r)$.*

Proof: The implication (3=>2) follows from Lemma 2.3.2. The other implications are obvious.

We denote by $z(X/S,r)$ (resp. $c(X/S,r)$, $z_{equi}(X/S,r)$, $c_{equi}(X/S,r)$) the subgroup of $Cycl(X/S,r)$ (resp. of $PropCycl(X/S,r)$, $Cycl_{equi}(X/S,r)$,

$PropCycl_{equi}(X/S,r))$ consisting of cycles satisfying the equivalent conditions of Lemma 3.3.9. It is clear that $z(X/S,r)$ (resp. ...) is a subpresheaf in the presheaf $Cycl(X/S,r)_{\mathbf{Q}}$. Moreover

$$c(X/S,r) = z(X/S,r) \cap PropCycl(X/S,r)_{\mathbf{Q}}$$

etc.

Lemma 3.3.10 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $T \to S$ be a Noetherian scheme over S and $\mathcal{W}$ be an element of* $\mathbf{Z}(\mathrm{Hilb}(X/S,r))$. *Then*

$$[cycl_X(\mathcal{W})]_T = cycl_{X\times_S T}(\mathcal{W}_T).$$

Proof: It is sufficient to treat the case $\mathcal{W} = Z$ where Z is a closed subscheme of X flat and equidimensional of relative dimension r over S. Consider any commutative diagram of the form

$$\begin{array}{ccccc} & & Spec(A) & \stackrel{y_1}{\to} & T \\ & \stackrel{y_0}{\nearrow} & & & \\ Spec(k) & & & & \downarrow f \\ & \stackrel{x_0}{\searrow} & & & \\ & & Spec(R) & \stackrel{x_1}{\to} & S \end{array}$$

in which (x_0,x_1) (resp. (y_0,y_1)) is a fat k-point of S (resp. of T). Corollary 3.2.4 shows that

$$(y_0,y_1)^*(cycl_{X\times_S T}(Z\times_S T)) = cycl_{X\times_S Spec(k)}(Z\times_S Spec(k)) =$$

$$= (x_0,x_1)^*(cycl_X(Z)).$$

Thus the cycle $cycl_{X\times_S T}(Z\times_S T)$ satisfies the requirements defining the cycle $[cycl_X(Z)]_T$.

Corollary 3.3.11 *The homomorphisms $cycl_X$ define a homomorphism of presheaves*

$$cycl : \mathbf{Z}(\mathrm{Hilb}(X/S,r)) \to \mathrm{z_{equi}}(X/S,r).$$

Lemma 3.3.12 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and $f : S' \to S$ be a flat morphism of Noetherian schemes. Assume further that the schemes S, S' are reduced. Then for any element $\mathcal{Z}$ in $Cycl(X/S, r)$ one has*

$$cycl(f)(\mathcal{Z}) = f_X^*(\mathcal{Z})$$

where $f_X = pr_1 : X \times_S S' \to X$ and f_X^* is the flat pull-back defined in section 2.3.

Proposition 3.3.13 *Let S be a Noetherian scheme of exponential characteristic n and $X \to S$ be a scheme of finite type over S. Then the subgroups $Cycl(X \times_S T/T, r)[1/n]$ in $Cycl(X \times_S T/T, r) \otimes \mathbf{Q}$ for Noetherian schemes T over S form a subpresheaf $Cycl(X/S, r)[1/n]$ in the presheaf $Cycl(X/S, r)_{\mathbf{Q}}$.*

Proof: It follows easily from Lemmas 3.3.2 and 3.3.9.

Proposition 3.3.14 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the quotient presheaf $Cycl(X/S, r)_{\mathbf{Q}}/z(X/S, r)$ is a presheaf of torsion abelian groups.*

Proof: We have to show that for any $\mathcal{Z} = \sum n_i z_i \in Cycl(X/S, r)$ there exists an integer $N > 0$ such that $N\mathcal{Z} \in z(X/S, r)$. Using Noetherian induction we may assume that for any closed subscheme $T \subset S$ such that $T \neq S$ there exists an integer $N(T)$ such that $N(T)\mathcal{Z}_T \in z(X \times_S T/T, r)$. Denote by Z_i the closure of the point z_i considered as an integral closed subscheme of X. Let $S' \to S_{red}$ be a blow-up such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S'. Let U be a dense open subset of S over which the morphism $S' \to S_{red}$ is an isomorphism and let T be the closed reduced subscheme $S - U$. We claim that $N(T)\mathcal{Z} \in z(X/S, r)$. By Lemma 3.3.9 it is sufficient to verify that $N(T)\mathcal{Z}_s \in Cycl(X_s, r)$ for any point $s \in S$. If $s \in T$ it follows from our choice of $N(T)$. If $s \in U$ it follows from Lemma 3.3.3.

Proposition 3.3.15 *Let S be a regular Noetherian scheme. Then for any scheme of finite type X over S and any $r \geq 0$ one has:*

$$Cycl(X/S, r) = z(X/S, r)$$

$$Cycl_{equi}(X/S, r) = z_{equi}(X/S, r)$$

etc.

Proof: Let $\mathcal{Z}$ be an element of $Cycl(X/S, r)$ and let $s \in S$ be a point of S. We have to show that $\mathcal{Z}_s \in Cycl(X_s, r)$. We proceed by induction on $n = dim(\mathcal{O}_{S,s})$.

If $n = 1$ then $\mathcal{O}_{S,s}$ is a discrete valuation ring so that we have a canonical fat point $Spec(k_s) \stackrel{x_0}{\to} Spec(\mathcal{O}_{S,s}) \stackrel{x_1}{\to} S$ over the point $Spec(k_s) \to S$. Thus $\mathcal{Z}_s = (x_0, x_1)^*(\mathcal{Z}) \in Cycl(X_s, r)$.

If $n > 1$ choose a regular system of parameters $t_1, \ldots, t_n$ in $\mathcal{O}_{S,s}$. Replacing S by an appropriate open neighborhood of s we may assume that S is affine , $t_1, \ldots, t_n \in \mathcal{O}(S)$ and the closed subscheme T of S defined by the equation $t_1 = 0$ is regular and irreducible. Let τ be the generic point of T. Since $dim(\mathcal{O}_{S,\tau}) = 1$ we conclude that $\mathcal{Z}_\tau \in Cycl(X_\tau, r)$ and hence $\mathcal{Z}_T \in Cycl(X \times_S T/T, r)$. The induction hypothesis shows now that $\mathcal{Z}_s \in Cycl(X_s, r)$.

3.4 *Relative cycles over geometrically unibranch schemes.*

Lemma 3.4.1 *Let k be a field, $X \to Spec(k)$, $S \to Spec(k)$ be two schemes of finite type over k and Z be a closed subscheme in $X \times_{Spec(k)} S$ defined by nilpotent sheaf of ideals which is flat over S. Let further E be an extension of k and s_1, s_2 be two E-points of S over k. If S is geometrically connected then the cycles associated with the closed subschemes $Z \times_{s_1} Spec(E)$ and $Z \times_{s_2} Spec(E)$ in $X \times_{Spec(k)} Spec(E)$ coincide.*

Proof: We may replace E by its algebraic closure and thus assume that E is algebraically closed. Next we may replace X by $X_E = X \times_{Spec(k)} Spec(E)$, S be $S_E = S \times_{Spec(k)} Spec(E)$ and Z by $Z_E = Z \times_{Spec(k)} Spec(E)$ and thus assume that $E = k$. The scheme S being connected we may find a chain of rational points of S starting with s_1 and ending with s_2 and such that any pair of consecutive points belongs to the same irreducible component of S. Thus we may assume that S is an integral scheme. Let $X_1, \ldots, X_n$ denote the irreducible components of X which are considered as closed integral subschemes of X. Since S is integral and the base field is algebraically closed we conclude that the schemes $X_i \times S$ are integral and coincide with the irreducible components of $X \times S$.

This shows that the cycle $cycl_{X \times S}(Z)$ may be written (uniquely) in the form $\sum n_i cycl_{X \times S}(X_i \times S)$. Proposition 3.2.2 shows now that the cycles $cycl_X(Z \times_{s_j} Spec(k))$ both coincide with $\sum n_i cycl_X(X_i)$.

Theorem 3.4.2 *Let S be a Noetherian geometrically unibranch scheme and $X \to S$ be a scheme of finite type over S. Let further $Z \subset X$ be a closed subscheme which is equidimensional of relative dimension r over S. Then $cycl_X(Z) \in Cycl_{equi}(X/S, r)$.*

Proof: Replacing Z by its irreducible components (which are also equidimensional over S) we may assume that the scheme Z is integral. Choose a blow-up $S' \to S_{red}$ such that the proper transform $\tilde{Z}$ of Z is flat over S'. Let further k be a field, $s : Spec(k) \to S$ be a k-point of S and $s_1, s_2 : Spec(k) \to S'$ be two liftings of s to S'. According to Proposition 3.1.5 we have to show that the cycles $cycl(\tilde{Z} \times_{s_1} Spec(k))$, $cycl(\tilde{Z} \times_{s_2} Spec(k))$ coincide. Note that according to Proposition 2.1.7 and Lemma 2.1.11 the closed subscheme $\tilde{Z}$ in $Z \times_S S'$ is defined by a nilpotent sheaf of ideals and hence $\tilde{Z} \times_S Spec(k)$ is a closed subscheme of $(Z \times_S S') \times_S Spec(k) = (Z \times_S Spec(k)) \times_{Spec(k)} (S' \times_S Spec(k))$ defined by a nilpotent sheaf of ideals. The scheme $S' \times_S Spec(k)$ is geometrically connected according to Proposition 2.1.6. Thus our statement follows from Lemma 3.4.1.

Corollary 3.4.3 *Let S be a Noetherian geometrically unibranch scheme and $X \to S$ be a scheme of finite type over S. Then the abelian group $Cycl_{equi}(X/S, r)$ (resp. $PropCycl_{equi}(X/S, r)$) is freely generated by cycles of integral closed subschemes Z in X which are equidimensional (resp. proper and equidimensional) of dimension r over S.*

Corollary 3.4.4 *Let S be a Noetherian geometrically unibranch scheme and $X \to S$ be a scheme of finite type over S. Then the abelian group $Cycl_{equi}(X/S, r)$ (resp. the abelian group $PropCycl_{equi}(X/S, r)$) is generated by the abelian monoid $Cycl^{eff}(X/S, r)$ (resp. by the abelian monoid $PropCycl^{eff}(X/S, r)$).*

Corollary 3.4.5 *Let S be a Noetherian regular scheme and X be a scheme of finite type over S. Then abelian group $z_{equi}(X/S, r)(S)$ (resp. the abelian monoid $z^{eff}(X/S, r)(S)$) is the free abelian group (resp. the free abelian monoid) generated by closed integral subschemes of X which are equidimensional of dimension r over S.*

Corollary 3.4.6 *Let S be a Noetherian regular scheme and X be a scheme of finite type over S. Then the abelian group $c_{equi}(X/S, r)(S)$ (resp. the abelian* *monoid $c^{eff}(X/S, r)(S)$) is the free abelian group (resp. the free abelian monoid) generated by closed integral subschemes of X which are proper and equidimensional of dimension r over S.*

Example 3.4.7 The statement of Corollary 3.4.4 is false for schemes S which are not geometrically unibranch. Let us consider the following situation.

Let $S = S_1 \cup S_2$ be a union of two copies of affine line (i.e. $S_1 \cong S_2 \cong \mathbf{A}^1$) such that the point $\{0\}$ (resp. the point $\{1\}$) of S_1 is identified with the point $\{0\}$ (resp. the point $\{1\}$) of S_2.

We take X to be abstractly isomorphic to S, i.e. $X = X_1 \cup X_2$ also is a union of two copies of affine line glued together in the same way. Consider the morphism $X \to S$ which maps X_1, X_2 identically on S_1. Using Proposition 3.1.5 one can easily see that:

$$Cycl^{eff}(X/S, 0) = 0$$

$$Cycl(X/S, 0) = \mathbf{Z}$$

and hence the abelian group of relative cycles is not generated by abelian monoid of effective relative cycles in this case.

Remarks:

1. The statement of Corollary 3.4.3 is false for the abelian groups $Cycl(X/S, r)$ and $PropCycl(X/S, r)$ since if $dim(S) > 1$ there exist elements $\mathcal{Z}$ in these groups such that $supp(\mathcal{Z})$ is not equidimensional over S (see exmple 3.1.9).

2. It is not true in general that for a geometrically unibranch scheme S the groups $z_{equi}(X/S, r)$, $c_{equi}(X/S, r)$ (or abelian monoids $z^{eff}(X/S, r)$, $c^{eff}(X/S, r)$ are generated by elements which correspond to integral closed subschemes of X (see example 3.5.10(2)) but in view of Proposition 3.3.15 it is true for regular schemes S.

Proposition 3.4.8 *Let S be a normal Noetherian scheme and $X \to S$ be a smooth scheme of finite type of dimension r over S. Then one has:*

$$z_{equi}(X/S, r-1) = Cycl_{equi}(X/S, r-1)$$

$$c_{equi}(X/S, r-1) = PropCycl_{equi}(X/S, r-1)$$

and the first group is isomorphic to the group of relative Cartier divisors on X over S.

Proof: Note first that any normal scheme is geometrically unibranch by [8]. By Corollary 3.4.3 the group $Cycl_{equi}(X/S, r-1)$ is generated by integral closed subschemes Z in X which are equidimensional of relative dimension $r-1$ over S. By [8, 21.14.3] any such Z is flat over S and our result follows from the definition of relative Cartier divisor (see [9]) and Corollary 3.3.11.

3.5 *Multiplicities of components of inverse images of equidimensional cycles over geometrically unibranch schemes.*

Let S be a Noetherian geometrically unibranch scheme and $X \to S$ be a scheme of finite type over S. Corollary 3.4.3 shows that the abelian group $Cycl_{equi}(X/S, r)$ is the free abelian group generated by generic points of closed integral subschemes $Z \subset X$ of X which are equidimensional of relative dimension r over S. Let now $T \to S$ be a morphism of Noetherian schemes. Lemma 3.3.6 shows that the cycle $[Cycl_X(Z)]_T$ is a formal linear combination of generic points of $Z \times_S T$ with certain multiplicities. The aim of this section is to give an explicit formula for these multiplicities. It is sufficient to consider the case $T = Spec(k_s)$ where s is a certain point of S. Moreover since the groups $Cycl_{equi}(X/S, r)$ and $Cycl_{equi}(X_{red}/S_{red}, r)$ coincide we may assume that the scheme S is reduced and since irreducible components of S do not intersect we may further assume that S is integral. The formula we are going to provide involves multiplicities of certain ideals so we start by recalling briefly the necessary definitions and results from commutative algebra (see [10]).

Let $\mathcal{O}$ be a local Noetherian ring of dimension r and M be a finitely generated $\mathcal{O}$ module. An ideal $I \subset \mathcal{O}$ is called an ideal of definition if I contains a certain power of the maximal ideal. For any ideal of definition I of $\mathcal{O}$ one may consider the so called Samuel function:

$$\chi^I_M(n) = length(M/I^{n+1}M).$$

It is known that for n big enough $\chi^I_M(n)$ is a polynomial in n of degree at most r. Furthermore it may be written in the form

$$\chi^I_M(n) = (e/r!)n^r + (\text{terms of lower degree})$$

where e is a nonnegative integer called the multiplicity of I with respect to M. We will denote it by $e(I, M)$. The integer $e(I, \mathcal{O})$ is called the multiplicity of I and is denoted $e(I)$.

Proposition 3.5.1 *Let $\mu_1, \ldots, \mu_k$ be all minimal prime ideals of $\mathcal{O}$ such that $dim(\mathcal{O}/\mu) = r$, then*

$$e(I, M) = \sum_{i=1}^{k} e(I, \mathcal{O}/\mu_i) length_{\mathcal{O}_{\mu_i}}(M_{\mu_i}) = \sum_{i=1}^{k} e(I(\mathcal{O}/\mu_i)) length_{\mathcal{O}_{\mu_i}} M_{\mu_i}.$$

Proof: See [10] Theorem 14.7.

Lemma 3.5.2 *Let $\mathcal{O} \to \mathcal{O}'$ be a local homomorphism of Noetherian local rings. Let $\mathcal{M}_{\mathcal{O}}$ be the maximal ideal of $\mathcal{O}$ and suppose that $\mathcal{O}'$ is a flat $\mathcal{O}$-algebra and $\mathcal{M}_{\mathcal{O}}\mathcal{O}'$ is an ideal of definition of $\mathcal{O}'$. Then for any ideal of definition I of $\mathcal{O}$ the following formula holds*

$$e(I\mathcal{O}') = e(I) length_{\mathcal{O}'}(\mathcal{O}'/\mathcal{M}_{\mathcal{O}}\mathcal{O}').$$

Proof: Flatness of $\mathcal{O}'$ over $\mathcal{O}$ implies that for any n we have

$$\chi_{\mathcal{O}'}^{I\mathcal{O}'}(n) = length_{\mathcal{O}'}(\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}/I^n) = length_{\mathcal{O}}(\mathcal{O}/I^n) length_{\mathcal{O}'}(\mathcal{O}'/\mathcal{M}_{\mathcal{O}}\mathcal{O}') =$$

$$= \chi_{\mathcal{O}}^{I}(n) length_{\mathcal{O}'}(\mathcal{O}'/\mathcal{M}_{\mathcal{O}}\mathcal{O}').$$

The following property of multiplicities is obvious.

Lemma 3.5.3 *Assume that $\mathcal{O}'$ is a finite local $\mathcal{O}$ algebra such that $dim(\mathcal{O}') = dim(\mathcal{O})$ and M is a finitely generated $\mathcal{O}'$-module. Let k and k' be the residue fields of $\mathcal{O}$ and $\mathcal{O}'$. Then for any ideal of definition I of $\mathcal{O}$ the following formula holds:*

$$e(I\mathcal{O}', M) = e(I, M)/[k' : k].$$

Let $Z \to S$ be a scheme equidimensional of relative dimension r over an integral Noetherian geometrically unibranch scheme S. Let further s be a point of S, I be an ideal of definition of the local ring $\mathcal{O}_{S,s}$ and z be a generic point of the fiber $Z_s = Z \times_{Spec(k_s)} S$. We set $n_I(z) = e(I\mathcal{O}_{Z,z})/e(I)$.

Lemma 3.5.4 *1. $n_I(z)$ is independent of the choice of I.*

2. If Z is flat over S then $n_I(z)$ coincides with the multiplicity of z in the cycle $(cycl_Z(Z))_s = cycl_{Z_s}(Z_s)$.

Proof: The second statement follows immediately from Lemma 3.5.2. Replacing Z by an appropriate neighborhood of z we may assume that the morphism $Z \to S$ admits a decomposition of the form $Z \stackrel{p_0}{\to} \mathbf{A}^r_S \to S$ where p_0 is an equidimensional quasi-finite morphism (see Proposition 2.1.3). Denote $p_0(z)$ by x. Then x is the (unique) generic point of the fiber of $\mathbf{A}^r_S$ over s and according to the part (2) of our proposition we have $e(I\mathcal{O}_{\mathbf{A}^r_S,x}) = e(I)$. This shows that $n_I(z) = e(I\mathcal{O}_{Z,z})/e(I\mathcal{O}_{\mathbf{A}^r_S,x})$ so that we may replace S by $\mathbf{A}^r_S$ and assume that $r = 0$.

Denote by S' the henselization of S at s and by s' the closed point of S'. Set also $Z' = Z \times_S S'$. The fiber of Z' over s' is isomorphic to the fiber of Z over s and thus there exists exactly one point z' lying over z. Since $\mathcal{O}_{S',s'} = \mathcal{O}^h_{S,s}$ is ind-etale over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{Z',z'}$ is ind-etale over $\mathcal{O}_{Z,z}$ we conclude from Lemma 3.5.2 that $e(I) = e(I\mathcal{O}_{S',s})$, $e(I\mathcal{O}_{Z,z}) = e(I\mathcal{O}_{Z',z'})$ and hence $n_I(z) = n_{I\mathcal{O}_{S',s'}}(z')$. Thus replacing S by S' and Z by Z' we may assume that S is a henselian local scheme (note that S' is integral since S is geometrically unibranch). In this case $\mathcal{O}_{Z,z}$ is a finite $\mathcal{O}_{S,s}$-algebra and according to Lemma 3.5.3 and Proposition 3.5.1 we have

$$n_I(z) = \frac{e(I\mathcal{O}_{Z,z})}{e(I)} = \frac{e(I,\mathcal{O}_{Z,z})}{[k_z : k_s]e(I)} = \frac{e(I)dim_{F(S)}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} F(S))}{e(I)[k_z : k_s]} =$$

$$= \frac{dim_{F(S)}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} F(S))}{[k_z : k_s]}$$

where $F(S)$ is the quotient field of the integral domain $\mathcal{O}_S$. Thus $n_I(z)$ does not depend on the choice of I.

We will use the notation $n(z)$ or $n_{Z/S}(z)$ for the common value of multiplicities $n_I(z)$. We will denote by $[Z/S]_s$ the element of $Cycl(Z_s, r)$ of the form $\sum n(z)z$ where the sum is taken over all generic points of $Z_s = Z \times_S Spec(k_s)$. In the course of the proof of Lemma 3.5.4 we have established some useful properties of multiplicities $n(z)$ which we would like to list now for future use.

Corollary 3.5.5 *1. Assume that the morphism $Z \to S$ is factorized in the form $Z \to \mathbf{A}^r_S \to S$ where the first morphism is quasi-finite and equidimensional. Then for any point $z \in Z$ generic in its fiber over S one has $n_{Z/S}(z) = n_{Z/\mathbf{A}^r_S}(z)$.*

2. *Let S' be the henselization of S at s, s' be the closed point of S' and z' be the only point of $Z' = Z \times_S S'$ lying over z. Then $n_{Z'/S'}(z') = n_{Z/S}(z)$.*

3. *Let $Z \to S$ be a finite equidimensional morphism. Assume that the fiber Z_s consists of only one point z. Then*

$$n(z) = \frac{dim_{F(S)}(\mathcal{O}_{Z,z} \otimes_{\mathcal{O}_{S,s}} F(S))}{[k_z : k_s]}$$

where $F(S)$ is the function field of the integral scheme S.

Proposition 3.5.6 *Let $p : Z \to S$ be an equidimensional scheme of relative dimension r over an integral Noetherian geometrically unibranch scheme S. Let S' be another integral Noetherian geometrically unibranch scheme and let $f : S' \to S$ be a dominant morphism. Set $Z' = Z \times_S S'$. Let finally s' be any point of S' and s be its image in S. Then $[Z'/S']_{s'} = [Z/S]_s \otimes_{k_s} k_{s'}$.*

In other words if z' is a generic point of the fiber $Z'_{s'}$ and z is its image in Z then $n(z') = n(z) length((k_{s'} \otimes_{k_s} k_z)_{\mu_{z'}})$ where $\mu_{z'}$ is the ideal $ker(k_{s'} \otimes_{k_s} k_z \to k_{z'})$.

Proof: Replacing Z by an appropriate neighborhood of z we may assume by Proposition 2.1.3 that the morphism $Z \to S$ admits a decomposition of the form $Z \stackrel{p_0}{\to} \mathbf{A}^r_S \to S$ where p_0 is an equidimensional quasi-finite morphism. Corollary 3.5.5(1) shows that $n_{Z/S}(z) = n_{Z/\mathbf{A}^r_S}(z)$ and $n_{Z'/S'}(z') = n_{Z'/\mathbf{A}^r_{S'}}(z')$. Furthermore denoting by x (resp. x') the image of z (resp. z') in $\mathbf{A}^r_S$ (resp. $\mathbf{A}^r_{S'}$) one checks easily that the local rings $(k_{s'} \otimes_{k_s} k_z)_{\mu_{z'}}$ and $(k_{x'} \otimes_{k_x} k_z)_{\mu_{z'}}$ coincide. Thus we may replace S by $\mathbf{A}^r_S$, S' by $\mathbf{A}^r_{S'}$ and assume that $r = 0$. We certainly may also assume S, S' to be local schemes and s, s' to be their closed points.

Consider first the special case $S' = Spec(\mathcal{O}^{sh}_{S,s})$ where $\mathcal{O}^{sh}_{S,s}$ is the strict henselization of $\mathcal{O}_{S,s}$. Since $\mathcal{O}_{S',s'}$ (resp. $\mathcal{O}_{Z',z'}$) is ind-etale over $\mathcal{O}_{S,s}$ (resp. $\mathcal{O}_{Z,z}$) we conclude from Lemma 3.5.2 that $e(I\mathcal{O}_{Z',z'}) = e(I\mathcal{O}_{Z,z})$, $e(I\mathcal{O}_{S',s'}) = e(I\mathcal{O}_{S,s})$ for any ideal of definition I of $\mathcal{O}_{S,s}$ and hence $n(z) = n(z')$. On the other hand $k_{s'}$ is a separable algebraic extension of k_s. Thus $(k_{s'} \otimes_{k_s} k_z)_{\mu_{z'}}$ is a reduced local Artinian ring and hence a field, i.e.

$$length((k_{s'} \otimes_{k_s} k_z)_{\mu_{z'}}) = 1.$$

In the general case there exists a morphism $f' : Spec(\mathcal{O}^{sh}_{S',s'}) \to Spec(\mathcal{O}^{sh}_{S,s})$ such that the diagram

$$\begin{array}{ccc} Spec(\mathcal{O}^{sh}_{S',s'}) & \xrightarrow{f'} & Spec(\mathcal{O}^{sh}_{S,s}) \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

commutes. In view of the special case considered above we may replace f by f' and assume that S and S' are strictly henselian local schemes. Finally replacing Z by $Spec(\mathcal{O}_{Z,z})$ we may assume that Z is finite over S and z is its only point over s. In this situation k_z is a purely inseparable extension of k_s and hence the Artinian ring $k_{s'} \otimes_{k_s} k_z$ is local (i.e. z' is the only point of Z' over s') and its residue field coincides with $k_{z'}$. Using Corollary 3.5.5(3) we conclude that

$$n(z) = \frac{dim_{F(S)}(\mathcal{O}_{Z,z} \otimes_{\mathcal{O}_{S,s}} F(S))}{[k_z : k_s]}$$

$$n(z') = \frac{dim_{F(S')}(\mathcal{O}_{Z',z'} \otimes_{\mathcal{O}_{S',s'}} F(S'))}{[k_{z'} : k_{s'}]}$$

$$length((k_{s'} \otimes_{k_s} k_z)_{\mu_{z'}}) = length(k_{s'} \otimes_{k_s} k_z) = \frac{[k_z : k_s]}{[k_{z'} : k_{s'}]}$$

Now it is sufficient to note that

$$\mathcal{O}_{Z',z'} \otimes_{\mathcal{O}_{S',s'}} F(S') = (\mathcal{O}_{Z,z} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}) \otimes_{\mathcal{O}_{S',s'}} F(S') =$$

$$= (\mathcal{O}_{Z,z} \otimes_{\mathcal{O}_{S,s}} F(S)) \otimes_{F(S)} F(S')$$

and hence

$$dim_{F(S)}(\mathcal{O}_{Z,z} \otimes_{\mathcal{O}_{S,s}} F(S)) = dim_{F(S')}(\mathcal{O}_{Z',z'} \otimes_{\mathcal{O}_{S',s'}} F(S')).$$

Lemma 3.5.7 *Under the assumptions of Proposition 3.5.6 let Z' be a closed subscheme of Z defined by a nilpotent sheaf of ideals and such that the closed embedding $Z' \to Z$ is an isomorphism over the generic point of S. Then for any point s of S the cycles $[Z'/S]_s$ and $[Z/S]_s$ coincide.*

Proof: Let z be a generic point of the fiber Z_s. We have to show that $n_{Z/S}(z) = n_{Z'/S}(z)$. Let I be an ideal of definition of the local ring $\mathcal{O}_{S,s}$, let $\mu_1, \ldots, \mu_n$ be all minimal prime ideals of $\mathcal{O}_{Z,z}$ and let ν be the kernel of the natural surjective homomorphism $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z',z}$. The ideal ν being nilpotent is contained in $\mu_1 \cap \ldots \cap \mu_n$ and the ideals $\mu_1/\nu, \ldots, \mu_n/\nu$ are the minimal prime ideals of $\mathcal{O}_{Z',z}$. Furthermore our assumptions imply that $(\mathcal{O}_{Z,z})_{\mu_i} = (\mathcal{O}_{Z',z})_{\mu_i/\nu}$. Proposition 3.5.1 shows that

$$e(I\mathcal{O}_{Z,z}) = \sum_i e(I(\mathcal{O}_{Z,z}/\mu_i))length((\mathcal{O}_{Z,z})_{\mu_i}) =$$

$$= \sum_i e(I(\mathcal{O}_{Z',z}/(\mu_i/\nu)))length((\mathcal{O}_{Z',z})_{\mu_i/\nu}) = e(I\mathcal{O}_{Z',z}).$$

Theorem 3.5.8 *Let $X \to S$ be a scheme of finite type over an integral Noetherian geometrically unibranch scheme S. Let further Z be a closed integral subscheme of X equidimensional of relative dimension r over S and let $\mathcal{Z}$ be the corresponding element of $Cycl(X/S, r)$. Then the cycles $\mathcal{Z}_s$ and $[Z/S]_s$ coincide for any point $s \in S$ of S.*

Proof: Let k be an extension of k_s and let (x_0, x_1, R) be a fat point of S over the k-point $Spec(k) \to Spec(k_s) \to S$. To prove the theorem we have to show that $(x_0, x_1)^*(\mathcal{Z}) = [Z/S]_s \otimes_{k_s} k$. Since $Z \times_S Spec(R)$ is equidimensional over $Spec(R)$ and the closed embedding $\phi_{x_1}(Z) \to Z \times_S Spec(R)$ is an isomorphism over the generic point of $Spec(R)$ we conclude by Lemma 2.1.10 that $\phi_{x_1}(Z)$ is defined by a nilpotent sheaf of ideals. Let s' be the closed point of $Spec(R)$. Since $\phi_{x_1}(Z)$ is flat over $Spec(R)$ we see from 3.5.4(2),3.5.7 and 3.5.6 that

$$(x_0, x_1)^*(\mathcal{Z}) = cycl(\phi_{x_1}(Z) \times_{Spec(R)} Spec(k)) = [\phi_{x_1}(Z)/Spec(R)]_{s'} =$$

$$= [Z \times_S Spec(R)/Spec(R)]_{s'} = [Z/S]_s \otimes_{k_s} k.$$

For cycles equidimensional over regular Noetherian schemes the above formula for multiplicities of components of the inverse image reduces to the usual Tor-formula as one sees from the following lemma.

Lemma 3.5.9 *Let $Z \to S$ be an equidimensional scheme of relative dimension r over a regular scheme S and $s \in S$ be a point of S. If z is a generic point of the fiber Z_s of Z over s then one has*

$$n_{Z/S}(z) = \sum_{i=0}^{dim(\mathcal{O}_{S,s})} (-1)^i length_{\mathcal{O}_{Z,z}}(Tor_i^{\mathcal{O}_{S,s}}(\mathcal{O}_{Z,z}, k_s)).$$

Proof: Let $t_1, \ldots, t_k$ be a regular system of parameters of the regular local ring $\mathcal{O}_{S,s}$. Take $I = \mathcal{M} = t_1\mathcal{O}_{S,s} + \ldots + t_n\mathcal{O}_{S,s}$ to be the maximal ideal of this ring. Since $\mathcal{O}_{S,s}$ is regular the multiplicity $e(I)$ equals one (see [10]). Thus $n(z) = e(I\mathcal{O}_{Z,z})$. Theorem of Serre ([15]) shows that

$$e(I\mathcal{O}_{Z,z}) = \sum_i (-1)^i length_{\mathcal{O}_{Z,z}}(H_i(K(\underline{t}, \mathcal{O}_{Z,z})))$$

where $K(\underline{t}, \mathcal{O}_{Z,z})$ is the Koszul complex corresponding to the sequence $\underline{t} = (t_1, \ldots, t_n)$. On the other hand $K(\underline{t}, \mathcal{O}_{S,s})$ is a projective resolution of k_s over $\mathcal{O}_{S,s}$ and hence

$$H_i(K(\underline{t}, \mathcal{O}_{Z,z})) = H_i(K(\underline{t}, \mathcal{O}_{S,s}) \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Z,z}) = Tor_i^{\mathcal{O}_{S,s}}(\mathcal{O}_{Z,z}, k_s).$$

Example 3.5.10

1. Let F be a field of characteristic $p > 0$ and let $a, b \in F^*$ be elements independent modulo $(F^*)^p$. Set

$$A = F[T_0, T_1, T_2]/(aT_0^p + bT_1^p - T_2^p)$$

   and let $S = Spec(A)$. One verifies easily that A is an integrally closed domain and hence S is a normal integral scheme. Let X be the normalization of S in the field $F(S)(\gamma)$ where $\gamma^p = b/a$. It is easy to check that X is isomorphic to $Spec(F(\alpha, \beta)[T_1, T_2])$ where $\alpha^p = a$, $\beta^p = b$ and the homomorphism $A = F[S] \to F[X] = F(\alpha, \beta)[T_1, T_2]$ maps T_0 to $\alpha^{-1}T_2 - \gamma^{-1}T_1$.

   Take s to be the only singular point of S (i.e. $T_0(s) = T_1(s) = T_2(s) = 0$) and x to be the only point of X over s. Take I to be the maximal ideal of $\mathcal{O}_{S,s}$. Then $I\mathcal{O}_{X,x}$ is the maximal ideal of the regular local ring $\mathcal{O}_{X,x}$ and hence $e(I\mathcal{O}_{X,x}) = 1$. On the other hand one checks easily that $e(I) = p$ (see [10, 14.5]). Thus $n(x) = 1/p$. Therefore cycles of the form $cycl(f)(\mathcal{Z})$ do not have in general integral coefficients[1].

[1] This example is due to A.S.Merkurjev.

2. In the notations of previous example let Y be the scheme obtained by gluing of p copies of X in the singular point. One can easily see that the fundamental cycle on Y is an element of $c(Y/S,0)$ which can not be represented as a sum of cycles which correspond to integral subschemes of Y.

3.6 *Functoriality of Chow presheaves.*

Let $f: S_1 \to S_2$ be a morphism of Noetherian schemes. We say that a closed subscheme Z of S_1 is proper with respect to f if the restriction of f to Z is a proper morphism. We say that a point s of S_1 is proper with respect to f if the closure of s in S_1 which we consider as a reduced closed subscheme is proper whith respect to f.

Let S be a Noetherian scheme and $f: X \to Y$ be a morphism of schemes of finite type over S. Let further $\mathcal{Z} = \sum n_i z_i$ be a cycle on X which lies over generic points of S. We say that $\mathcal{Z}$ is proper with respect to f if all the points z_i are proper with respect to f. We define then a cycle $f_*(\mathcal{Z})$ on Y as the sum $\sum n_i m_i f(z_i)$ where m_i is the degree of the field extension $k_{f(z_i)}/k_{z_i}$ if this extension is finite and zero otherwise.

Theorem 3.6.1 *Let S be a Noetherian scheme, $p: X_1 \to X_2$ be a morphism of schemes of finite type over S, and $f: S' \to S$ be a Noetherian scheme over S. Set $X_i' = X_i \times_S S'$ $(i = 1,2)$ and denote by $p': X_1' \to X_2'$ be the corresponding morphism over S'. Let further $\mathcal{Z} = \sum n_i Z_i$ (resp. $\mathcal{W} = \sum m_j W_j$) be an element of $\mathbf{Z}(Hilb(X_1/S,r))$ (resp. of $\mathbf{Z}(Hilb(X_2/S,r))$). Assume that the closed subschemes Z_i are proper with respect to p and*

$$p_*(cycl_{X_1}(\mathcal{Z})) = cycl_{X_2}(\mathcal{W}).$$

Then the cycle $cycl_{X_1'}(\mathcal{Z} \times_S S')$ is proper with respect to p' and we have

$$p'_*(cycl_{X_1'}(\mathcal{Z} \times_S S')) = cycl_{X_2'}(\mathcal{W} \times_S S').$$

Proof: Replacing X_1 by $\cup Z_i$ and X_2 by $(\cup W_j) \cup (\cup p(Z_i))$ we may assume that p is proper, X_1 is equidimensional of relative dimension r over S and all fibers of X_2 over S are of dimension $\leq r$.

Both cycles in question are linear combinations of points of X_2' which lie over generic points of S', are generic in their fibers over S' and are of dimension r in these fibers. Let $\eta' \in X'_2$ be any of such points and let η be its image in X_2. Computing the multiplicities of η' in our cycles we may replace X_2 by any open neighborhood of η. Let V be an irreducible component of X_2 which is not equidimensional of dimension r over S. Then $V \subset p(Z_i)$ for a certain i and hence V is dominated by a component of Z_i. Proposition 2.1.9 shows that all fibers of V over S are of dimension $< r$ and hence η does not belong to V. Thus throwing away bad components we may assume that X_2 is equidimensional of relative dimension r over S. Proposition 2.1.9 shows also that we may assume that p is a finite morphism. According to Proposition 2.1.3 we may assume further that the morphism $X_2 \to S$ has a factorization of the form $X_2 \to \mathbf{A}^r_S \to S$ where the first arrow is an equidimensional quasi-finite morphism. Since the morphism p maps components of X_1 onto components of X_2 we conclude that the composition $X_1 \to X_2 \to \mathbf{A}^r_S$ is also equidimensional (and quasifinite). Let Z_i^0 (resp. W_j^0) be the closed subset of Z_i (resp. of W_j) consisting of points where Z_i (resp. W_j) is not flat over $\mathbf{A}^r_S$. Lemma 3.2.3 shows that Z_i^0 and W_j^0 contain no points generic in their fibers over S. Thus η does not belong to $(\cup W_j^0) \cap (\cup p(Z_i^0))$ and shrinking X_2 further around η we may assume that Z_i and W_j are flat over $\mathbf{A}^r_S$. This shows that we may replace S by $\mathbf{A}^r_S$ and assume that $r = 0$.

Let τ' (resp. τ) be the image of η' (resp. η) in S' (resp. S). We may replace S' by $Spec(\mathcal{O}_{S',\tau'})$ and assume that S' is a local Artinian scheme. Let $\mathcal{O}^{sh}_{S',\tau'}$ and $\mathcal{O}^{sh}_{S,\tau}$ be the strict henselizations of the local rings $\mathcal{O}_{S',\tau'}$ and $\mathcal{O}_{S,\tau}$ respectively. Find a morphism $f_0 : Spec(\mathcal{O}^{sh}_{S',\tau'}) \to Spec(\mathcal{O}^{sh}_{S,\tau})$ making the following diagram commute

$$\begin{array}{ccc} Spec(\mathcal{O}^{sh}_{S',\tau'}) & \stackrel{f_0}{\to} & Spec(\mathcal{O}^{sh}_{S,\tau}) \\ \downarrow & & \downarrow \\ Spec(\mathcal{O}_{S',\tau'}) = S' & \to & S. \end{array}$$

In view of Lemma 2.3.1 it is sufficient to check that the flat pull-backs of cycles in question to the scheme $X'_2 \times_{S'} Spec(\mathcal{O}^{sh}_{S',\tau'})$ coincide. Proposition 2.3.4 shows that these pull-backs are equal to $p'_*(cycl(\mathcal{Z} \times_S Spec(\mathcal{O}^{sh}_{S',\tau'})))$ and $cycl(\mathcal{W} \times_S Spec(\mathcal{O}^{sh}_{S',\tau'}))$ respectively. Thus we may replace S' by $Spec(\mathcal{O}^{sh}_{S',\tau'})$ in the same way we may replace S by $Spec(\mathcal{O}^{sh}_{S,\tau})$ and thus assume that S and S' are strictly henselian local schemes and τ, τ' are their closed points. Finally replacing X_2 by $Spec(\mathcal{O}_{X_2,\eta})$ we may assume that X_2 is a local scheme finite over S. Since S is strictly local it implies that X'_2 is also local i.e. η' is the only point of X_2'. Let $\alpha_1', \ldots, \alpha_k'$ be all the points of X_1'. Using Lemma 3.2.1 we see that the multiplicity of η' in the cycle $p'_*(cycl_{X_1'}(\mathcal{Z} \times_S S'))$ is equal to

$$\sum_{l=1}^{k} [k_{\alpha_l'} : k_{\eta'}] \sum_i length \mathcal{O}_{Z_i', \alpha_l'} =$$

$$= \sum_i \frac{n_i}{[k_{\eta'} : k_{\tau'}]} \sum_l [k_{\alpha_l'} : k_{\tau'}] length \mathcal{O}_{Z_i', \alpha_l'} =$$

$$= \frac{length \mathcal{O}_{S', \tau'}}{[k_{\eta'} : k_{\tau'}]} \sum_i n_i deg(Z_i/S).$$

On the other hand the multiplicity of η' in the cycle $cycl_{X_2'}(\mathcal{W} \times_S S')$ is equal to

$$\sum_j m_j length \mathcal{O}_{W_j', \eta'} = \frac{length \mathcal{O}_{S', \tau'}}{[k_{\eta'} : k_{\tau'}]} \sum_j m_j deg(W_j/S).$$

Thus we have to show that $\sum n_i deg(Z_i/S) = \sum m_j deg(W_j/S)$. To do so choose a generic point τ^0 of S, let $\alpha_1^0, \ldots, \alpha_n^0$ be all points of X_2 over τ^0 and for each $s = 1, \ldots, n$ let $\alpha_{st}^0 (t = 1, \ldots, n_s)$ be the points of X_1 over α_s^0. Our assumption imply that for every $s = 1, \ldots, n$ we have the following equality

$$\sum_t [k_{\alpha_{st}^0} : k_{\alpha_s^0}] \sum_i length \mathcal{O}_{Z_i, \alpha_{st}^0} = \sum_j m_j length \mathcal{O}_{W_j, \alpha_s^0}.$$

Taking the sum of this equalities with coefficients $[k_{\alpha_s^0} : k_{\tau^0}]$ and using once again Lemma 3.2.1 we get what we wanted.

Proposition 3.6.2 *Let $p : X \to Y$ be a morphism of schemes of finite type over a Noetherian scheme S and $\mathcal{Z} = \sum n_i z_i$ be an element of $Cycl(X/S, r)$ such that the points z_i are proper with respect to p. Then the following statements hold:*

1. *The cycle $p_*(\mathcal{Z})$ on Y belongs to $Cycl(Y/S, r)$.*

2. *For any morphism $f : S' \to S$ of Noetherian schemes the cycle $cycl(f)(\mathcal{Z})$ has the form $\sum m_j z_j'$ where the points z_j' are proper with respect to $p' = p \times_S S'$ and moreover*

$$p'_*(cycl(f)(\mathcal{Z})) = cycl(f)(p_*(\mathcal{Z})).$$

Proof: Denote by Z_i (resp. W_i) the closure of z_i (resp. $p(z_i)$) considered as an integral closed subscheme of X (resp. of Y). Replacing X by $\cup Z_i$ we may assume that the morphism p is proper. Let k be a field, $x : Spec(k) \to S$ be a k-point of S and (x_0, x_1, R) be a fat point of S over x. Set

$$\mathcal{Z}_0 = \sum n_i \phi_{x_1}(Z_i) \in \mathbf{Z}(\mathrm{Hilb(X \times_S Spec(R)/Spec(R), r)})$$

$$\mathcal{W}_0 = \sum n_i m_i \phi_{x_1}(W_i) \in \mathbf{Z}(\mathrm{Hilb(Y \times_S Spec(R)/Spec(R), r)})$$

where $m_i = [k_{z_i} : k_{p(z_i)}]$ if this field extension is finite and zero otherwise.

It is clear from the definitions that $cycl(\mathcal{W}_0) = (p \times_S Spec(R))_*(cycl(\mathcal{Z}_0))$. Theorem 3.6.1 implies now that

$$(x_0, x_1)^*(p_*(\mathcal{Z})) = cycl(\mathcal{W}_0 \times_{Spec(R)} Spec(k)) =$$

$$= (p \times_S Spec(k))_* cycl(\mathcal{Z}_0 \times_{Spec(R)} Spec(k)) = (p \times_S Spec(k))_*((x_0, x_1)^*(\mathcal{Z})).$$

Thus the cycle $(x_0, x_1)^*(p_*(\mathcal{Z}))$ is independent of the choice of fat point (x_0, x_1, R) over x. The same argument shows now that for any morphism $f : S' \to S$ of Noetherian schemes the cycle $p'_*(cycl(f)(\mathcal{Z}))$ meets the property defining the cycle $cycl(f)(p_*(\mathcal{Z}))$ and hence is equal to this cycle.

Corollary 3.6.3 *Let S be a Noetherian scheme and $f : X \to Y$ be a morphism (resp. a proper morphism) of schemes of finite type over S. Then there are homomorphisms:*

$$f_* : c(X/S, r) \to c(Y/S, r)$$

$$f_* : c_{equi}(X/S, r) \to c_{equi}(Y/S, r)$$
$$f_* : c^{eff}(X/S, r) \to c^{eff}(Y/S, r)$$

(resp. homomorphisms

$$f_* : z(X/S, r) \to z(Y/S, r)$$

$$f_* : z_{equi}(X/S, r) \to z_{equi}(Y/S, r)$$
$$f_* : z^{eff}(X/S, r) \to z^{eff}(Y/S, r))$$

such that for any composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of schemes of finite type over S one has $(gf)_* = g_* f_*$.

Let us consider now the contravariant functoriality of Chow presheaves.

Lemma 3.6.4 *Let S be a Noetherian scheme and $f : X \to Y$ be a flat equidimensional morphism of dimension n of schemes of finite type over S. Then for any element $\mathcal{Z}$ in $Cycl(Y/S,r)\otimes\mathbf{Q}$ one has $f^*(\mathcal{Z}) \in Cycl(X/S, r+n)$ and for any Noetherian scheme $g : S' \to S$ we have*

$$cycl(g)(f^*(\mathcal{Z})) = (f \times_S S')^*(cycl(g)(\mathcal{Z})).$$

Proof: Easy.

Let S be a Noetherian scheme, $f : X \to Y$ be a flat (resp. flat and proper) equidimensional morphism of relative dimension n of schemes of finite type over S and $F(-,-)$ be one of the presheaves $z(-,-)$, $z^{eff}(-,-)$, $z_{equi}(-,-)$ (resp. $c(-,-)$, $c^{eff}(-,-)$, $c_{equi}(-,-)$). If $\mathcal{Z}$ is a cycle on Y which belongs to $F(Y/S,r)$ then by Lemma 3.6.4 the cycle $f^*(\mathcal{Z})$ belongs to $F(X/S, r+n)$ and this construction gives us homomorphisms of presheaves

$$f^* : F(Y/S,r) \to F(X/S, r+n).$$

For any composable pair $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$ of flat (resp. flat and proper) equidimensional morphisms of schemes of finite type over S we obviously have $(gf)^* = f^*g^*$.

Proposition 3.6.5 *Let S be a Noetherian scheme. Consider a pull-back square of schemes of finite type over S of the form:*

$$\begin{array}{ccc} Y' & \stackrel{g}{\to} & Y \\ p' \downarrow & & \downarrow p \\ X' & \stackrel{f}{\to} & X \end{array}$$

such that the morphism f is flat and equidimensional of dimension r. Assume further that f is proper and $F(-,-)$ is one of the presheaves $c(-,-)$, $c^{eff}(-,-)$, $c_{equi}(-,-)$ or that p is proper and $F(-,-)$ is one of the presheaves $z(-,-)$, $z^{eff}(-,-)$, $z_{equi}(-,-)$.

Then the following diagram of presheaves commutes:

$$\begin{array}{ccc} F(Y/S,n) & \stackrel{g^*}{\to} & F(Y'/S, n+d) \\ p_* \downarrow & & \downarrow p'_* \\ F(X/S,n) & \stackrel{f^*}{\to} & F(X'/S, n+d) \end{array}$$

Proof: It follows immediately from our definitions and 2.3.4.

As an application of our construction of push-forward homomorphisms we will show that for finite cycles over normal schemes our construction of base change homomorphisms gives the same answer as the one used in [16]. To start we will recall briefly the latter construction. For an integral scheme X we denote by $F(X)$ its field of functions.

Definition 3.6.6 *A finite surjective morphism of integral Noetherian schemes $f : Y \to S$ is called a pseudo-Galois covering if the field extension $F(Y)/F(S)$ is normal and the canonical homomorphism*

$$Aut_S(Y) \to Aut_{F(S)}(F(Y)) = Gal(F(Y)/F(S))$$

is an isomorphism.

Let S be a normal integral Noetherian scheme, X be an integral scheme and $p : X \to S$ be a finite surjective morphism. Let $g : S' \to S$ be any Noetherian integral scheme over S. Denote by X'_i the irreducible components of $X' = X \times_S S'$ and by x_i (resp. x) denote the generic point of X'_i (resp. of X). Since any normal scheme is geometrically unibranch Theorem 3.4.2 implies that $x \in Cycl_{equi}(X/S, 0)$. Consider the cycle $cycl(g)(x) = \sum n_i x'_i$.

Assume that there exists[2] a pseudo-Galois covering $f : Y \to S$ and an S-morphism $q : Y \to X$. Let G be the Galois group $Gal(F(Y)/F(S)) = Aut_S(Y)$. Denote by Y'_j the irreducible components of $Y' = Y \times_S S'$. It is easy to check that the action of G permutes the components Y'_j transitively so that in particular the field extensions $F(Y'_j)/F(S')$ are all isomorphic. Denote by $l(i)$ the number of components Y'_j lying over X'_i and by l the total number of components Y'_j.

Proposition 3.6.7 *In the above notations one has:*

$$n_i = \frac{[F(X) : F(S)]l(i)}{[F(X'_i) : F(S')]l}.$$

[2]Such a covering always exists. For an excellent scheme S it follows trivially from the finitness of normalizations of S in finite extensions of its field of functions. The proof in general case is a little more complicated.

Proof: Denote the generic point of Y (resp. of Y'_j) by y (resp. by y'_j). The cycle y is in $Cycl(Y/S, 0)$ by Theorem 3.4.2 and has the following obvious properties:

1. $f_*(y) = [F(Y) : F(S)]s$ where s is the generic point of S.
2. $q_*(y) = [F(Y) : F(X)]x$
3. $\sigma_*(y) = y$ for any $\sigma \in G$.

Consider the cycle $cycl(g)(y) = \sum m_j y'_j$. Proposition 3.6.2 shows that $(f \times_S S')_*(cycl(g)(y)) = [F(Y) : F(S)]s$ i.e. $\sum m_j [F(Y'_j) : F(S')] = [F(Y) : F(S)]$. Moreover for any $\sigma \in G$ we have

$$(\sigma \times_S S')_*(cycl(g)(y)) = cycl(g)(\sigma_*(y)) = cycl(g)(y).$$

Since the action of G on the set $y'_1, \dots, y'_l$ is transitive we conclude that all multiplicities m_j are the same and equal to $\frac{[F(Y):F(S)]}{l[F(Y'_j):F(S')]}$. Finally $cycl(g)(x) = \frac{1}{[F(Y):F(X)]}(q \times_S S')_*(cycl(g)(y))$ and hence

$$n_i = \frac{1}{[F(Y) : F(X)]} \sum_{y'_j/x'_i} m_j [F(Y'_j) : F(X'_i)] = \frac{[F(X) : F(S)]l(i)}{[F(X'_i) : F(S')]l}.$$

3.7 *Correspondence homomorphisms.*

Let $Y \to X$ be a scheme of finite type over a Noetherian scheme X. For any cycle $\mathcal{Y} \in Cycl(Y/X, r) \otimes \mathbf{Q}$ define a homomorphism

$$Cor(\mathcal{Y}, -) : Cycl(X) \otimes \mathbf{Q} \to \mathrm{Cycl(Y)} \otimes \mathbf{Q}$$

as follows. Let $\mathcal{X} = \sum n_i x_i$ be an element of $Cycl(X) \otimes \mathbf{Q}$. Denote by X_i the closure of the point x_i which we consider as an integral closed subscheme of X. Let $i_{X_i} : X_i \to X$ be the corresponding embedding. We set

$$Cor(\mathcal{Y}, \mathcal{X}) = \sum n_i (X_i \times_X Y \to Y)_*(cycl(i_{X_i})(\mathcal{Y})).$$

Lemma 3.7.1 *Consider a pull-back square of Noetherian schemes of the form*

$$\begin{array}{ccc} Y' & \stackrel{q}{\to} & Y \\ \downarrow & & \downarrow \\ X' & \stackrel{p}{\to} & X. \end{array}$$

Assume that the morphism $Y \to X$ is of finite type and let $\mathcal{Y}$ be an element of $Cycl(Y/X, r) \otimes \mathbf{Q}$ and $\mathcal{X}'$ be a cycle on X' which is proper with respect to p. Then one has:

$$Cor(\mathcal{Y}, p_*(\mathcal{X}')) = q_*(Cor(cycl(p)(\mathcal{Y}), \mathcal{X}')).$$

Proof: We may assume that $\mathcal{X}' = cycl_{X'}(Z')$ where Z' is an integral closed subscheme of X' and $p_{Z'} : Z' \to X$ is a proper morphism. We have

$$q_*(Cor(cycl(p)(\mathcal{Y}), \mathcal{X}')) = (Z' \times_X Y \to Y)_* cycl(Z' \to X)(\mathcal{Y})$$

and

$$Cor(\mathcal{Y}, p_*(\mathcal{X}')) = n(p(Z') \times_X Y \to Y)_* cycl(p(Z') \to X)(\mathcal{Y})$$

where $n = [F(Z') : F(p(Z'))]$ if this extension is finite and zero otherwise. We have further

$$(Z' \times_X Y \to Y)_* cycl(Z' \to X)(\mathcal{Y}) =$$

$$= (p(Z') \times_X Y \to Y)_* (q \times_X Y)_* cycl(q) cycl(p(Z') \to X)(\mathcal{Y}).$$

where q is the morphism $Z' \to p(Z')$. It is sufficient to show that

$$(Z' \times_X Y \to p(Z') \times_X Y)_* cycl(Z' \to p(Z'))(\mathcal{W}) = n(\mathcal{W})$$

for any element $\mathcal{W}$ in $Cycl(p(Z') \times_X Y/p(Z'))$. To prove the last statement we may replace $p(Z')$ by its generic point $Spec(F(Z'))$ and assume that $\mathcal{W} = cycl(W)$ where W is an integral closed subscheme of $Y \times_X Spec(F(Z'))$. For an infinite field extension $F(Z')/F(p(Z'))$ we immediately conclude that the left hand side of our equality equals zero (as well as the right hand side). For finite field extension our equality follows from Lemmas 3.3.12 and 2.3.5.

The following lemma is straightforward.

Lemma 3.7.2 *Consider a pull-back square of Noetherian schemes of the form*

$$\begin{array}{ccc} Y' & \stackrel{q}{\to} & Y \\ \downarrow & & \downarrow \\ X' & \stackrel{p}{\to} & X. \end{array}$$

Assume that the morphism $Y \to X$ is of finite type, schemes X' and X are reduced and the morphism p is flat. Then for any cycle $\mathcal{X}$ in $Cycl(X) \otimes \mathbf{Q}$ and any cycle $\mathcal{Y}$ in $Cycl(Y/X, r) \otimes \mathbf{Q}$ one has:

$$q^* Cor(\mathcal{Y}, \mathcal{X}) = Cor(q^*(\mathcal{Y}), p^*(\mathcal{X})).$$

Theorem 3.7.3 *Let S be a Noetherain scheme and $f : Y \to X$ be a morphism of schemes of finite type over S. Let further $\mathcal{Y} = \sum n_i y_i$ be an element of $Cycl(Y/X, n) \otimes \mathbf{Q}$ and $\mathcal{X}$ be an element of $Cycl(X/S, m) \otimes \mathbf{Q}$. Then the element $Cor(\mathcal{Y}, \mathcal{X})$ belongs to the group $Cycl(Y/S, n+m)$. Moreover for any Noetherian scheme $g : S' \to S$ over S one has*

$$cycl(g)(Cor(\mathcal{Y}, \mathcal{X})) = Cor(cycl(g \times_S X)(\mathcal{Y}), cycl(g)(\mathcal{X})).$$

Proof: We will first consider the following particular case of our theorem.

Lemma 3.7.4 *The statement of the theorem holds if $\mathcal{Y} = cycl_Y(Y_0)$, $\mathcal{X} = cycl_X(X_0)$ where Y_0 (resp. X_0) is a closed subscheme of Y (resp. of X) which is flat over X (resp. over S).*

Proof: Set $X' = X \times_S S'$, $Y' = Y \times_S S'$. By Lemma 3.3.10 $Cor(\mathcal{Y}, \mathcal{X})$ coincides with the cycle associated with the closed subscheme $Y_0 \times_X X_0$ in Y which is clearly flat over S. Therefore $Cor(\mathcal{Y}, \mathcal{X})$ is a relative cycle over S by 3.2.5. We have further

$$cycl(g)(Cor(\mathcal{Y}, \mathcal{X})) = cycl(g)(cycl_Y(Y_0 \times_X X_0)) = cycl_{Y'}((Y_0 \times_X X_0) \times_S S')$$

and

$$Cor(cycl(g \times_S X)(\mathcal{Y}), cycl(g)(\mathcal{X})) = Cor(cycl_{Y'}(Y_0 \times_S S'), cycl_{X'}(X_0 \times_S S')) =$$

$$= cycl_{Y'}((Y_0 \times_S S') \times_{X'} (X_0 \times_S S'))$$

Since

$$(Y_0 \times_S S') \times_{X'} (X_0 \times_S S') \cong (Y_0 \times_X X_0) \times_S S'$$

we conclude that our equality holds.

Let k be a field, $x : Spec(k) \to S$ be a k-valued point of S and (x_0, x_1, R) be a fat point of S over x. It is clearly sufficient to show that

$$(x_0, x_1)^*(Cor(\mathcal{Y}, \mathcal{X})) = Cor(cycl(x \times_S X)(\mathcal{Y}), cycl(x)(\mathcal{X})).$$

Let (x_0, Id) be the obvious fat point of $Spec(R)$. By Lemma 3.7.2 (and Lemma 3.3.12) we have

$$(x_0, x_1)^*(Cor(\mathcal{Y}, \mathcal{X})) = (x_0, Id)^*(Cor(cycl(x_1 \times_S X)(\mathcal{Y}), cycl(x_1)(\mathcal{X}))).$$

We may assume now that $S = Spec(R)$ where R is a discrete valuation ring. In this case all cycles which are formal linear combinations of points over the generic point of S are relative cycles over S (by Corollary 3.2.6). Thus we may assume that $\mathcal{X} = cycl_X(X_0)$ where X_0 is a closed integral subscheme of X which is equidimensional of relative dimension m over S and we have to show that

$$cycl(x)(Cor(\mathcal{Y}, \mathcal{X})) = (Cor(cycl(x \times_S X)(\mathcal{Y}), cycl(x)(\mathcal{X}))).$$

Denote by i the closed embedding $X_0 \to X$. We set

$$\begin{array}{ll} Y_0 = Y \times_Y X_0 & \\ X_x = X \times_S Spec(k) & Y_x = Y \times_S Spec(k) \\ (X_0)_x = X_0 \times_S Spec(k) & (Y_0)_x = Y_0 \times_S Spec(k) \\ u = pr_1 : Y_0 \to Y & u_x = u \times_S Spec(k) : (Y_0)_x \to Y_x \\ v = pr_1 : X_x \to X & v_0 = pr_1 : (X_0)_x \to X_0 \end{array}$$

We have

$$x^*(Cor(\mathcal{Y}, \mathcal{X})) = x^*(u_*(cycl(i)(\mathcal{Y}))) =$$

$$= ((u_x)_*(cycl(x)(cycl(i)(\mathcal{Y}))) = (u_x)_*(cycl(x)(Cor(cycl(i)(\mathcal{Y}), cycl_{X_0}(X_0))))$$

and

$$Cor(cycl(v)(\mathcal{Y}), cycl(x)(\mathcal{X})) =$$

$$= (u_x)_*(Cor(cycl(v_0)cycl(i)(\mathcal{Y}), cycl(x)(cycl_{X_0}(X_0)))).$$

We may replace now X by X_0 and assume further that X is integral and equidimensional of relative dimension m over S and $\mathcal{X}$ is the fundamental cycle of X.

Denote by Y_i the closures of the points y_i which we consider as integral closed subschemes of Y. Let $f_X : \tilde{X} \to X$ be a blow-up of X_{red} such that the proper transforms of Y_i with respect to f_X are flat and equidimensional of relative dimension n over $\tilde{X}$. Consider the pull-back square

$$\begin{array}{ccc} \tilde{Y} & \stackrel{f_Y}{\to} & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \stackrel{f_X}{\to} & X. \end{array}$$

Let $\tilde{\mathcal{X}}$ be the fundamental cycle of $\tilde{X}$. Since S is the spectrum of a discrete valuation ring we have $\tilde{\mathcal{X}} \in Cycl(\tilde{X}/S, m)$. Consider the following diagram of abelian groups (we write $Cycl(-)$ instead of $Cycl(-,-) \otimes \mathbf{Q}$):

$$\begin{array}{ccccccc} Cycl(Y/X) & \stackrel{cycl(f_X)}{\to} & Cycl(\tilde{Y}/\tilde{X}) & \stackrel{Cor(-,\tilde{\mathcal{X}})}{\to} & Cycl(\tilde{Y}/S) & \stackrel{(f_Y)_*}{\to} & Cycl(Y/S) \\ \downarrow cycl(j) & & \downarrow cycl(\tilde{j}) & & \downarrow cycl(x) & & \downarrow cycl(x) \\ Cycl(Y_x/X_x) & \stackrel{cycl(f_{X_x})}{\to} & Cycl(\tilde{Y}_x/\tilde{X}_x) & \stackrel{Cor(-,\tilde{\mathcal{X}}_x)}{\to} & Cycl(\tilde{Y}_x/S_x) & \stackrel{(f_{Y_x})_*}{\to} & Cycl(Y_x/S_x) \end{array}$$

where

$$\begin{array}{ll} S_s = Spec(k) & \\ X_x = X \times_S Spec(k) & Y_x = Y \times_S Spec(k) \\ \tilde{X}_x = \tilde{X} \times_S Spec(k) & \tilde{Y}_x = \tilde{Y} \times_S Spec(k) \end{array}$$

and

$$j : X_x \to X \quad \tilde{j} : \tilde{X}_x \to \tilde{X}$$

are the obvious morphisms and $\tilde{\mathcal{X}}_x = cycl(x)(\tilde{\mathcal{X}})$.

The composition of the upper horizontal arrows equals $Cor(-, \mathcal{X})$ by Lemma 3.7.1. By Proposition 3.6.2

$$(\tilde{X}_x \to \tilde{X})_*(\tilde{\mathcal{X}}_x) = cycl(x)(\mathcal{X})$$

and thus the composition of lower horizontal arrows equals $Cor(-, cycl(x)(\mathcal{X}))$ by Lemma 3.7.1. We have only to show now that our diagram is commutative on $\mathcal{Y} \in Cycl(Y/X)$. The first square is commutative since $f_X \circ \tilde{j} = j \circ f_{X_x}$. The last square is commutative by Proposition 3.6.2. Finally the middle square is commutative on $cycl(f_X)(\mathcal{Y})$ by our choice of the blow-up $\tilde{X} \to X$ and Lemma 3.7.4.

Corollary 3.7.5 *Let S be a Noetherian scheme and $X_1 \to X_2 \stackrel{p}{\to} S$ be a morphism of schemes of finite type over S. Let $F(-,-)$ be one of the presheaves $z(-,-), z^{eff}(-,-), z_{equi}(-,-), c(-,-), c_{equi}(-,-), c^{eff}(-,-)$. Then for any $n, m \geq 0$ there is a canonical morphism of presheaves of the form*

$$Cor_{Y/X} : p_*(F(X_1/X_2, n)) \otimes F(X_2/S, m) \to F(X_1/S, m+n)$$

Lemma 3.7.1 implies immediately the following proposition.

Proposition 3.7.6 *Let S be a Noetherian scheme. Consider a pull-back square of schemes of finite type over S of the form:*

$$\begin{array}{ccc} Y' & \stackrel{g}{\to} & Y \\ \downarrow & & \downarrow \\ X' & \stackrel{f}{\to} & X \end{array}$$

Denote by $p : X \to S$, $p' : X' \to S$ the structural morphisms of X and X' respectively. Assume further that $F(-)$ is one of the presheaves $c(-,-)$, $c^{eff}(-,-)$, $c_{equi}(-,-)$ or that f is proper and $F(-,-)$ is one of the presheaves $z(-,-)$, $z^{eff}(-,-)$, $z_{equi}(-,-)$.

Then the following diagram of morphisms of presheaves commutes:

$$\begin{array}{ccccc} p_*F(Y/X) \otimes F(X'/S) & \stackrel{Id \otimes f_*}{\to} & p_*F(Y/X) \otimes F(X/S) & \searrow & \\ \downarrow & & & & F(Y/S) \\ p'_*F(Y'/X') \otimes F(X'/S) & \stackrel{Cor_{Y'/X'}}{\to} & F(Y'/S) & \nearrow & \end{array}$$

Proposition 3.7.7 *Let $Z \to Y \to X$ be a composable pair of morphisms of finite type of Noetherian schemes. Let further $\mathcal{X}$ be an element of $Cycl(X) \otimes$ $\mathbf{Q}$ and $\mathcal{Z}, \mathcal{Y}$ be elements of $Cycl(Z/Y, n) \otimes \mathbf{Q}$ and $Cycl(Y/X, m) \otimes \mathbf{Q}$ respectively. Then one has*

$$Cor(Cor(\mathcal{Z}, \mathcal{Y}), \mathcal{X}) = Cor(\mathcal{Z}, Cor(\mathcal{Y}, \mathcal{X})).$$

Proof: It follows easily from definitions and Theorem 3.7.3.

Corollary 3.7.8 *Let S be a Noetherian scheme and $Z \stackrel{f}{\to} Y \stackrel{g}{\to} X$ be a composable pair of morphisms of schemes of finite type over S. Denote by*

$h : Z \to S$ the structural morphism and let $F(-)$ be one of the presheaves $z(-,-), z^{eff}(-,-), z_{equi}(-,-), c(-,-), c_{equi}(-,-), c^{eff}(-,-)$.
Then the following diagram of morphisms commutes:

$$\begin{array}{ccc} h_*g_*F(Z/Y) \otimes h_*F(Y/X) \otimes F(X/S) & \to & h_*F(Z/X) \otimes F(X/S) \\ \downarrow & & \downarrow \\ h_*g_*F(Z/Y,n) \otimes F(Y/S) & \to & F(Z/S). \end{array}$$

Existence of correspondence homomorphisms for relative cycles allows us to construct the homomorphism of external product (see [4]). Let $p_X : X \to S$, $p_Y : Y \to S$ be two schemes of finite type over a Noetherian scheme S. Let further $F(-,-)$ be one of the groups $z(-,-)$, $z_{equi}(-,-)$, $z^{eff}(-,-)$, $c(-,-)$, $c_{equi}(-,-)$, $c^{eff}(-,-)$. We define the external product homomorphism

$$F(X/S,n) \otimes F(Y/S,m) \to F(X \times_S Y/S, n+m)$$

as the following composition

$$F(X/S,n)\otimes F(Y/S,m) \stackrel{cycl(p_Y)\otimes Id}{\to} F(X\times_S Y/Y,n)\otimes F(Y/S,m) \stackrel{Cor_{X\times_S Y/Y}}{\to} F(X\times_S Y/S,n+m).$$

One can verify easily using results of this section that it satisfies all the standard properties of external products of cycles and defines a homomorphism of the corresponding Chow presheaves.

4 Chow sheaves in the h-topologies.

4.1 *The h-topologies.*

In this section we will remind briefly the definitions and some basic properties of three Grothendieck topologies (the h-topology, the qfh-topology and the cdh-topology) on the categories of schemes. For more information on the h- and the qfh-topologies see [18] or [16].

A morphism of schemes $p : X \to Y$ is called a topological epimorphism if the underlying Zariski topological space of Y is a quotient space of the underlying Zariski topological space of X (i.e. p is surjective and a subset A of Y is open if and only if $p^{-1}(A)$ is open in X), p is called a universal topological epimorphism if for any Z/Y the morphism $p_Z : X \times_Y Z \to Z$ is a topological epimorphism.

An h-covering of a scheme X is a finite family of morphisms of finite type $\{p_i : X_i \to X\}$ such that $\coprod p_i : \coprod X_i \to X$ is a universal topological epimorphism.

A qfh-covering of a scheme X is an h-covering $\{p_i\}$ such that all the morphisms p_i are quasi-finite.

h-coverings (resp. qfh-coverings) define a pretopology on the category of schemes, the h-topology (resp. the qfh-topology) is the associated topology.

The definition of the cdh-topology is a little less natural. Namely the cdh-topology on the category of schemes is the minimal Grothendieck topology such that the following two types of coverings are cdh-coverings.

1. Nisnevich coverings, i.e. etale coverings $\{U_i \stackrel{p_i}{\to} X\}$ such that for any point x of X there is a point x_i on one of the U_i such that $p_i(x_i) = x$ and the morphism $Spec(k_{x_i}) \to Spec(k_x)$ is an isomorphism.

2. Coverings of the form $X' \coprod Z \stackrel{p_{X'} \coprod p_Z}{\to} X$ such that $p_{X'}$ is a proper morphism, p_Z is a closed embedding and the morphism $p_{X'}^{-1}(X - p_Z(Z)) \to X - p_Z(Z)$ is an isomorphism.

Obviously the h-topology is stronger than both the qfh- and the cdh-topology, the cdh-topology is stronger than the Zariski topology and standard results on flat morphisms show that qfh-topology is stronger than the flat topology.

Lemma 4.1.1 *Let $\{U_i \xrightarrow{p_i} X\}$ be an h-covering of a Noetherian scheme X. Denote by $\coprod V_j$ the disjoint union of all irreducible components of $\coprod U_i$ which dominate an irreducible component of X. Then the morphism $q : \coprod V_j \to X$ is surjective.*

Proof: See [18].

Remark: In fact the property of h-coverings considered in Lemma 4.1.1 is characteristic. Namely one can show that a morphism of finite type $X \to S$ of Noetherian schemes is an h-covering if and only if for any Noetherian scheme T over S the union of irreducible components of $X \times_S T$ which dominate irreducible components of T is surjective over T.

Proposition 4.1.2 *Let X be a Noetherian scheme and $\mathbf{U} = \{U_i \to X\}$ be an h-covering of X. Then there is a refinement $\{V_j \to X\}$ of $\mathbf{U}$ such that each morphism $q_j : V_j \to X$ admits a decomposition of the form $V_j \xrightarrow{q_j^f} W_j \xrightarrow{q_j^p} X$ such that q_j^f is a faithfully flat morphism, W_j is irreducible and q_j^p is an abstract blow-up (see Defenition 2.2.4) of an irreducible component of X.*

Proof: To prove our proposition we may assume that X is integral and our covering is of the form $U \to X$. By Theorem 2.2.2 there is a blow-up $p : W \to X$ such that the proper transform $\tilde{U}$ of U is flat over W. It is sufficient to show that the morphism $\tilde{U} \to W$ is surjective. Let Z be a closed subscheme in X such that $Z \neq X$ and the morphism $p : W \to X$ is an isomorphism outside Z. Since $W \times_X U \to W$ is an h-covering and the closure of the complement $W \times_X U - \tilde{U}$ lies over $p^{-1}(Z)$ and, therefore is not dominant over any irreducible component of W, the surjectivity of the morphism $\tilde{U} \to W$ follows from Lemma 4.1.1.

Lemma 4.1.3 *Let S be a Noetherian scheme and $p : X \to S$ be a scheme of finite type over S. Suppose that there is an h-covering $f : S' \to S$ such that the scheme $X' = X \times_S S'$ is proper over S'. Then X is proper over S.*

Proof: Denote the projections $X \times_S S' \to X$, $X \times_S S' \to S'$ by f' and p' respectively. Let Z be a closed subset in X. It is sufficient to show that $p(Z)$ is closed in S. Since $f : S' \to S$ is an h-covering $p(Z)$ is closed in S if and only if $f^{-1}(p(Z))$ is closed in S'. We obviously have $f^{-1}(p(Z)) = p'((f')^{-1}(Z))$. Since p' is proper we conclude that this set is closed.

Lemma 4.1.4 *Let S be a Noetherian scheme and $\phi : F \to G$ be a morphism of presheaves on the category of Noetherian schemes over S. Assume further that for any integral Noetherian scheme T over S and any section $a \in G(T)$ of G over T there is an abstract blow-up $f : T' \to T$ such that $f^*(a)$ belongs to the image of $\phi_{T'} : F(T') \to G(T')$. Then $\phi_{cdh} : F_{cdh} \to G_{cdh}$ is an epimorphism of the associated cdh-sheaves.*

Proof: It is sufficient to show that for any section $a \in G(S)$ of G over S there is a cdh-covering $\{X_i \stackrel{p_i}{\to} S\}$ such that $p_i^*(a)$ belongs to the image of $\phi_{X_i} : F(X_i) \to G(X_i)$. Our condition implies that there is a closed subscheme S_1 in S such that $S_1 \neq S$ and a proper surjective morphism $q : X_1 \to S$ such that $X_1 - q^{-1}(S_1) \to S - S_1$ is an isomorphism and $q^*(a)$ belongs to $Im(\phi_{X_1})$. Since $X_1 \coprod S_1 \to S$ is a cdh-covering of S we reduced our problem to S_1. Repeating this construction we get a sequence of closed subschemes $\ldots S_i \subset S_{i-1} \subset \ldots \subset S_1 \subset S$ such that $S_i \neq S_{i-1}$. Since the scheme S is Noetherian this sequence must be finite, i.e. $S_i = \emptyset$ for $i > n$. The family of morphisms $\{X_i \to S_i \to S\}_{i=1,\ldots,n}$ is then a cdh-covering of S with the required property.

4.2 *Sheaves in the h-topologies associated with Chow presheaves.*

Lemma 4.2.1 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and $\mathcal{Z}$ be an element of $Cycl(X/S, r)$. Then there is an abstract blow-up $f : S' \to S$ such that*

$$cycl(f)(\mathcal{Z}) = \sum n_i cycl(Z'_i) = \sum n_i cycl((Z'_i)_{red})$$

where Z'_i are irreducible closed subschemes of $X \times_S S'$ which are flat and equidimensional of relative dimension r over S'.

Proof: It follows immediately from Theorem 2.2.2.

Theorem 4.2.2 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves $Cycl(X/S, r)_{\mathbf{Q}}$ and $Cycl^{eff}(X/S, r)_{\mathbf{Q}}$ are h-sheaves and if S is a scheme of exponential characteristic p the same holds for the presheaves $z(X/S, r) \otimes \mathbf{Z}[1/\mathrm{p}]$ and $z^{eff}(X/S, r) \otimes \mathbf{Z}[1/\mathrm{p}]$.*

Proof: We will only consider the case of $Cycl(X/S,r)_{\mathbf{Q}}$. The proof for $Cycl^{eff}(X/S,r)_{\mathbf{Q}}$ is similar.

Note first that the presheaves $Cycl(X/S,r)_{\mathbf{Q}}$ are separated with respect to the h-topology, i.e. the canonical morphisms of presheaves $Cycl(X/S,r)_{\mathbf{Q}} \to (Cycl(X/S,r)_{\mathbf{Q}})_h$ are monomorphisms. Therefore according to the standard construction of the sheaf associated with a presheaf (see [11],[1]) it is sufficient to show that for any cofinial class of h-coverings $\{U_i \to S\}_{i=1,\dots,n}$ of S the following sequence of abelian groups is exact:

$$Cycl(X/S,r)_{\mathbf{Q}}(S) \to \bigoplus_i Cycl(X/S,r)_{\mathbf{Q}}(U_i) \to \bigoplus_{i,j} Cycl(X/S,r)_{\mathbf{Q}}(U_i \times_S U_j)$$

We may obviously replace the covering $\{U_i \to S\}$ by the covering

$$p : U = \coprod U_i \to S$$

and hence assume that $n = 1$. By Lemma 4.1.1 we may also assume that any irreducible component of U dominates an irreducible component of S.

We will use the following simple lemma.

Lemma 4.2.3 *Let $p_Y : Y \to Spec(k)$, $p_X : X \to Spec(k)$ be two schemes of finite type over a field k. Then the sequence of abelian groups*

$$Cycl(X/Spec(k),r) \otimes \mathbf{Q} \stackrel{cycl(p_Y)}{\to} Cycl(X \times_{Spec(k)} Y/Y,r) \otimes \mathbf{Q} \stackrel{cycl(pr_1)-cycl(pr_2)}{\to}$$

$$\to Cycl(X \times_{Spec(k)} Y \times_{Spec(k)} Y/Y \times_{Spec(k)} Y,r) \otimes \mathbf{Q}$$

(where $pr_i : Y \times_{Spec(k)} Y \to Y$ are the projections) is exact.

Let $\mathcal{Y}$ be an element of $Cycl(X/S,r)_{\mathbf{Q}}(U)$ such that $cycl(pr_1)(\mathcal{Y}) = cycl(pr_2)(\mathcal{Y})$. Since any irreducible component of U dominates an irreducible component of S Lemma 4.2.3 implies that there exists a cycle $\mathcal{Z}$ on X such that for any generic point $\eta : Spec(L) \to U$ of U one has $cycl(p \circ \eta)(\mathcal{Z}) = cycl(\eta)(\mathcal{Y})$. It is obviously sufficient to show that $\mathcal{Z}$ belongs to $Cycl(X/S,r) \otimes \mathbf{Q}$.

Let k be a field and (x_0, x_1, R), (y_0, y_1, Q) be two fat k points of S over a k-point $s : Spec(k) \to S$ of S. We have to show that $(x_0, x_1)^*(\mathcal{Z}) = (y_0, y_1)^*(\mathcal{Z})$.

Lemma 4.2.4 *Let R be a discrete valuation ring and $p : X \to Spec(R)$ be an h-covering of $Spec(R)$. Then there exist a discrete valuation ring Q, a dominant morphism $p_0 : Spec(Q) \to Spec(R)$ which takes the closed point of $Spec(Q)$ to the closed point of $Spec(R)$ and a morphism $s : spec(R) \to X$ such that $p \circ s = p_0$.*

Proof: By Lemma 4.1.1 there is an irreducible component X_0 of X which is dominant over S and whose image contains the closed point of $Spec(R)$ i.e. X_0 is irreducible and surjective over S. Then by [6] there is a discrete valuation ring Q and a dominant morphism $Spec(Q) \to X_0$ such that the image of the closed point of $Spec(Q)$ lies in the closed fiber of X_0 over $Spec(R)$.

By Lemma 4.2.4 we may construct a commutative diagram of the form

$$
\begin{array}{ccccccc}
 & & Spec(R') & & \stackrel{x_1'}{\to} & & U \\
 & & & \stackrel{f}{\searrow} & & & \\
 & \stackrel{x_0'}{\nearrow} & & & Spec(R) & & \downarrow p \\
 & & & \stackrel{x_0}{\nearrow} & & \stackrel{x_1}{\searrow} & \\
Spec(L) & \stackrel{r}{\to} & Spec(k) & & & & S \\
 & & & \stackrel{y_0}{\searrow} & & \stackrel{y_1}{\nearrow} & \\
 & \stackrel{y_0'}{\searrow} & & & Spec(Q) & & \uparrow p \\
 & & & \stackrel{g}{\nearrow} & & & \\
 & & Spec(Q') & & \stackrel{y_1'}{\to} & & U
\end{array}
$$

such that (x_0', x_1', R'), (y_0', y_1', Q') are fat points of U and f, g are dominant morphisms. It is obviously sufficient to show that $(x_0, x_1)^*(\mathcal{Z}) \otimes_k L = (y_0, y_1)^*(\mathcal{Z}) \otimes_k L$. We clearly have:

$$(x_0, x_1)^*(\mathcal{Z}) \otimes_k L = (x_0', x_1')^*(\mathcal{Y})$$

$$(y_0, y_1)^*(\mathcal{Z}) \otimes_k L = (y_0', y_1')^*(\mathcal{Y}).$$

It is sufficient to show that the right hand sides of these two equalities coincide. Denote the L points $x_1' \circ x_0'$ and $y_1' \circ y_0'$ of U by x' and y' respectively. Since $\mathcal{Y}$ is an element of $Cycl(X \times_S U/U, r)$ we have

$$(x_0', x_1')^*(\mathcal{Y}) = cycl(x')(\mathcal{Y}) = cycl(x' \times_S y')(cycl(pr_1)(\mathcal{Y}))$$

$$(y_0', y_1')^*(\mathcal{Y}) = cycl(y')(\mathcal{Y}) = cycl(x' \times_S y')(cycl(pr_2)(\mathcal{Y}))$$

(where $pr_i : U \times_S U \to U$ are the projections) and since

$$cycl(pr_1)(\mathcal{Y}) = cycl(pr_2)(\mathcal{Y})$$

by our condition on $\mathcal{Y}$ we conclude that $(x_0, x_1)^*(\mathcal{Z}) = (y_0, y_1)^*(\mathcal{Z})$.

Proposition 4.2.5 *Let $X \to S$ be a scheme of finite type over a scheme S. Then the presheaves $Cycl_{equi}(X/S, r)_{\mathbf{Q}}$ are qfh-sheaves and if S is a scheme of exponential characteristic p the same holds for the presheaves $z_{equi}(X/S, r) \otimes$ $\mathbf{Z}[1/\mathrm{p}]$.*

Proof: It is clearly sufficient to consider the case of $Cycl_{equi}(X/S, r)_{\mathbf{Q}}$. Let $\mathcal{Z}$ be an element of $Cycl(X/S, r) \otimes \mathbf{Q}$. In view of Theorem 4.2.2 it is sufficient to show that if there exists a qfh-covering $f : S' \to S$ of S such that $supp(cycl(f)(\mathcal{Z}))$ is equidimensional of dimension r over S' then $supp(\mathcal{Z})$ is equidimensional of dimension r over S. By Theorem 2.1.1 we have only to show that $dim(supp(\mathcal{Z})/S) \leq r$. By Lemma 4.1.1 we may assume that any irreducible component of S' dominates an irreducible component of S and $dim(supp(cycl(f)(\mathcal{Z}))/S') \leq r$. Then $supp(cycl(f)(\mathcal{Z}))$ is the closure in $supp(\mathcal{Z}) \times_S S'$ of the fibers of this scheme over generic points of S'. Since the projection $supp(\mathcal{Z}) \times_S S' \to supp(\mathcal{Z})$ is a qfh-covering using again Lemma 4.1.1 we conclude that the morphism $g : supp(cycl(f)(\mathcal{Z})) \to supp(\mathcal{Z})$ is surjective and hence $dim(supp(\mathcal{Z})/S) \leq r$ since the morphisms g and f are quasi-finite.

Proposition 4.2.6 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves $PropCycl(X/S, r)_{\mathbf{Q}}$ and $PropCycl^{eff}(X/S, r)_{\mathbf{Q}}$ are h-sheaves and if S is a scheme of exponential characteristic p the same holds for the presheaves $c(X/S, r) \otimes \mathbf{Z}[1/\mathrm{p}]$ and $c^{eff}(X/S, r) \otimes \mathbf{Z}[1/\mathrm{p}]$.*

Proof: In view of Theorem 4.2.2 it is sufficient to show that if $\mathcal{Z}$ is an element of $Cycl_{equi}(X/S, r) \otimes \mathbf{Q}$ and there exists an h-covering $f : S' \to S$ such that the support $supp(cycl(f)(\mathcal{Z}))$ is proper over S' then $supp(\mathcal{Z})$ is proper over S.

We may obviously assume that S is reduced. By 4.1.2 we may assume further that f admits a decomposition of the form $S' \stackrel{f_0}{\to} S'' \stackrel{f_1}{\to} S$ such that f_1 is an abstract blow-up and f_0 is faithfully flat. Let $cycl(f_1)(\mathcal{Z}) = \sum n_i cycl(Z''_i)$ where Z''_i are irreducible closed subschemes of $X'' = X \times_S S''$.

Since f_0 is flat the closed subsets $supp(cycl(f_1 f_0)(\mathcal{Z}))$ and $supp(cycl(f_1)(\mathcal{Z})) \times_{S''} S'$ coincide. Since f_0 in particular is an h-covering we conclude by Lemma 4.1.3 that $supp(cycl(f_1 f_0)(\mathcal{Z}))$ is proper over S''.

Let $\mathcal{Z} = \sum m_j cycl(Z_j)$ where Z_j are integral closed subschemes of X and $m_j \neq 0$. We have to show that the morphism $\cup Z_j \to S$ is proper. Consider the commutative diagram:

$$\begin{array}{ccc} \cup Z''_i & \to & \cup Z_i \\ \downarrow & & \downarrow \\ S'' & \to & S \end{array}$$

The upper horizontal arrow is proper since it is a composition of a closed embedding $Z''_i \to Z_i \times_S S''$ with a proper morphism $Z_i \times_S S'' \to Z_i$. Being an isomorphism in generic points it is surjective. Therefore $\cup Z_i \to S$ is proper since f_1 is proper and $\cup Z''_i \to S''$ is proper.

The following proposition follows immediately from Propositions 4.2.5, 4.2.6.

Proposition 4.2.7 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves $PropCycl_{equi}(X/S,r)_{\mathbf{Q}}$ are qfh-sheaves and if S is a scheme of exponential characteristic p the same holds for the presheaves* $c_{equi}(X/S,r) \otimes \mathbf{Z}[1/p]$.

Proposition 4.2.8 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then for any $r \geq 0$ the h-sheaves $Cycl(X/S,r)_{\mathbf{Q}}$, $PropCycl(X/S,r)_{\mathbf{Q}}$ are the h-sheaves of abelian groups associated in the obvious way with the h-sheaves of abelian monoids $Cycl^{eff}(X/S,r)_{\mathbf{Q}}$, $PropCycl^{eff}(X/S,r)_{\mathbf{Q}}$. The same holds for the corresponding p-divisible sheaves if S is a scheme of exponential characteristic p.*

Proof: By 4.2.2, 4.2.6 the presheaves $Cycl(X/S,r)_{\mathbf{Q}}$, $PropCycl(X/S,r)_{\mathbf{Q}}$ and their effective versions are h-sheaves. By Lemma 4.1.4 it is sufficient to show that for any element $\mathcal{Z}$ of $Cycl(X/S,r)_{\mathbf{Q}}(S)$ there is an abstract blow-up $f : S' \to S$ such that $cycl(f)(\mathcal{Z})$ is a linear combination of elements of $Cycl^{eff}(X/S,r)(S')$. It follows immediately from Lemma 4.2.1.

Theorem 4.2.9 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then one has:*

1. *For any $r \geq 0$ the presheaves $z(X/S, r)$, $c(X/S, r)$, $z^{eff}(X/S, r)$, $c^{eff}(X/S, r)$ are sheaves in the cdh-topology.*

2. *The cdh-sheaves associated with the presheaves $z_{equi}(X/S, r)$, $c_{equi}(X/S, r)$ are isomorphic to $z(X/S, r)$ and $c(X/S, r)$ respectively.*

Proof: Since we know already that $Cycl(X/S, r)_{\mathbf{Q}}$ etc. are h-sheaves to prove the first part of the theorem it is sufficient to show that for any element $\mathcal{Z}$ of $Cycl(X/S, r)_{\mathbf{Q}}(S)$ such that there exists a cdh-covering $p : S' \to S$ such that $cycl(p)(\mathcal{Z}) \in z(X/S, r)(S')$ one has $\mathcal{Z} \in z(X/S, r)(S)$.

It follows immediately from 3.3.9 and the fact that any cdh-covering of a spectrum of a field splits.

The second part follows trivially from the first part and Lemma 4.2.1.

Proposition 4.2.10 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then for any $r \geq 0$ the cdh-sheaf $z(X/S, r)$ (resp. $c(X/S, r)$) is the cdh-sheaf of abelian groups associated in the obvious way with the cdh-sheaf of abelian monoids $z^{eff}(X/S, r)$ (resp. $c^{eff}(X/S, r)$).*

Proof: It follows from Lemma 4.1.4 and Lemma 4.2.1.

Let $p : X \to S$ be a morphism of finite type of Noetherian schemes. For any Noetherian scheme T over S consider an equivalence relation $R_T \subset \mathbf{N}(Hilb(X/S, r))(T) \times \mathbf{N}(Hilb(X/S, r))(T)$ such that $(\sum n_i Z_i, \sum m_j W_j)$ belongs to R_T if and only if $cycl_{X_T}(\sum n_i Z_i) = cycl_{X_T}(\sum m_j W_j)$ where $X_T = X \times_S T$. Proposition 3.2.2 implies that it gives us equivalence relations R on the presheaves $\mathbf{N}(Hilb(X/S, r))$, $\mathbf{N}(PropHilb(X/S, r))$, $\mathbf{Z}(Hilb(X/S, r))$, $\mathbf{Z}(PropHilb(X/S, r))$ which are obviously consistent with the additive structure of these presheaves.

Theorem 4.2.11 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then one has canonical isomorphisms of cdh-sheaves:*

$$z(X/S, r) = (\mathbf{Z}(Hilb(X/S, r))/R)_{cdh}$$

$$z^{eff}(X/S, r) = (\mathbf{N}(Hilb(X/S, r))/R)_{cdh}$$

$$c(X/S,r) = (\mathbf{Z}(PropHilb(X/S,r))/R)_{cdh}$$
$$c^{eff}(X/S,r) = (\mathbf{N}(PropHilb(X/S,r))/R)_{cdh}$$

Proof: We will only consider the first isomorphism. Proof in the other cases is similar. Note first that by 3.3.11 there is a morphism of presheaves $\mathbf{Z}(Hilb(X/S,r)) \to z(X/S,r)$ and clearly the presheaf $\mathbf{Z}(Hilb(X/S,r))/R$ is the image of this morphism. The fact that the corresponding associated cdh-sheaf coincides with $z(X/S,r)$ follows immediately from Lemma 4.1.4 and Lemma 4.2.1.

Theorem 4.2.12 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then one has:*

1. *The sheaf $c_{equi}(X/S,0)_{qfh}$ is canonically isomorphic to the qfh-sheaf $\mathbf{Z}_{\mathrm{qfh}}(\mathrm{X/S})$ of abelian groups freely generated by the sheaf of sets representable by X.*

2. *The sheaf $c^{eff}(X/S,0)_{qfh}$ is canonically isomorphic to the qfh-sheaf $\mathbf{N}_{\mathrm{qfh}}(\mathrm{X/S})$ of abelian monoids freely generated by the presheaf of sets representable by X.*

Proof: We will only consider the first statement. The prove of the second one is similar.

Let $\delta \in c^{eff}(X/S,0)(X) \subset Cycl(X \times_S X/X,0)$ be the element which corresponds to the diagonal $\Delta : X \to X \times_S X$. By the universal property of the freely generated sheaves it gives us a morphism of sheaves $\mathbf{Z}_{\mathrm{qfh}}(\mathrm{X}) \to \mathrm{c^{eff}(X/S,0)_{qfh}}$. We have to show that it is an isomorphism. Note that it is a monomorphism since the functor of associated sheaf is exact and the corresponding morphism from the presheaf of abelian groups freely generated by X to the presheaf $c(X/S,0)$ is obviously a monomorphism.

To show that it is an epimorphism it is sufficient to verify that for any element $\mathcal{Z}$ of $c^{eff}(X/S,0)(S)$ there is a qfh-covering $p : S' \to S$ such that $cycl(p)(\mathcal{Z})$ is a formal linear combination of S'-points of $X \times_S S'$ over S'. We may assume that S is integral and $\mathcal{Z} = \sum n_i cycl(Z_i)$ where $Z_i \neq Z_j$ are integral closed subscheme of X which are finite and surjective over S. We will use induction on $deg(\mathcal{Z}/S) = \sum |n_i| deg(Z_i/S)$. If $N = 0$ then $\mathcal{Z} = 0$ and there is nothing to prove.

Set $Z = \coprod Z_i$. Since $p_Z : Z \to S$ is finite and surjective it is a qfh-covering. It is sufficient (by the induction hypothesis) to show that there exists a cycle $\mathcal{Z}_1$ in $c_{equi}(X \times_S Z_1/Z_1, 0)$ which is a linear combination of Z_1-points of $X \times_S Z_1$ over Z_1 and such that

$$deg(cycl(p_{Z_1})(\mathcal{Z}) - \mathcal{Z}_1) < deg(\mathcal{Z}/S).$$

The cycle $cycl(p_{Z_1})(\mathcal{Z})$ is of the form $\sum n_i(\sum m_j W_{ij})$ where W_{ij} are the irreducible components of the schemes $Z_1 \times_S Z_i$. We obviously have

$$\sum m_j deg(W_{ij}/Z_1) = deg(Z_i/S)$$

and hence

$$\sum n_i m_j deg(W_{ij}/Z_1) = deg(\mathcal{Z}/S).$$

Let W_{11} be the irreducible component of $Z_1 \times_S Z_1$ which is the image of the diagonal embedding $Z_1 \to Z_1 \times_S Z_1$. One can easily see that $cycl(W_{11}) \in c_{equi}(X \times_S Z_1/Z_1, 0)$ and hence we may set $\mathcal{Z}_1 = n_1 m_1 cycl(W_{11})$.

Lemma 4.2.13 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the h-sheaf $z_{equi}(X/S, r)_h$ (resp. the sheaf $c_{equi}(X/S, r)_h$) is isomorphic to the h-sheaf $z(X/S, r)_h$ (resp. to the sheaf $c(X/S, r)_h$).*

Proof: It follows trivially from Lemma 4.2.1

Proposition 4.2.14 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then one has:*

1. *The sheaf $c(X/S, 0)_h$ is canonically isomorphic to the h-sheaf $\mathbf{Z}_{\mathrm{h}}(\mathrm{X/S})$ of abelian groups freely generated by the sheaf of sets representable by X.*

2. *The sheaf $c^{eff}(X/S, 0)_h$ is canonically isomorphic to the h-sheaf $\mathbf{N}_{\mathrm{h}}(\mathrm{X/S})$ of abelian monoids freely generated by the presheaf of sets representable by X.*

Proof: It follows immediately from Theorem 4.2.12 and Lemma 4.2.13.

Proposition 4.2.15 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the canonical morphisms of presheaves:*

$$z(X/S,r)_{qfh} \to z(X/S,r)_h$$

$$c(X/S,r)_{qfh} \to c(X/S,r)_h$$

etc.
are isomorphisms.

Proof: To prove this proposition we need the following two lemmas.

Lemma 4.2.16 *Let S be a Noetherian scheme and $\{x_1,\dots,x_n\}$ be a finite set of points of S. Let $E_1,\dots,E_n$ be finite extensions of the fields $k_{x_1},\dots,k_{x_n}$. Then there is a finite surjective morphism $S' \to S$ such that for any $i = 1,\dots,n$ and any point y over x_i the field extension k_y/k_i contains E_i.*

Proof: Replacing S by the disjoint union of its irreducible components we may assume that S is integral. Using Zariski's main theorem [11] one can reduce the problem to the case when S is a semi-local scheme and x_i are the closed points of S. We may obviously assume that E_i are normal extensions of k_{x_i} and since E_i are finitely generated over k_{x_i} using induction on the minimal number of generators we may further assume that $E_i = k_{x_i}(z_i)$ for some elements z_i in E_i. Let

$$f_i(z) = z_i^d + a_{i1}z^{d_i-1} + \dots + a_{id_i}$$

be the minimal polinomial of z_i over k_{x_i} and

$$g_i = f_i^{d_1\dots d_n/d_i}.$$

Then g_i are of the same degree $d = d_1\dots d_n$. Let b_{ij}, $i = 1,\dots,n$, $j = 1,\dots,d$ be the coefficients of g_i. We will find then elements B_j in $\mathcal{O}(S)$ such that the reduction of B_j in the point x_i equals b_{ij}. We set $R = \mathcal{O}(S)[z]/(Z^d + B_1z^{d-1} + \dots + B_d)$. It is a finite algebra over $\mathcal{O}(S)$ and the morphism $Spec(R) \to S$ satisfies the required conditions.

Lemma 4.2.17 *Let $p : X \to S$ be a scheme of finite type over a Noetherian scheme S. Assume further that p is surjective. Then there exists a finite surjective morphism $S' \to S$ such that for any field k and any k-point $s' : Spec(k) \to S'$ there is a lifting of s' to a k-point of $X' = X \times_S S'$.*

Proof: Let Z be a closed subscheme of S. We say that our lemma holds outside Z if there is a finite surjective morphism $S_Z \to S$ such that for any field k and any k-point s' of $S_Z \times_S (S - Z)$ there is a lifting of s' to $S_Z \times_S X$.

Our lemma obviously holds for $Z = X$. By Noetherian induction it is sufficient to show that if our lemma holds for Z then it holds for a closed subset $Z' \subset Z$ such that $Z' \neq Z$.

Let $\eta_1, \ldots, \eta_k$ be all generic points of Z. Since the morphism p is of finite type there are finite extensions $E_1, \ldots, E_k$ of the fields $k_{\eta_1}, \ldots, k_{\eta_k}$ such that the points $Spec(E_i) \to S$ of S admit liftings to X. By Lemma 4.2.16 there is a finite surjective morphism $f : S' \to S$ such that for any field k and any k-point s' of S' over one of the points η_i there is a lifting of s' to $X \times_S S'$. Since p is of finite type this condition also holds for any point s' of S' which belongs to $f^{-1}(U)$ for a dense open subset U of Z. Let $Z' = Z - U$. Setting $S_{Z'} = S_Z \times_S S'$ we conclude that our lemma holds for Z'.

Let us now prove Proposition 4.2.15. We will only consider the case of $z(X/S, r)$. Note first that by Proposition 4.2.2 both $z(X/S, r)_{qfh}$ and $z(X/S, r)_h$ are subpresheaves in $Cycl(X/S, r)_{\mathbf{Q}}$. We have only to show that if $\mathcal{Z}$ is an element of $Cycl(X/S, r) \otimes \mathbf{Q}$ and there is an h-covering $f : S' \to S$ such that $cycl(f)(\mathcal{Z})$ belongs to $z(X/S, r)(S')$ then there is a qfh-covering with the same property. It follows from 3.3.9 and Lemma 4.2.17.

Corollary 4.2.18 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then for any $r \geq 0$ the qfh-sheaves $Cycl_{equi}(X/S, r)_{\mathbf{Q}}$, $PropCycl_{equi}(X/S, r)_{\mathbf{Q}}$ are isomorphic on the category of excellent Noetherian schemes over S to the qfh-sheaves of abelian groups associated in the obvious way with the qfh-sheaves of abelian monoids $Cycl^{eff}(X/S, r)_{\mathbf{Q}}$, $PropCycl^{eff}(X/S, r)_{\mathbf{Q}}$ and the same holds for the corresponding p-divisible sheaves if S is a scheme of exponential characteristic p.*

Proof: Note first that one has by 3.1.7:

$$Cycl^{eff}(X/S, r)_{\mathbf{Q}} \subset Cycl_{equi}(X/S, r)_{\mathbf{Q}}$$

$$PropCycl^{eff}(X/S, r)_{\mathbf{Q}} \subset PropCycl_{equi}(X/S, r)_{\mathbf{Q}}$$

As in the proof of 4.2.8 it is sufficient to show that for any element $\mathcal{Z} = \sum n_i z_i$ in $Cycl_{equi}(X/S, r)_{\mathbf{Q}}$ there is a qfh-covering $f : S' \to S$ such that $cycl(f)(\mathcal{Z})$

is a linear combination of elements of $Cycl^{eff}(X/S,r)_{\mathbf{Q}}(S')$. Since S is an excellent scheme its normalization is a qfh-covering and our statement follows from Corollary 3.4.3.

Remark: Corollary 4.2.18 remains true without the excellency assumption.

4.3 *Fundamental exact sequences for Chow sheaves.*

Theorem 4.3.1 *Let $p : X \to S$ be a scheme of finite type over a Noetherian scheme S, $i : Z \to X$ be a closed subscheme of X and $j : U \to X$ be the complement to Z in X. Then for any $r \geq 0$ the following sequence of presheaves is left exact and it is also right exact as a sequence of cdh-sheaves:*

$$0 \to z(Z/S,r) \stackrel{i_*}{\to} z(X/S,r) \stackrel{j^*}{\to} z(U/S,r) \to 0.$$

Proof: This sequence is obviously left exact as a sequence of presheaves. It is sufficient to show that the last arrow is a surjection in the cdh-topology. Let $\mathcal{Z} = \sum n_i z_i$ be an element of $z(U/S,r)$. By Lemma 4.1.4 it is sufficient to show that there is an abstract blow-up (2.2.4) of the form $f : S' \to S$ such that $cycl(f)(\mathcal{Z})$ belongs to the image of $z(X/S,r)(S')$ in $z(U/S,r)(S')$. Let Z_i be the closures of the points z_i in X wich we consider as closed integral subschemes. We may assume that S is reduced. Then by Theorem 2.2.2 there is a blow-up $f : S' \to S$ such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S'. We set $\mathcal{Z}' = \sum n_i cycl_{X \times_S S'}(\tilde{Z}_i)$. Then by Corollary 3.3.11 one has $\mathcal{Z}' \in z(X/S,r)(S')$ and its restriction to $U \times_S S'$ obviously equals $cycl(f)(\mathcal{Z})$.

Corollary 4.3.2 *Let $p : X \to S$ be a scheme of finite type over a Noetherian scheme S and $X = X_1 \cup X_2$ be an open covering of X. Denote the inclusions $X_i \subset X$,$X_1 \cap X_2 \subset X_i$ by f_i and g_i respectively. Then for any $r \geq 0$ the following sequence of presheaves is left exact and it is also right exact as a sequence of cdh-sheaves:*

$$0 \to z(X/S,r) \stackrel{(f_1)^*+(f_2)^*}{\to} z(X_1/S,r) \oplus z(X_2/S,r) \stackrel{(g_1)^*-(g_2)^*}{\to}$$

$$\to z((X_1 \cap X_2)/S,r) \to 0.$$

Remark: Note that the sequence of Theorem 4.3.1 is in fact already exact in the topology where coverings are proper cdh-coverings.

Proposition 4.3.3 *Let S be a Noetherian scheme, $p : X \to S$ be a scheme of finite type over S, $i : Z \to X$ be a closed subscheme of X and $f : X' \to X$ be a proper morphism such that the morphism $f^{-1}(X - Z) \to X - Z$ is an isomorphism.*

Consider the pull-back square

$$\begin{array}{ccc} f^{-1}(Z) & \stackrel{i'}{\to} & X' \\ f_Z \downarrow & & \downarrow f \\ Z & \stackrel{i}{\to} & X \end{array}$$

Denote by $F(-, r)$ one of the cdh-sheaves $z(-, r)$, $c(-, r)$. Then the following sequence of presheaves is left exact and it is also right exact as a sequence of cdh-sheaves

$$0 \to F(f^{-1}(Z)/S, r) \stackrel{i'_* \oplus (f_Z)_*}{\to} F(X'/S, r) \oplus F(Z/S, r) \stackrel{f_* \oplus (-i_*)}{\to}$$

$$\to F(X/S, r) \to 0.$$

Proof: It is clearly sufficient to consider the case of the sheaves $z(-, -)$.

Lemma 4.3.4 *In the notations of Proposition 4.3.3 the following sequence of abelian groups is exact:*

$$0 \to Cycl_{equi}(f^{-1}(Z)/S, r) \stackrel{i'_* \oplus (f_Z)_*}{\to} Cycl_{equi}(X'/S, r) \oplus Cycl_{equi}(Z/S, r) \stackrel{f_* \oplus (-i_*)}{\to} Cycl_{equi}(X/S, r).$$

Proof: The first arrow is a monomorphism since $i' : f^{-1}(Z) \to X'$ is a monomorphism.

Let us show that the sequence is exact in the middle term. Let $\mathcal{Y} = \sum n_i y_i$ be an element of $Cycl_{equi}(Z/S, r)$ and $\mathcal{W} = \sum m_j w_j$ be an element of $Cycl_{equi}(X'/S, r)$ such that $f_*(\mathcal{W}) = i_*(\mathcal{Y})$. The cycle $\mathcal{W}$ can be represented (uniquely) as a sum $\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_1$ such that $supp(\mathcal{W}_0) \subset f^{-1}(Z)$ and $supp(\mathcal{W}_1) \cap f^{-1}(Z) = \emptyset$. By our condition on $\mathcal{W}$ we have

$$f_*(\mathcal{W}_0) = i_*(\mathcal{Y})$$

$$f_*(\mathcal{W}_1) = 0$$

and since f is an isomorphism outside $f^{-1}(Z)$ we conclude that $\mathcal{W}_1 = 0$ and hence $\mathcal{W} \oplus \mathcal{Y}$ belongs to the image of the homomorphism $i'_* \oplus (f_Z)_*$.

One can easily see that Lemma 4.3.4 implies that the sequence of abelian groups

$$0 \to z_{equi}(f^{-1}(Z)/S, r) \stackrel{i'_* \oplus (f_Z)_*}{\to} z_{equi}(X'/S, r) \oplus z_{equi}(Z/S, r) \stackrel{f_* \oplus (-i_*)}{\to}$$
$$\to z_{equi}(X/S, r).$$

is also exact. Hence, our sequence of sheaves is left exact and it is sufficient to show that the homomorphism

$$z_{equi}(X'/S, r) \oplus z_{equi}(Z/S, r) \stackrel{f_* \oplus (-i_*)}{\to} z_{equi}(X/S, r)$$

is surjective as a homomorphism of cdh-sheaves. By Lemma 4.1.4 we may assume that S is integral and it is sufficient to show that for any element $\mathcal{W}$ of the group $z_{equi}(X/S, r)$ there is a blow-up $g : S' \to S$ such that $cycl(g)(\mathcal{W})$ belongs to the image of $z_{equi}(X' \times_S S'/S', r) \oplus z_{equi}(Z \times_S S'/S', r)$. By Lemma 4.2.1 we may assume that $\mathcal{W} = cycl_X(W)$ where W is an integral closed subscheme of X which is equidimensional of relative dimension r over S. If $W \subset Z$ our statement is obvious. Let w be the generic point of W and W' be the closure of $f^{-1}(w)$ in X'. By Theorem 2.2.2 there is a blow-up $g : S' \to S$ such that the proper transform $\tilde{W}'$ of W' is flat over S'. Denote by f' the morphism $f \times_S S' : X' \times_S S' \to X \times_S S'$. Since g is an isomorphism in generic points and f is an isomorphism ouside Z we clearly have $f'_*(cycl(\tilde{W}')) = cycl(g)(\mathcal{W})$. To finish the proof it is sufficient to note that $\tilde{W}'$ belongs to $z(X' \times_S S'/S', r)$ by Corollary 3.3.11.

Corollary 4.3.5 *Let S be a Noetherian scheme, $p : X \to S$ be a scheme of finite type over S and $X = Z_1 \cup Z_2$ be a covering of X by two closed subschemes. Denote the inclusions $Z_1 \cap Z_2 \subset Z_i$, $Z_i \subset X$ by f_i and g_i respectively and let $F(-, r)$ be one of the cdh-sheaves $z(-, r)$, $c(-, r)$.*

Then the following sequence of cdh-sheaves is exact

$$0 \to F(Z_1 \cap Z_2, r) \stackrel{(f_1)_* + (f_2)_*}{\to} F(Z_1, r) \oplus F(Z_2, r)) \stackrel{(g_1)_* - (g_2)_*}{\to} F(X, r) \to 0.$$

Proof: It is sufficient to apply Proposition 4.3.3 in the case $X' = Z_1 \coprod Z_2$, $Z = Z_1 \cap Z_2$.

Proposition 4.3.6 *In the notations of Proposition 4.3.3 assume in addition that f is a finite morphism and S is a geometrically unibranch scheme. Then the following sequence of abelian groups is exact:*

$$0\to Cycl_{equi}(f^{-1}(Z)/S,r)\stackrel{i'_*\oplus(f_Z)_*}{\to}Cycl_{equi}(X'/S,r)\oplus Cycl_{equi}(Z/S,r)\stackrel{f_*\oplus(-i_*)}{\to}Cycl_{equi}(X/S,r)\to 0.$$

The same statement holds for the groups $PropCycl_{equi}(-,-)$.

Proof: It is clearly sufficient to consider the case of $Cycl_{equi}(-,-)$. By Lemma 4.3.4 our sequence is left exact.

Let $\mathcal{W}$ be an element of $Cycl_{equi}(X/S,r)$. By Corollary 3.4.3 we may assume that $\mathcal{W} = cycl_X(W)$ where W is an integral closed subscheme of X which is equidimensional of relative dimension r over S. If $W \subset Z$ then $\mathcal{W}$ belongs to the image of the homomorphism i_*. Otherwise let w be the generic point of W and let W' be the closure of $w' = f^{-1}(w)$ in X'. Since w belongs to $X - Z$ we have $f_*(cycl_{X'}(W')) = \mathcal{W}$. To finish the proof of our proposition it is sufficient to show that $cycl_{X'}(W') \in Cycl_{equi}(X'/S,r)$. It follows from the fact that f is finite and Theorem 3.4.2.

Remark: Note that in Proposition 4.3.6 one can not replace in general the groups $Cycl_{equi}(-,-)$ by the groups $z_{equi}(-,-)$. An example of the situation when the corresponding sequence is not right exact for the groups $z_{equi}(-,-)$ can be easily deduced from example 3.5.10(2).

Proposition 4.3.7 *Let S be a Noetherian scheme, $p : X \to S$ be a scheme of finite type over S and $X = U_1 \cup U_2$ be an open covering of X. Denote the inclusions $U_1 \cap U_2 \subset U_i$, $U_i \subset X$ by f_i and g_i respectively.*

Then the sequence of presheaves

$$0 \to c_{equi}(U_1 \cap U_2/S,0) \stackrel{(f_1)_*+(f_2)_*}{\to} c_{equi}(U_1/S,0) \oplus c_{equi}(U_2/S,0) \stackrel{(g_1)_*-(g_2)_*}{\to}$$
$$\to c_{equi}(X/S,0) \to 0.$$

is exact in the Nisnevich topology.

Proof: Note that our sequence is left exact as a sequence of presheaves by obvious reason. To prove that the last arrow is a surjection in the Nisnevich topology it is sufficient to show that the map

$$c_{equi}(U_1/S,0) \oplus c_{equi}(U_2/S,0) \stackrel{(g_1)_*-(g_2)_*}{\to} c_{equi}(X/S,0)$$

is surjective if S is a local henselian scheme (see [12]). It follows trivially from the fact that for any element $\mathcal{Z}$ of $c_{equi}(X/S,0)$ its support is finite over S and the existence of decomposition of schemes finite over local henselian schemes into a disjoint union of local schemes (see [11, I.4.2.9(c)]).

Corollary 4.3.8 *In the notations of Proposition 4.3.7 the sequence of cdh-sheaves*

$$0 \to c(U_1 \cap U_2, 0) \stackrel{(f_1)_* + (f_2)_*}{\to} c(U_1, 0) \oplus c(U_2, 0) \stackrel{(g_1)_* - (g_2)_*}{\to} c(X, 0) \to 0.$$

is exact.

Proof: It follows immediately from Proposition 4.3.7, Theorem 4.2.9 and exactness of the functor of associated sheaf.

Remark: Note that the exact sequence of Proposition 4.3.7 is quite different from the exact sequence of Corollary 4.3.2. In particular while the sequence for finite cycles requires only Nisnevich coverings to be exact the sequence for general cycles requires only abstract blow-ups, i.e. proper cdh-coverings to be exact.

Proposition 4.3.9 *Let $p : X \to S$ be a scheme of finite type over a Noetherian scheme S, $Z \to X$ be a closed subscheme of X and $f : X' \to X$ be an etale morphism such that the morphism $f^{-1}(Z) \to Z$ is an isomorphism. Let further U be the complement to Z in X and U' be the complement to $f^{-1}(Z)$ in X'. Then the canonical morphism of quotient sheaves in the Nisnevich topology*

$$c_{equi}(X'/S,0)/c_{equi}(U'/S,0) \to c_{equi}(X/S,0)/c_{equi}(U/S,0)$$

is an isomorphism.

Proof: It is sufficient to show that for a local henselian scheme S the morphism of abelian groups

$$c_{equi}(X'/S,0)/c_{equi}(U'/S,0) \stackrel{f_*}{\to} c_{equi}(X/S,0)/c_{equi}(U/S,0).$$

is an isomorphism. The following lemma is an easy corollary of the standard properties of henselian schemes (see [11]).

Lemma 4.3.10 *Let $q : X \to S$ be an etale morphism such that the scheme S is henselian and let x be a closed point of X over the closed point s of S such that the morphism $Spec(k_x) \to Spec(k_s)$ is an isomorphism. Then q is an isomorphism in a neighborhood of x.*

(**Injectivity.**) Let $\mathcal{W}'$ be an element of $c_{equi}(X'/S, 0)$ and assume that $f_*(\mathcal{W}')$ belongs to $c_{equi}(U/S, 0)$. Since S is henselian the support $W' = supp(\mathcal{W}')$ of $\mathcal{W}'$ is a disjoint union of local henselian schemes and we may assume that the closed points of W belong to $f^{-1}(Z)$ and $f_*(\mathcal{W}) = 0$.

Since f is an isomorphism on $f^{-1}(Z)$ we may further assume that W' is local. Then by Lemma 4.3.10 the morphism $supp(\mathcal{W}') \to supp(f_*(\mathcal{W}'))$ is an isomorphism and hence $\mathcal{W}' = 0$.

(**Surjectivity.**) Let $\mathcal{W}$ be an element of $c_{equi}(X/S, 0)$. As above we may assume that $W = supp(\mathcal{W})$ is local and its closed point belongs to Z. Let W' be the local scheme of the closed point of $f^{-1}(W)$ over the closed point of W. Then by Lemma 4.3.10 the morphism $W' \to W$ is an isomorphism. Denote by $\mathcal{W}'$ the cycle on W' (and hence on X') which corresponds to $\mathcal{W}$. We obviously have $f_*(\mathcal{W}') = \mathcal{W}$ and $\mathcal{W}' \in c_{equi}(X'/S, 0)$.

Corollary 4.3.11 *In notations of Proposition 4.3.9 there is a canonical isomorphism of quotient sheaves in the cdh-topology of the form:*

$$c(X'/S, 0)/c(U'/S, 0) \to c(X/S, 0)/c(U/S, 0).$$

Proof: It follows from Proposition 4.3.9 and Theorem 4.2.9.

4.4 *Representability of Chow sheaves.*

In this section we consider representability of Chow sheaves of effective proper cycles on quasi-projective schemes over a Noetherian scheme S. Let us begin with the following definition.

Definition 4.4.1 *Let S be a Noetherian scheme and F be a presheaf of sets on the category of Noetherian schemes over S. We say that F is h-representable by a scheme X over S if there is an isomorphism $F_h \to L_h(X)$ of the h-sheaf associated with F with the h-sheaf associated with the presheaf represented by X.*

Note that the scheme X which h-represents a presheaf F is not uniquely defined up to isomorphism since the h-topology is not subcanonical. Nevertheless as was shown in [18] X is well defined up to a universal homeomorphism. In particular if for some X which h-represents F all generic points of X are of characteristic zero then there exists a unique semi-normal scheme which h-represents F.

Let S be a Noetherian scheme and F be a presheaf on the category of Noetherian schemes over S. A k-point of F is a pair of the form $(Spec(k) \to S, \phi)$ where k is a field and $\phi \in F(Spec(k)/S)$ is a section of F over $Spec(k)$. We say that a k-point ϕ of F is equivalent to a k'-point ϕ' of F if there is a field k'' a morphism $Spec(k'') \to S$ and morphisms $u : Spec(k'') \to Spec(k)$, $u' : Spec(k'') \to Spec(k')$ over S such that the sections $u^*(\phi)$ and $(u')^*(\phi')$ of F over $Spec(k'')$ coincide. A point of F is by definition an equivalence class of k-points of F. Denote the set of points of F by $Top(F)$. One can easily see that for any morphism of presheaves $f : F \to G$ one has a map of sets $Top(f) : Top(F) \to Top(G)$ which is a monomorphism (resp. an epimorphism) if f is a monomorphism (resp. an epimorphism). In particular for any subpresheaf F_0 in F we get a subset $Top(F_0)$ in $Top(F)$.

The following lemma is trivial.

Lemma 4.4.2 *Let t be a Grothendieck topology on the category of Noetherian schemes over S such that for any algebraically closed field k any morphism $Spec(k) \to S$ and any t-covering $p : U \to Spec(k)$ there exists a section $s : Spec(k) \to U$ of p. For a presheaf F on the category of Noetherian schemes over S denote by F_t the associated t-sheaf. Then the map $Top(F) \to Top(F_t)$ is a bijection.*

Note that all topologies we use in this paper satisfty the condition of Lemma 4.4.2.

For any presheaf of sets F on the category of Noetherian schemes over S, any Noetherian scheme S' over S and any section ϕ of F over S' denote by $Top(\phi)$ the obvious map of sets $S' \to Top(F)$.

Let now $A \subset Top(F)$ be a subset in $Top(F)$. Denote by F_A the subpresheaf in F such that for any Noetherian scheme S' over S the subset $F_A(S') \subset F(S')$ consists of sections ϕ such that $Top(\phi)(S') \subset A$. If a subpresheaf F_0 in F is of the form F_A for a subset A in $Top(F)$ we say that F_0 is defined by a pointwise condition. One can easily see that F_0 is defined by a pointwise condition if and only if for any Noetherian scheme S' over S and a section ϕ of F over S' such that $Top(\phi)(S') \subset Top(F_0)$ one has $\phi \in F_0(S')$.

Let A be a subset in $Top(F)$. We say that A is open (resp. closed, constructible) if for any Noetherian scheme S' over S and any section ϕ of T on S' the set $Top(\phi)^{-1}(A)$ is open (resp. closed, constructible) in S'. We will say further that a subpresheaf F_0 of F is open (resp. closed, constructible) if F_0 is of the form F_A for an open (resp. closed, constructible) subset in $Top(F)$.

One can easily see that open subsets of $Top(F)$ form a topology on this set and that a subset is closed if and only if its complement is open. Note also that for any morphism of presheaves $f : F \to G$ the corresponding map $Top(f) : Top(F) \to Top(G)$ is a continuous map with respect to this topology. The following lemma is straightforward.

Lemma 4.4.3 *Let $X \to S$ be a Noetherian scheme over a Noetherian scheme S. Denote by $L(X/S)$ the presheaf of sets represented by X on the category of Noetherian schemes over S.*

1. *The map $Top(\phi) : X \to Top(L(X/S))$ defined by the tautological section of $L(X/S)$ over X is a homeomorphism of the corresponding topological spaces.*

2. *Let t be a topology on the category of Noetherian schemes over S satisfying the condition of Lemma 4.4.2 and such that for any Noetherian scheme S' over S the morphism $S'_{red} \to S'$ is a t-covering. Then for any open (resp. closed) subset $A \subset Top(L_t(X/S))$ the subpresheaf $L_t(X/S)_A$ is t-representable by the corresponding open (resp. closed) subscheme in X.*

Note that the h-topologies (h-,cdh-,qfh-) satisfy the conditions of Lemma 4.4.3(2).

We say that a presheaf F on the category of Noetherian schemes over a Noetherian scheme S is topologically separated if for any Noetherian scheme T over S and any dominant morphism $T' \to T$ the map $F(T) \to F(T')$ is injective. Note that all the Chow presheaves considered in this paper are topologically separated.

Lemma 4.4.4 *Let S be a Noetherian scheme, F be a topologically separated presheaf on the category of Noetherian schemes over S, $X \to S$ be a* *scheme of finite type over S and $f : z(X/S,r) \to F$ be a monomorphism of presheaves. Consider a closed subset A in $Top(z(X/S,r))$. Then the subsheaf $f(z(X/S,r)_A)_h$ of F_h is a closed subpresheaf of F if and only if $Top(f)(A)$ is a constructible subset in $Top(F) = Top(F_h)$.*

Proof: We obviously have $Top(f)(A) = Top(f(z(X/S,r)_A)) \subset Top(F)$ which proves the "only if" part.

Assume that $Top(f)(A)$ is a constructible subset in $Top(F)$. Let us show first that $f(z(X/S,r)_A)_h$ is a subpresheaf in F_h given by a pointwise condition. Let T be a Noetherian scheme over S and $\phi \in F_h(T)$ be a section of F_h such that for any geometrical point $x : Spec(k) \to T$ of T we have $x^*(\phi) \in z(X/S,r)_A$. We have to show that $\phi \in f(z(X/S,r)_A)_h(T)$. Our problem is h-local with respect to T. Replacing T by the union of its irreducible components we may assume that T is an integral scheme. Let η be the generic point of T, $\bar{k}_\eta$ be an algebraic closure of the function field of T and $\bar{\eta} : Spec(\bar{k}_\eta) \to T$ be the corresponding geometrical point of T. Then $\bar{\eta}^*(\phi)$ corresponds to an element $\mathcal{Z}$ in $z(X \times_T Spec(\bar{k}_\eta)/Spec(\bar{k}_\eta), r)$. Since X is of finite type over S and A is a closed subset of $Top(Z(X/S,r))$ there is a quasi-finite dominant morphism $p : U \to T$ and an elememt $\mathcal{Z}_U$ in $z(X/S,r)_A(U)$ such that the restriction of $\mathcal{Z}_U$ to $Spec(\bar{k}_\eta)$ equals $\mathcal{Z}$. Theorem 2.2.2 implies easily now that there is an h-covering $\bar{p} : \bar{U} \to T$ which is finite over the generic point of T and a section $\mathcal{Z}_{\bar{U}}$ of $z(X/S,r)_A$ over $\bar{U}$ such that its restriction to $Spec(\bar{k}_\eta)$ equals $\mathcal{Z}$. Since F is topologically separated we conclude that $\bar{p}^*(\phi) = \mathcal{Z}_{\bar{U}}$ and hence $f(z(X/S,A))_h$ is indeed defined by a pointwise condition. To prove that $Top(f(z(X/S,A))_h)$ is a closed subset in $Top(F) = Top(F_h)$ is trivial.

Proposition 4.4.5 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and X_0 be a closed subscheme in X. Then the sheaf $z(X_0/S,r)$ is a closed subpresheaf in the sheaf $z(X/S,r)$.*

Proof: Note first that the sheaf $z(X_0/S,r)$ is a subpresheaf in $z(X/S,r)$ given by a pointwise condition. Moreover a section of $z(X/S,r)$ belongs to $z(X_0/S,r)$ if and only if it belongs to $z(X_0/S,r)$ over the generic points of S. It implies easily that the only thing we have to show is that for any cycle $\mathcal{Z}$ in $z(X/S,r)$ which does not belong to $z(X_0/S,r)$ in a generic point η of S there is a neighborhood U of η in S such that $\mathcal{Z}_s$ does not belong to $z((X_0)_s/Spec(k_s),r)$ for any point s in U. It is obvious.

Proposition 4.4.6 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and X_0 be an open subscheme in X. Then the sheaf $c^{eff}(X_0/S,r)$ is an open subpresheaf in the sheaf $c^{eff}(X/S,r)$.*

Proof: Let us show first that the subpresheaf $c^{eff}(X_0/S,r)$ is given by a pointwise condition. Let $\mathcal{Z}$ be an element of $c^{eff}(X/S,r)$. Assume that for any geometrical point $x : Spec(k) \to S$ of S we have $cycl(x)(\mathcal{Z}) \in c^{eff}(X_0 \times_S Spec(k)/Spec(k),r)$. We have to show that $\mathcal{Z} \in c^{eff}(X_0/S,r)$. Since $c^{eff}(X_0/S,r)$ is a cdh-subsheaf in $c^{eff}(X/S,r)$ it is sufficient to show that there is a blow-up $f : S' \to S$ such that $cycl(f)(\mathcal{Z}) \in c^{eff}(X_0 \times_S S'/S',r)$. We may assume therefore that $\mathcal{Z} = \sum n_i cycl(Z_i)$ where Z_i are irreducible closed subschemes of X which are flat over S. It is sufficient to show that our condition on $\mathcal{Z}$ implies that $supp(\mathcal{Z}) \subset X_0$. It follows immediately from the fact that $\mathcal{Z}$ is effective since $supp(cycl(x)(cycl(Z_i))) = Z_i \times_S Spec(k)$ for any geometrical point $Spec(k) \to S$ of S.

Let now $\mathcal{Z}$ be an arbitrary element of $c^{eff}(X/S,r)$. We have to show that the set of points s of S such that $\mathcal{Z}_{\bar{k}_s} \in c^{eff}(X_0 \times_S Spec(\bar{k}_s)/Spec(\bar{k}_s),r)$ is open in S. Again we may replace S by its blow-up and assume that $\mathcal{Z} = \sum n_i cycl(Z_i)$ where Z_i are irreducible closed subschemes of X which are flat over S. Then clearly our subset is the intersection of subsets U_i where $s \in U_i$ if and only if $Z_i \times_S Spec(k_s) \subset X_0 \times_S Spec(k_s)$. Then $U_i = S - p(Z_i \cap (X - X_0))$ and since $p : Z_i \to S$ is proper we conclude that U_i are open subsets of S.

Example 4.4.7 The analog of Proposition 4.4.6 for the sheaves $c(X/S,r)$ is false. Let us consider the scheme $X = \mathbf{P}^2_k \times \mathbf{A}^1 \to \mathbf{A}^1$ over the affine line. Let further Z_1 (resp. Z_2) be the closed subscheme given by the equation $x_0+x_1+tx_2 = 0$ (resp. $x_0+x_1-tx_2 = 0$) where x_i are the coordinates on $\mathbf{P}^2$ and t is the coordinate on $\mathbf{A}^1$. Let $\mathcal{Z} = Z_1 - Z_2$ and $X_0 = \mathbf{A}^2 \times \mathbf{A}^1 \subset \mathbf{P}^2_k \times \mathbf{A}^1$. Then the set of points $t \in \mathbf{A}^1$ where $\mathcal{Z}_t \in (X_0)_t$ consists of one point $t = 0$.

Let S be a Noetherian scheme. For any multi-index $I = (i_1, \ldots, i_k)$ denote by $\mathbf{P}^I_S$ the product of projective spaces $\mathbf{P}^{i_1}_S \times_S \ldots \times_S \mathbf{P}^{i_k}$ over S. Let $\mathcal{Z}$ be an element of $Cycl^{eff}(\mathbf{P}^I_S/S,r)$. For any point s of S denote by $deg_s(\mathcal{Z})$ the (multi-)degree of the cycle $\mathcal{Z}_s$ on $\mathbf{P}^I_{k_s}$ (note that apriory $deg_s(\mathcal{Z})$ is sequence of rational numbers).

Proposition 4.4.8 *Let S be a Noetherian scheme and $\mathcal{Z}$ be an element of $Cycl^{eff}(\mathbf{P}^I_S/S,r)$. Then the function $s \to deg_s(\mathcal{Z})$ is locally constant on S.*

Proof: It is sufficient to show that if η is a generic point of S and s is a point in the closure of η then $deg_{\eta}(\mathcal{Z}) = deg_s(\mathcal{Z})$. Since for any cycle $\mathcal{W}$ on $\mathbf{P}^I_k$ and any field extension k'/k we have $deg(\mathcal{Z}) = deg(\mathcal{Z} \otimes_k k')$ it is sufficient to show that for some field extensions L, E of k_η and k_s respectively the cycles $\mathcal{Z}_{Spec(L)}$ and $\mathcal{Z}_{Spec(E)}$ have the same degree. Let (x_0, x_1, R) be a fat point on S such that the image of x_1 is $\{\eta, s\}$. Replacing S by $Spec(R)$ we may assume that S is the spectrum of a discrete valuation ring. In this case $\mathcal{Z} = \sum n_i cycl(Z_i)$ where Z_i are closed subschemes of $\mathbf{P}^I_S$ which are flat and equidimensional over S and our statement follows from the invariance of (multi-)degree in flat families.

Corollary 4.4.9 *Let S be a connected Noetherian scheme. Then for any cycle $\mathcal{Z}$ in $Cycl(\mathbf{P}^I_S/S, r)$ and any point s of S the degree $deg_s(\mathcal{Z})$ is a sequence of integers which does not depend on s.*

For a Noetherian scheme S, a multi-index $I = (i_1, \ldots, i_k)$ and a sequence of nonnegative integers $D = (d_1, \ldots, d_n)$ denote by $z^{eff}_D(\mathbf{P}^I_S/S, r)$ the subset in $z(\mathbf{P}^I_S/S, r)$ which consists of cycles $\mathcal{Z}$ such that for any point s of S one has $deg_s(\mathcal{Z}) = D$. One can easily see that $z^{eff}_d(\mathbf{P}^I_S/S, r)$ is in fact a cdh-subsheaf in $z^{eff}(\mathbf{P}^I_S/S, r)$. Proposition 4.4.8 implies further that for a connected Noetherian scheme one has

$$z^{eff}(\mathbf{P}^I_S/S, r) = \bigcup_D z^{eff}_D(\mathbf{P}^I_S/S, r).$$

The proof of the following lemma is standard.

Lemma 4.4.10 *Let S be a Noetherian scheme. Then for any multi-index $I = (i_1, \ldots, i_k)$ and any sequence of nonnegative integers $D = (d_1, \ldots, d_k)$ the sheaf $z^{eff}_D(\mathbf{P}^I_S/S, (\sum i_j) - 1)$ is h-representable by the projective space P^N_S for some $N = N(I, D)$.*

Denote by G the product $(\mathbf{P}^n)^* \times_{Spec(\mathbf{Z})} \ldots_{Spec(\mathbf{Z})} (\mathbf{P}^n)^*$ of $r+1$-copies of the projective space dual to the standard projective space (i.e $(\mathbf{P}^n)^*$ is the scheme which parameterizes hyperplanes in $\mathbf{P}^n$). Let further $L \in \mathbf{P}^n \times_{Spec(\mathbf{Z})} G$ be the closed subscheme of points $(x, H_1, \ldots, H_{r+1})$ such that $x \in H_i$ for all $i = 1, \ldots, r+1$. It is smooth over $\mathbf{P}^n$ and fibers of the projection $f : L \to \mathbf{P}^n$ are isomorphic to $(\mathbf{P}^{n-1})^{r+1}$.

Denote further by $Div_d(G)$ the projective space which parametrizes cycles of codimension 1 and degree $(d, \ldots, d)$ on $((\mathbf{P}^n)^*)^{r+1}$ (see Lemma 4.4.10). Let $Div_d^{irr}(G)$ be the open subspace in $Div_d(G)$ which parametrizes irreducible divisors on G and let $\Gamma \in G \times Div_d^{irr}(G)$ be the support of the corresponding relative cycle on $G \times Div_d^{irr}(G)$ over $Div_d^{irr}(G)$. Set

$$U = L \times Div_d^{irr}(G) - (L \times Div_d^{irr}(G) \cap \mathbf{P}^n \times \Gamma)$$

Let finally Φ be the subset $(pr : L \times Div_d^{irr}(G) \to \mathbf{P}^n \times Div_d^{irr}(G))(U)$. Since L is smooth (and hence universally open) over $\mathbf{P}^n$ this subset is open in $\mathbf{P}^n \times Div_d^{irr}(G)$. We define $C_{r,d}^{irr}$ to be the subset in $Div_d^{irr}(G)$ which consists of points s such that $dim((\mathbf{P}^n \times Div_d^{irr}(G) - U) \times_{Div_d^{irr}(G)} Spec(k_s)) \geq r$. Since the projection $\mathbf{P}^n \times Div_d^{irr}(G) \to Div_d^{irr}(G)$ is proper Chevalley theorem (2.1.1) implies that $C_{r,d}^{irr}$ is a closed subset in $Div_d^{irr}(G)$.

Theorem 4.4.11 *For any Noetherian scheme S and any $r, d \geq 0$ the sheaf of sets $z_d^{eff}(\mathbf{P}^n_S/S, r)$ is h-representable by a projective scheme $C_{r,d}$ over S.*

Proof: Note that it is obviously sufficient to prove our theorem for $S = Spec(\mathbf{Z})$. We will use the notations which we introduced in the construction of $C_{r,d}^{irr}$ above. Let $\mathcal{Z}$ be an element of $z^{eff}(\mathbf{P}^n_S/S, r)$. Let

$$f : L \to \mathbf{P}^n$$

$$p : L \to G$$

be the obvious morphisms. Since f is smooth and p is proper we get a homomorphism of presheaves

$$Chow = p_* f^* : z^{eff}(\mathbf{P}^n/Spec(\mathbf{Z}), \mathrm{r}) \to \mathrm{z^{eff}(G/Spec(\mathbf{Z}), (n-1)(r+1)+r)}$$

The following lemma is straighforward.

Lemma 4.4.12 *1. The homomorphism $Chow$ is a monomorphism.*

2. The homomorphism $Chow$ takes the subsheaf $z_d^{eff}(\mathbf{P}^n_S/S, r)$ to the subsheaf $z_D^{eff}(((\mathbf{P}^n_S)^*)^{r+1}/S, (n-1)(r+1)+r)$ where $D = (d, \ldots, d)$.

In view of Lemma 4.4.12 and Lemma 4.4.10 the homomorphism $Chow$ gives us an embedding of the sheaf $z_d^{eff}(\mathbf{P}^n_S/S, r)$ to the cdh-sheaf representable by the projective space $Div_d(G)$. Since $z_d^{eff}(\mathbf{P}^n/Spec(\mathbf{Z}),\mathrm{r})$ is clearly a closed subpresheaf in the sheaf $z(\mathbf{P}^n/Spec(\mathbf{Z}),\mathrm{r})$ it is sufficient by 4.4.3 and 4.4.4 to show that the subset $C_{r,d} = Im(Top(Chow))$ in $Div^d(G)$ is constructible.

Denote by F_d the subpresheaf in $z_d^{eff}(\mathbf{P}^n/Spec(\mathbf{Z}),\mathrm{r})$ such that for a Noetherian scheme S the subset $F_d(S)$ in $z_d^{eff}(\mathbf{P}^n_S/S,r)$ consists of cycles $\mathcal{Z}$ such that for any algebraically closed field k and a k-point $x : Spec(k) \to S$ the cycle $cycl(x)(\mathcal{Z})$ on $\mathbf{P}^n_k$ is of the form $cycl(Z)$ for a closed integral subscheme Z in $\mathbf{P}^n_k$ of dimension r and degree d. Let further

$$\mathbf{F}_d = \coprod_{k=1}^{d} \coprod_{d_1+\ldots+d_k=d} \prod_{i=1,\ldots,k} F_{d_i}.$$

We have the following diagram of morphisms of presheaves

$$\begin{array}{ccc} \mathbf{F}_d & \stackrel{Chow}{\to} & \coprod_{k=1}^{d} \coprod_{d_1+\ldots+d_k=d} \prod_{i=1,\ldots,k} L(Div_{d_i}^{irr}(G)) \\ \downarrow & & \downarrow \\ z_d^{eff}(\mathbf{P}^n/Spec(\mathbf{Z}),\mathrm{r}) & \to & L(Div_d(G)) \end{array}$$

The first vertical arrow is a surjection on the corresponding topological spaces and the second vertical arrow being induced by a morphism of schemes of finite type takes constructible sets to constructible sets. It is sufficient therefore to show that the image of $Top(Chow^{irr})$ where $Chow^{irr}$ is the morphism

$$F_d \to L(Div_d^{irr}(G))$$

is constructible. Let us show that in fact $Im(Top(Chow^{irr})) = C_{r,d}^{irr}$.

(" $\mathbf{Im(Top(Chow^{irr}))} \subset \mathbf{C^{irr}_{r,d}}$") Let k be an algebraically closed field and $\mathcal{Z} \in F_d(\mathbf{P}^n_k)$. We have to show that the point x on $Div_d^{irr}(G)_k$ which corresponds to $Chow^{irr}(\mathcal{Z})$ belongs to $C_{r,d}^{irr}$. It follows immediately from our definition of $C_{r,d}^{irr}$ since in this case the fiber of the projection $\mathbf{P}^n \times Div_d^{irr}(G) - U \to Div_d^{irr}(G)$ over x contains the support of $\mathcal{Z}$.

("$\mathbf{C^{irr}_{r,d}} \subset \mathbf{Im(Top(Chow^{irr}))}$ ") Let k be an algebraically closed field and $x : Spec(k) \to Div_d^{irr}(G)$ be a point of $Div_d^{irr}(G)$ which belongs to $C_{r,d}^{irr}$, i.e. such that $dim((\mathbf{P}^n \times Div_d^{irr}(G) - U) \times_{Div_d^{irr}(G)} Spec(k)) \geq r$. Let D_x be the divisor on G_k which corresponds to x. One can verify easily that fibers of $\mathbf{P}^n \times Div_d(G) - U$ over $Div_d(G)$ are of dimension $\leq r$. Denote by Z_i the irreducible components of the fiber of $\mathbf{P}^n \times Div_d(G) - U$ over x which have dimension r. Then $p(f^{-1}(Z_i))$ is an irreducible divisor in G which is obviously contained in $supp(D)$. Since $p(f^{-1}(Z_i)) \neq p(f^{-1}(Z_j))$ for $i \neq j$ (Lemma 4.4.12(1)) and $supp(D)$ is irreducible we conclude that there is only one component Z of dimension r and $p(f^{-1}(Z)) = supp(D)$. It implies easily that $D = Chow^{irr}(cycl(Z))$. Theorem is proven.

Let S be a Noetherian scheme and $i : X \to \mathbf{P}^n_S$ be a projective scheme over S. Denote by $z_d^{eff}((X,i)/S,r)$ the subpresheaf in the Chow presheaf $z^{eff}(X/S,r)$ such that for a Noetherian scheme T over S the subset $z_d^{eff}((X,i)/S,r)(T)$ in $z^{eff}(X/S,r)(T)$ consists of relative cycles of degree d with respect to the embedding $i \times_S Id_T$. If $U \subset X$ is an open subset of X we denote by $c_d^{eff}((U,i)/S,r)$ the preimage of $z_d^{eff}((X,i)/S,r)$ with respect to the obvious morphism $c^{eff}(U/S,r) \to z^{eff}(X/S,r)$.

Corollary 4.4.13 *For any Noetherian scheme S and a projective scheme $i : X \to \mathbf{P}^n_S$ over S the presheaf $z_d^{eff}((X,i)/S,r)$ is h-representable by a projective scheme $C_{r,d}(X,i)$ over S and*

$$z^{eff}(X/S,r) = \coprod_{d\geq 0} z_d^{eff}((X,i)/S,r).$$

If U is an open subscheme in X then $c_d^{eff}((U,i)/S,r)$ is representable by an open subscheme $C_{r,d}(U,i)$ in $C_{r,d}(X,i)$ and

$$c^{eff}(U/S,r) = \coprod_{d\geq 0} c_d^{eff}((U,i)/S,r).$$

Proof: It follows immediately from Theorem 4.4.11, Propositions 4.4.5, 4.4.6 and Lemma 4.4.3.

Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. A (closed) equivalence relation on X is a closed subscheme $R \subset X \times_S X$ such that for any Noetherian scheme T over S the subset $Hom_S(T,R)$ of the set $Hom_S(T, X \times_S X) = Hom_S(T,X) \times Hom_S(T,X)$ is an equivalence relation on $Hom_S(T,X)$. An equivalence relation R is called a proper equivalence relation if the projections $R \to X$ are proper morphisms.

Let R be an equivalence relation on X and Y be a Noetherian scheme over S. Let $\Gamma \to Y \times_S X$ be a closed subset of $Y \times_S X$. Denote by $p_Y : \Gamma \to Y$, $p_X : \Gamma \to X$ the obvious morphisms. We call Γ a graph-like closed subset (with respect to R) if the following conditions hold:

1. The morphism $p_Y : \Gamma \to Y$ is a universal topological epimorphism (i.e. an h-covering).

2. For any algebraically closed field $\bar{k}$ and a $\bar{k}$-valued point $\bar{y} : Spec(\bar{k}) \to Y$ all the elements of the subset $p_X(p_Y^{-1}(\bar{y}))$ in $X(\bar{k})$ are equivalent with respect to $R(\bar{k})$.

Any such Γ defines for any algebraically closed field $\bar{k}$ a map of sets $f_\Gamma : Y(\bar{k}) \to X(\bar{k})/R(\bar{k})$. We say that two graph-like subsets Γ_1, Γ_2 are equivalent if for any algebraically closed field $\bar{k}$ the corresponding maps f_{Γ_1} and f_{Γ_2} coincide. A continuous algebraic map from Y to X/R (over S) is an equivalence class of graph-like closed subschemes of $Y \times_S X$ with respect to this equivalence relation[3]. We denote the set of continuous algebraic maps from Y to X/R over S by $Hom_S^{a.c.}(Y, X/R)$.

Let now R be a proper equivalence relation. Denote the projections $R \to X$ by pr_1 and pr_2 respectively. For any closed subset Γ in $Y \times_S X$ which is of finite type over Y consider the subset $\Gamma_R \subset Y \times_S X$ of the form $(Id_Y \times_S pr_2)(Id_Y \times_S pr_1)^{-1}(\Gamma)$. Since R is proper Γ_R is a closed subset. One can verify easily that if Γ is a graph-like subset with respect to R then Γ_R is graph-like and Γ is equivalent to Γ_R. Moreover two graph-like closed subsets Γ_1 and Γ_2 are equivalent if and only if $(\Gamma_1)_R = (\Gamma_2)_R$.

Lemma 4.4.14 *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and $R \subset X \times_S X$ be a proper equivalence relation on X. Let $L_h(X)$ be the h-sheaf represented by X on the category of Noetherian schemes over S and let $L_h(X)/R$ be the quotient sheaf (in the h-topology) of $L_h(X)$ with respect to the equivalence relation defined by R. Then for any Noetherian scheme Y over S there is a canonical bijection:*

$$Hom_S^{a.c.}(Y, X/R) = (L_h(X)/R)(Y).$$

Proof: Clearly $Hom_S^{a.c.}(Y, X/R)$ is a presheaf on the category of Noetherian schemes over S with respect to Y. Let us show that it is in fact an h-sheaf. Consider an h-covering $p : U \to Y$ of Y. Note first that if Γ_1, Γ_2 are two graph-like closed subsets in $Y \times_S X$ such that $(p \times_S Id_X)^{-1}(\Gamma_1)$ is equivalent to $(p \times_S Id_X)^{-1}(\Gamma_2)$ then Γ_1 is equivalent to Γ_2.

[3]Our definition of continuous algebraic maps is a natural generalization of the definition given in [3].

Let now $\Gamma \subset U \times_S X$ be a graph-like closed subset such that $pr_1^{-1}(\Gamma)$ is equivalent to $pr_2^{-1}(\Gamma)$ where $pr_i : U \times_Y U \times_S X \to U \times_S X$ are the projections. We may assume that $\Gamma = \Gamma_R$. Then the same obviously holds for $pr_i^{-1}(\Gamma)$ and we conclude that $pr_1^{-1}(\Gamma) = pr_2^{-1}(\Gamma)$. Since p is a universal topological epimorphism it implies trivially that $\Gamma = (p \times_S Id_X)^{-1}(\Gamma_0)$ for a closed subset Γ_0 in $Y \times_S X$. Since Γ_0 is obvioulsy graph-like with respect to R it proves that $Hom_S^{a.c.}(-, X/R)$ is indeed an h-sheaf.

Let us construct a morphism of sheaves

$$\phi : Hom_S^{a.c.}(-, X/R) \to L_h(X)/R.$$

Let Γ be a graph-like closed subset in $Y \times_S X$. We have a morphism $\Gamma \to X$. Since $\Gamma \to Y$ is an h-covering our definition of a graph-like subset implies trivially that the corresponding section of $L_h(X)/R$ on Γ can be descended to a section of $L_h(X)/R$ on Y which does not depend on the choice of Γ in its equivalence class.

The proof of the fact that ϕ is an isomorphism is trivial.

Let $X \to S$ be a quasi-projective scheme over a Noetherian scheme S and $i : \bar{X} \to \mathbf{P}_S^n$ be a projective scheme over S such that there is an open embedding $X \subset \bar{X}$ over S. Consider the scheme $C_{r,d}(\bar{X}, i)$ which h-represents the presheaf $z_d^{eff}(\bar{X}/S, r)$ by Corollary 4.4.13. Set

$$C_{r,\leq d}(\bar{X}, i) = \coprod_{i \leq d} C_{r,i}(\bar{X}, r).$$

Consider the canonical section $\mathcal{Z}$ of $z(\bar{X}/S, r)_h$ over $C_{r,\leq d}(\bar{X}, i)$ and let $\tilde{\mathcal{Z}}$ be the section $(pr_1^*(\mathcal{Z}) - pr_2^*(\mathcal{Z})) - (pr_3^*(\mathcal{Z}) - pr_4^*(\mathcal{Z}))$ of $z(\bar{X}/S, r)_h$ over the product $(C_{r,\leq d}(\bar{X}, i))_S^4$. Denote by R_d^X the closed subset of points of $(C_{r,\leq d}(\bar{X}, i))_S^4$ where $\tilde{\mathcal{Z}}$ belongs to $z((\bar{X} - X)/S, r) \subset z(\bar{X}/S, r)$ (see 4.4.5). One can easily see that R_d^X is an equivalence relation on $(C_{r,\leq d}(\bar{X}, i))_S^2$ and since $C_{r,\leq d}(\bar{X}, i)$ is proper over S it is a proper equivalence relation. Note further that the obviuos embeddings $C_{r,\leq d}(\bar{X}, i) \to C_{r,\leq (d+1)}(\bar{X}, i)$ take R_d^X to R_{d+1}^X and hence for any Noetherian scheme T over S there is a family of maps

$$Hom_S^{a.c.}(T, (C_{r,\leq d}(\bar{X}, i))_S^2 / R_d^X) \to Hom_S^{a.c.}(T, (C_{r,\leq (d+1)}(\bar{X}, i))_S^2 / R_{d+1}^X).$$

Proposition 4.4.15 *For any Noetherian scheme T over S there is a canonical bijection*

$$colim_d Hom_S^{a.c.}(T, (C_{r,\leq d}(\bar{X}, i) \times_S C_{r,\leq d}(\bar{X}, i))/R_d^X) = z(X/S, r)_h(T).$$

Proof: It follows immediately from Lemma 4.4.14 and our definition of the equivalence relations R_d^X.

References

[1] M. Artin. *Grothendieck topologies.* Harvard Univ., Cambridge, 1962.

[2] Eric M. Friedlander. Some computations of algebraic cycle homology. *K-theory*, 8:271–285, 1994.

[3] Eric M. Friedlander and O. Gabber. Cycle spaces and intersection theory. *Topological methods in modern mathematics*, pages 325–370, 1993.

[4] W. Fulton. *Intersection Theory.* Springer-Verlag, Berlin, 1984.

[5] A. Grothendieck. *Revetements etale et groupe fondamental(SGA 1).* Lecture Notes in Math. 224. Springer, Heidelberg, 1971.

[6] A. Grothendieck and J. Dieudonne. *Etude Globale Elementaire de Quelques Classes de Morphismes (EGA 2).* Publ. Math. IHES,8, 1961.

[7] A. Grothendieck and J. Dieudonne. *Etude Cohomologique des Faisceaux Coherents (EGA 3).* Publ. Math. IHES,11,17, 1961,1963.

[8] A. Grothendieck and J. Dieudonne. *Etude Locale des Schemas et des Morphismes de Schemas (EGA 4).* Publ. Math. IHES,20,24,28,32, 1964-67.

[9] A. Grothendiek. Les schemas de Picard: theoremes d'existence. In *Seminaire Bourbaki*, n.232 1961/62.

[10] H. Matsumura. *Commutative ring theory.* Cambridge University Press, 1986.

[11] J.S. Milne. *Etale Cohomology.* Princeton Univ. Press, Princeton, NJ, 1980.

[12] Y. Nisnevich. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. In *Algebraic K-theory: connections with geometry and topology*, pages 241–342. Kluwer Acad. Publ., Dordrecht, 1989.

[13] M. Raynaud and L. Gruson. Criteres de platitude et de projectivite. *Inv. Math.*, 13:1–89, 1971.

[14] P. Samuel. *Methodes d'algebre abstraite en geometrie algebrique*. Ergebnisse der Mathematik, N.F.4. Springer-Verlag, Berlin, 1955.

[15] J.-P. Serre. Algebre locale. Multiplicities. *Lecture Notes in Math.*, 11, 1965.

[16] Andrei Suslin and Vladimir Voevodsky. Singular homology of abstract algebraic varieties. *Invent. Math.*, 123(1):61–94, 1996.

[17] V. Voevodsky. Cohomological theory of presheaves with transfers. *This volume.*

[18] V. Voevodsky. Homology of schemes. *Selecta Mathematica, New Series*, 2(1):111–153, 1996.