MOTIVIC COHOMOLOGY WITH $\mathbf{Z}/2$-COEFFICIENTS

by VLADIMIR VOEVODSKY⋆

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1. Introduction

Let k be a field and l a prime number different from the characteristic of k. Fix a separable closure k_{sep} of k and let μ_l denote the group of l-th roots of unity in k_{sep}. One may consider μ_l as a $\mathrm{Gal}(k_{sep}/k)$-module. By definition of μ_l one has a short exact sequence

$$1 \longrightarrow \mu_l \longrightarrow k^*_{sep} \xrightarrow{z^l} k^*_{sep} \longrightarrow 1$$

which is called the Kummer sequence. The boundary map in the associated long exact sequence of Galois cohomology is a homomorphism

$$(\mathbf{1}) \qquad k^* \to \mathrm{H}^1(k, \mu_l).$$

In [1], Bass and Tate proved that for $a \in k^* - \{1\}$ the cohomology class $(a) \wedge (1-a)$ lying in $\mathrm{H}^2(k, \mu_l^{\otimes 2})$ is zero i.e. that the homomorphism (1) extends to a homomorphism of rings

$$(\mathbf{2}) \qquad \mathrm{T}(k^*)/\mathrm{I} \to \mathrm{H}^*\big(k, \mu_l^{\otimes *}\big)$$

where $\mathrm{T}(k^*)r$ is the tensor algebra of the abelian group k^* and I the ideal generated by elements of the form $a \otimes b$ for $a, b \in k^*$ such that $a+b=1$. The graded components of the quotient $\mathrm{T}(k^*)/\mathrm{I}$ are known as the Milnor K-groups of k and the homomorphism (2) is usually written as

$$(\mathbf{3}) \qquad \mathrm{K}^{\mathrm{M}}_*(k) \to \mathrm{H}^*\big(k, \mu_l^{\otimes *}\big).$$

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Let $K_*^M(k)/l$ be the quotient of $K_*^M(k)$ by the ideal of elements divisible by l. It is clear that (3) factors through a map

$$(\mathbf{4}) \qquad K_*^M(k)/l \to H^*\big(k, \mu_l^{\otimes *}\big)$$

which is called the *norm residue homomorphism*. One of the main objectives of this paper is to prove the following conjecture for $l = 2$.

Conjecture **1.1.** — *The map (4) is an isomorphism for any field k of characteristic $\neq l$.*

This conjecture has a long and convoluted history. The map (4) is clearly an isomorphism in degree zero. In degree one, (4) is a monomorphism and its cokernel is the group of l-torsion elements in $H^1(k, k_{sep}^*)$ which is known to be zero as a corollary of the classical Hilbert 90 Theorem.

For fields which contain a primitive l-th root of unit, the homomorphism (4) in degree 2 has an interpretation in terms of central simple algebras. The question about central simple algebras which is equivalent to surjectivity of (4) in degree 2 seems to be very old. The question of injectivity of (4) in degree 2 was explicitly stated by John Milnor in [22, Remark on p. 147].

In [21] Milnor considered the homomorphism (4) in all degrees for $l = 2$ as a part of his investigation of the relations between $K_*^M(k)/2$ and quadratic forms over k. He mentioned that he does not know of any fields k for which (4) fails to be an isomorphism and gave several examples of classes of fields k for which (4) is an isomorphism in all degrees. His examples extend a computation made by Bass and Tate in the same paper where they introduce (4). Because of [21] the $l = 2$ case of Conjecture 1.1 is called the Milnor Conjecture.

The general case of Conjecture 1.1 was first formulated by Kazuya Kato in [13, p. 608] and, in the particular case of function fields over $\mathbf{C}$, by Spencer Bloch in [3]. Today it is know as the Bloch–Kato Conjecture.

In 1981 Alexander Merkurjev published a paper [18] where he proved that (4) is an isomorphism in degree 2 for $l = 2$ and any field k such that $char(k) \neq 2$. This paper is the starting point of all the further work on bijectivity of (4). In 1982 Merkurjev together with Andrei Suslin proved that (4) is an isomorphism in degree 2 for all l (see [19]).

In degree 3 and $l = 2$ the bijectivity of (4) was proved by Merkurjev and Suslin in [20] and independently by Markus Rost in [27].

Our approach to Conjecture 1.1 is based on the ideas introduced by Stephen Lichtenbaum and Alexander Beilinson in the early 1980-ies. In [15] and [2, 5.10.D] they formulated a set of conjectures describing properties of hypothetical (at that time) complexes of sheaves $\mathbf{Z}(n)$ which they called the motivic complexes. This set contained in particular the following:

1. $\mathbf{H}^n_{\mathrm{Zar}}(\mathrm{Spec}(k), \mathbf{Z}/l(n)) = \mathrm{K}^n_{\mathrm{M}}(k)/l$
2. if $l \neq char(k)$ then $\mathbf{H}^n_{et}(\mathrm{Spec}(k), \mathbf{Z}/l(n)) = \mathrm{H}^n\big(k, \mu_l^{\otimes n}\big)$
3. let X be a smooth variety over a field k, then the map

$$\mathbf{H}^p_{\mathrm{Zar}}(\mathrm{X}, \mathbf{Z}/l(q)) \to \mathbf{H}^p_{et}(\mathrm{X}, \mathbf{Z}/l(q))$$

is an isomorphism for $p \leq q$.

4. for any $n \geq 0$ one has

$$\mathbf{H}^{n+1}_{et}(\mathrm{Spec}(k), \mathbf{Z}(n)) = 0$$

Statements (3) and (4) on this list are known as the *Belinson–Lichtenbaum* Conjectures or, rather, as the protion of these conjectures related to the torsion effects. In terms of the motivic cohomology (the cohomlogy with coefficients in the motivic complexes), the norm residue homomorphism becomes, after the identifications (1) and (2), the natural homomorphism from the cohomology in the Zariski topology to the cohomology in the etale topology and Conjecture 1.1 becomes a particular case of (3).

We use the motivic complexes constructed in [43] (see also [17]). For these complexes (1) is proved in [33] (or [17]) and (2) in [17] (we also give a short proof here).

Unlike Conjecture 1.1, the statement (3) also makes sense for fields of characteristic equal to l. In this context it was proved in [7] by Thomas Geisser and Marc Levine. They use the version of motivic cohomology based on the higher Chow groups which is now known to be equivalent to the version used here by [32], [6] and [41].

The relation between (3) and Conjecture 1.1 for $l \neq char(k)$ was clarified in [33] and [8] where it was shown that (3), in a given weight n, is in fact equivalent to Conjecture 1.1 in the same weight and moreover that it is sufficient to show only the surjectivity of (4) and the injectivity follows. In particular it was shown that the known cases of Conjecture 1.1 imply the corresponding cases of (3) and (4).

In this paper we prove the 2-local version of (4), and (3) for $l = 2$ and all n and, as a corollary, Conjecture 1.1 for $l = 2$ and all n. We use the results of [33] and [8] in Theorem 6.6 to show that the l-local version of (4) implies (3) (for a given weight n). We then proceed to prove the 2-local version of (4) using an inductive argument where to get (4) in weight n one needs to know (3) in weight $(n - 1)$.

The paper is organized as follows. In Sections 2–5 we prove several results about motivic cohomology which are used in the proof of our main theorem but which are not directly related to the Beilinson–Lichtenbaum conjecturs. In Section 3 this is Corollary 3.8, in Section 4 Theorems 4.4 and 4.9 and in Section 5 Theorem 5.9. The proof of Corollary 3.8 uses Theorem 2.11 of Section 2. There are no other connections of these four sections to each other or to the remaining sections of the paper.

In Section 6 we show that (3) is a corollary of (4) and that Conjecture 1.1 is a corollary (3). This section is independent of the previous four sections.

In Section 7 we prove the 2-local version of (4) – Theorem 7.4. This section also contains some corollaries of this theorem. Using similar techniques one can also prove the Milnor Conjecture which asserts that the Milnor ring modulo 2 is isomorphic to the graded Witt ring of quadratic forms. For the proof of this result together with more detailed computations of motivic cohomology groups of norm quadrics see [25].

Two appendices contain the material which is used throughout the paper and which I could not find good references for.

All through the paper we use the Nisnevich topology [24] instead of the Zariski one. Since all the complexes of sheaves considered in this paper have transfers and homotopy invariant cohomology sheaves [38, Theorem 5.7, p. 128] implies that one can replace Nisnevich hypercohomology by Zariski ones everywhere in the paper without changing the answers.

The first version of the proof of Theorem 7.4 appeared in [35]. It was based on the idea that there should exist algebraic analogs of the higher Morava K-theories and that the m-th algebraic Morava K-theory can be used for the proof of Conjecture 1.1 for $l = 2$ and $n = m + 2$ in the same way as the usual algebraic K-theory is used in Merkurjev–Suslin proof of Conjecture 1.1 for $l = 2$, $n = 3$ in [20]. This approach was recently validated by Simone Borghesi [5] who showed how to construct algebraic Morava K-theories (at least in characteristic zero).

The second version of the proof appeared in [36]. Instead of algebraic Morava K-theories it used small pieces of these theories which are easy to construct as soon as one knows some facts about the cohomological operations in motivic cohomology and their interpretation in terms of the motivic stable homotopy category.

The main difference between the present paper and [36] is in the proof of Theorem 3.2 ([36, Theorem 3.25]). The approach used now was outlined in [36, Remark on p. 39]. It is based on the connection between cohomological operations and characteristic classes and circumvents several technical ingredients of the older proof. The most important simplification is due to the fact that we can now completely avoid the motivic *stable* homotopy category and the topological realization functor.

Another difference between this paper and [36] is that we can now prove all the intermediate results for fields of any characteristic. Several developments made this possible. The new proof of the suspension theorem for the motivic cohomology [42, Theorem 2.4] based on the comparison between the motivic cohomology and the higher Chow groups established in [32], [6] and [41] does not use resolution of singularities. The same comparison together with the new proof of the main result of [33] by Thomas Geisser and Marc Levine in [8] allows one to drop the resolution of singularities assumption in the proof of Theorem 6.6. Finally, the new approach to the proof of Theorem 3.2 does not require the topological realization functor which only exists in characteristic zero.

I am glad to be able to use this opportunity to thank all the people who answered a great number of my questions during my work on the Beilinson–Lichtenbaum conjectures. First of all I want to thank Andrei Suslin who taught me the techniques used in [19], [20]. Quite a few of the ideas of the first part of the paper are due to numerous conversations with him. Bob Thomason made a lot of comments on the preprint [35] and in particular explained to me why algebraic K-theory with $\mathbf{Z}/2$-coefficients has no multiplicative structure, which helped to eliminate the assumption $\sqrt{-1} \in k^*$ in Theorem 7.4. Jack Morava and Mike Hopkins answered a lot of my (mostly meaningless) topological questions and I am in debt to them for not being afraid of things like the Steenrod algebra anymore. The same applies to Markus Rost and Alexander Vishik for not being afraid anymore of the theory of quadratic forms. Dmitri Orlov guessed the form of the distinguished triangle in Theorem 4.4 which was a crucial step to the understanding of the structure of motives of Pfister quadrics. I would also like to thank Fabien Morel, Chuck Weibel, Bruno Kahn and Rick Jardine for a number of discussions which helped me to finish this work. Finally, I would like to thank Eric Friedlander who introduced me to Anderei Suslin and helped me in many ways during the years when I was working on Conjecture 1.1.

Most of the mathematics of this paper was invented when I was a Junior Fellow of the Harvard Society of Fellows and I wish to express my deep gratitude to the society for providing a unique opportunity to work for three years without having to think of things earthly. The first complete version was written during my stay in the Max-Planck Institute in Bonn. Further work was done when I was at the Northwestern University and in its final form the paper was written when I was a member of the Institute for Advanced Study in Princeton.

2. The degree map

In this section and Section 3 we use the motivic homotopy theory of algebraic varieties developed in [23]. For an introduction to this theory see also [37] and [40]. For a smooth projective variety X of pure dimension d we define the degree map $H^{2d,d}(X, \mathbf{Z}) \to \mathbf{Z}$ ("evaluation on the fundamental class") and prove Theorem 2.11 where we show that this map can be described as the composition of the Thom isomorphism for an appropriate vector bundle and a map defined by a morphism in the *pointed motivic homotopy category* $H_\bullet$. This theorem is a formal corollary of Spanier–Whitehead duality in the motivic stable homotopy theory but since the details of this duality are not worked out yet we are forced to give a very direct but not very conceptual proof here. Everywhere in this section motivic cohomology are considered with integral coefficients.

Recall that the Thom space $Th_X(E)$ of a vector bundle E over X with the zero section $i : X \to E$ is the pointed sheaf $E/(E - i(X))$. Let X be a smooth scheme and Z a smooth closed subscheme in X. We let $N_{X,Z}$ denote the normal bundle to Z in X considered as a vector bundle over Z. In [23, Th. 2.23, p. 115] we constructed the *purity isomorphism* in $H_\bullet(k)$ of the form

$$\rho_{X,Z} : X/(X-Z) \cong Th_Z(N_{X,Z}). \tag{5}$$

This isomorphism is constructed as follows. Let $B(X,Z)$ be the blow-up of the closed subscheme $Z \times \{0\}$ in $X \times \mathbf{A}^1$. Let $f : Z \times \mathbf{A}^1 \to B(X,Z)$ be the unique section of the projection $B(X,Z) \to X \times \mathbf{A}^1$ over $Z \times \mathbf{A}^1$. The fiber of the pair $(B(X,Z), f(Z \times \mathbf{A}^1))$ over $(X \times \{1\}, Z \times \{1\})$ is (X,Z) and the fiber over $(X \times \{0\}, Z \times \{0\})$ contains $(N_{X,Z}, Z)$. It was shown in [23] that the corresponding morphisms

$$X/(X-Z) \to B(X,Z)/(B(X,Z) - f(Z \times \mathbf{A}^1)) \tag{6}$$

and

$$N_{X,Z}/(N_{X,Z} - Z) \to B(X,Z)/(B(X,Z) - f(Z \times \mathbf{A}^1)) \tag{7}$$

are isomorphisms in $H_\bullet(k)$ and one defines (5) as the composition of the first one with the inverse to the second. We let $B^o(X,Z)$ denote the complement in $B(X,Z)$ to the proper transform of $X \times \{0\}$. This is an open neighborhood of $f(Z \times \mathbf{A}^1)$ and its fiber over $X \times \{0\}$ is exactly $N_{X,Z}$. Therefore, one can also define the purity isomorphisms using the analogs of the maps (6), (7) with the scheme $B(X,Z)$ replaced by $B^o(X,Z)$.

We will use the following three basic facts about the purity isomorphism. The first of them immediately follows from the construction.

Lemma **2.1.** — *Let* (X,Z) *be as above and* $f : X' \to X$ *a morphism such that* X' *and* $Z' = f^{-1}(Z)$ *are smooth and the map* $\phi : N_{X', f^{-1}(Z)} \to f^*N_{X,Z}$ *is an isomorphism. Let* ϕ' *be the map of the Thom spaces defined by* ϕ*. Then the square*

$$\begin{array}{ccc} X'/(X'-Z') & \longrightarrow & X/(X-Z) \\ \downarrow & & \downarrow \\ Th_{Z'}(N_{X',Z'}) & \xrightarrow{\phi'} & Th_Z(N_{X,Z}) \end{array}$$

commutes.

Lemma **2.2.** — *Let* $E \to X$ *be a vector bundle with the zero section* $X \subset E$*. Then the purity morphism*

$$Th_X(E) = E/(E-X) \to Th_X(N_{E,X})$$

coincides with the morphism defined by the natural isomorphism $E \cong N_{E,X}$.

Proof. — The scheme $B(E,X)$ is the blow-up of the zero section of the vector bundle $E \oplus \mathcal{O}$. It projects naturally to the projective bundle $\mathbf{P}(E \oplus \mathcal{O})$ such that $B(E,X) - f(X \times \mathbf{A}^1)$ maps to $\mathbf{P}(\mathcal{O})$. The composition of the resulting map

$$B(E,X)/(B(E,X) - f(X \times \mathbf{A}^1)) \to \mathbf{P}(E \oplus \mathcal{O})/(\mathbf{P}(E \oplus \mathcal{O}) - \mathbf{P}(\mathcal{O}))$$

with each of the maps (6), (7) is an isomorphism and the composition of the first with the inverse to the second is the map defined by the identification $E \cong N_{E,X}$. This proves the lemma.

Note that Lemmas 2.1 and 2.2 uniquely characterize the purity maps.

Lemma **2.3.** — *Let* X *be a smooth scheme and* Z_1, Z_2 *smooth closed subschemes in* X *which intersect transversally. Set* $Z_{12} = Z_1 \cap Z_2$. *Let*

$$Th_{Z_{12}}(N_{X,Z_{12}}) \to Th_{Z_1}(N_{X,Z_1}) \wedge Th_{Z_2}(N_{X,Z_2})$$

be the map defined by the isomorphism

$$(Z_{12} \to Z_1 \times Z_2)^*(N_{X,Z_1} \times N_{X,Z_2}) \cong N_{X,Z_{12}}.$$

Then the diagram

$$\begin{array}{ccc} X/(X-Z_{12}) & \longrightarrow & Th_{Z_{12}}(N_{X,Z_{12}}) \\ \downarrow & & \downarrow \\ X/(X-Z_1)\wedge X/(X-Z_2) & \longrightarrow & Th_{Z_1}(N_{X,Z_1}) \wedge Th_{Z_2}(N_{X,Z_2}) \end{array}$$

commutes.

Proof. — For (X, Z) as above and a smooth closed subscheme Y of X which contains Z we have a morphism $\pi : B^o(X,Z) \to B^o(X,Y)$ over $X \times \mathbf{A}^1$ such that $\pi^{-1}(f(Y \times \mathbf{A}^1))$ is isomorphic to $B^o(Y,Z)$ embedded in $B^o(X,Z)$ in the natural way. For $Z_{12} = Z_1 \cap Z_2$ we get two projections $\pi_i : B^o(X, Z_{12}) \to B^o(X, Z_i)$ and

$$\pi_1^{-1}\big(f\big(Z_1 \times \mathbf{A}^1\big)\big) \cap \pi_2^{-1}\big(f\big(Z_2 \times \mathbf{A}^1\big)\big) = f\big(Z_{12} \times \mathbf{A}^1\big).$$

We let U_{12} denote $B^o(X, Z_{12}) - f(Z_{12} \times \mathbf{A}^1)$ and similarly for U_1, U_2. The product $\pi_1 \times \pi_2$ defines the middle arrow in the commutative diagram

$$\begin{array}{ccc} X/(X-Z_{12}) & \longrightarrow & X/(X-Z_1)\wedge X/(X-Z_2) \\ \downarrow & & \downarrow \\ B^o(X,Z_{12})/U_{12} & \longrightarrow & B^o(X,Z_1)/U_1 \wedge B^o(X,Z_2)/U_2 \\ \uparrow & & \uparrow \\ N_{X,Z_{12}}/(N_{X,Z_{12}} - Z_{12}) & \longrightarrow & N_{X,Z_1}/(N_{X,Z_1} - Z_1) \wedge N_{X,Z_2}/(N_{X,Z_2} - Z_2) \end{array}$$

which implies the statement of the lemma.

For a vector bundle E on X of dimension d we let t_E denote the Thom class in $H^{2d,d}(Th(E), \mathbf{Z})$ constructed in [42]. For a smooth Z in X of pure codimension c the pull-back with respect to the purity map of the Thom class of the normal bundle is a well defined class in $H^{2c,c}(X/(X-Z))$ which we denote by $a_{X,Z}$ or a_Z. Sometimes we let the same symbol denote the image of this class in $H^{2c,c}(X)$. The classes a_Z are natural with respect to maps $(X', Z') \to (X, Z)$ satisfying the conditions of Lemma 2.1 and have the property that for a vector bundle E over X, $a_{E,X}$ is the Thom class.

Lemma **2.4.** — *For a sequence of smooth embeddings* $Z \subset Y \subset X$ *let* $\pi : X/(X-Y) \to X/(X-Z)$ *be the obvious map. Then*

$$\pi^*(a_{X,Z}) = \rho^*_{X,Y}(t_{N_{X,Y}} \wedge a_{Y,Z})$$

where ρ *is the purity map and the product is defined by the natural morphism*

$$Th(N_{X,Y}) \to Th(N_{X,Y}) \wedge Y.$$

Proof. — Consider first the commutative diagram

$$\begin{array}{ccc}
X/(X-Y) & \xrightarrow{1=\pi} & X/(X-Z) \\
{\scriptstyle 2}\downarrow & & {\scriptstyle 3}\downarrow \\
B(X,Y)/(B(X,Y)-f(Y\times\mathbf{A}^1)) & \xrightarrow{4} & B(X,Y)/(B(X,Y)-f(Z\times\mathbf{A}^1)) \\
{\scriptstyle 5}\uparrow & & {\scriptstyle 6}\uparrow \\
Th(N_{X,Y}) & \xrightarrow{7} & N_{X,Y}/(N_{X,Y}-Z).
\end{array}$$

The morphisms 2, 3, 5 and 6 come from morphisms of pairs satisfying the conditions of Lemma 2.1. The morphisms 2 and 5 are weak equivalences and hence define isomorphisms on the motivic cohomology. We want to show that $5^*(2^{-1})^*1^*(a_Z) = t \wedge a_{Y,Z}$. Since $a_Z = 3^*(a_{f(Z\times\mathbf{A}^1)})$ the left hand side equals $7^*6^*(a_{f(Z\times\mathbf{A}^1)}) = 7^*(a_Z)$. Consider now the commutative diagram of pointed sheaves:

$$\begin{array}{ccc}
Th(N_{X,Y}) & \xrightarrow{7} & N_{X,Y}/(N_{X,Y}-Z) \\
 & & {\scriptstyle 8}\downarrow \\
{\scriptstyle 9}\downarrow & & Th(N_{X,Y}) \wedge N_{X,Y}/(N_{X,Y}-(N_{X,Y})_{|Z}) \\
 & & {\scriptstyle 10}\downarrow \\
Th(N_{X,Y}) \wedge Y & \xrightarrow{11} & Th(N_{X,Y}) \wedge Y/(Y-Z).
\end{array}$$

By Lemma 2.3 and Lemma 2.2 we have $a_Z = 8^*(t \wedge a)$ where t is the Thom class and

$$a = a_{N_{X,Y},(N_{X,Y})_{|Z}}$$

is the class of the closed subscheme $(N_{X,Y})_{|Z}$ in $N_{X,Y}$. Lemma 2.1 applied to the projection

$$(N_{X,Y}, (N_{X,Y})_{|Z}) \to (Y, Z)$$

shows that $a_Z = 8^* 10^*(t \wedge a_{Y,Z})$ and we conclude that $7^*(a_Z) = 9^*(t \wedge a_{Y,Z})$.

Proposition **2.5.** — *There exists a unique family of homomorphisms*

$$deg : H^{2d,d}(X, \mathbf{Z}) \to \mathbf{Z}$$

given for all fields k, integers $d \geq 0$ and smooth projective schemes X *over k of pure dimension d, which satisfies the following two conditions:*

1. deg commutes with the maps defined by field extensions $k \subset E$
2. for a rational point x of X *one has $deg(a_x) = 1$.*

Proof. — One can easily see that it suffices to prove the proposition for algebraically closed fields. Since the motivic complex of weight n has no cohomology sheaves in degree $> n$, Lemma 2.6 below implies that any class in $H^{2d,d}$ vanishes on the complement to a closed subset Z of dimension zero. If the base field is algebraically closed any such subset is the union of a finite number of rational points. Together with the purity isomorphism and the Thom isomorphism for the motivic cohomology this implies that $H^{2d,d}(X)$ is generated by classes a_x of rational points. This immediatley implies the uniqueness part. Extending the definition of a_Z to zero cycles $\mathcal{Z} = \sum n_i z_i$ we get an epimorphism $C_0(X) \to H^{2d,d}(X)$ from the group of zero cycles on X to $H^{2d,d}$. It remains to check that for a cycle $\mathcal{Z}$ such that $a_{\mathcal{Z}} = 0$ one has $deg(\mathcal{Z}) = 0$.

Consider first the case $X = \mathbf{P}^d$. Let x be a rational point of $\mathbf{P}^d$ and $a = a_x$. Then for any other rational point x' one has $a_{x'} = a$ since any two points can be transformed to each other by an automorphism which is homotopic to the identity. Hence for a cycle $\mathcal{Z}$ one has $a_{\mathcal{Z}} = deg(\mathcal{Z})a$. The map $\mathbf{Z} \to H^{2d,d}(\mathbf{P}^d, \mathbf{Z})$ defined by $n \mapsto na$ is a monomorphism because $H^{2d,d}(\mathbf{P}^d - x, \mathbf{Z}) = H^{2d,d}(\mathbf{P}^{d-1}, \mathbf{Z}) = 0$. Therefore, for $\mathcal{Z}$ such that $a_{\mathcal{Z}} = 0$ we get $deg(\mathcal{Z}) = 0$.

For a general projective X choose a closed embedding $i : X \to \mathbf{P}^N$ and let $\gamma : \mathbf{P}^N \to Th(N_{\mathbf{P}^N,X})$ be the composition of the natural projection with the purity map. Then by Lemma 2.4 we have

$$\gamma^*(t_N \wedge a_{\mathscr{Z}}) = a_{i_*(\mathscr{Z})}$$

and if $a_{\mathscr{Z}} = 0$ we conclude that $a_{i_*(\mathscr{Z})} = 0$ and hence $deg(\mathscr{Z}) = deg(i_*(\mathscr{Z})) = 0$.

Lemma **2.6.** — *Let* X *be a noetherian scheme of finite dimension,* F *a sheaf on the Nisnevich site of* X *and* $\alpha \in H^n_{Nis}(X, F)$ *a cohomology class. Then there exists a closed subset* Z *in* X *of codimension at least n such that the restriction of* α *to* $X - Z$ *is zero.*

Proof. — Consider the morphism of sites $p : X_{Nis} \to X_{Zar}$. The fiber of the higher direct image $R^ip_*(F)$ in a point x of X is the Nisnevich cohomology group $H^i_{Nis}(X_x, F)$ where X_x is the local scheme of x in X. By the cohomological dimension theorem in the Nisnevich topology we conclude that this fiber is zero for $i > dim(X_x)$ i.e. for $codim(x) < i$. Hence, $R^ip_*(F)$ is a sheaf supported in codimension i. A standard argument shows that for a Noetherian topological space T of finite dimension, a sheaf F on T supported in codimension i and a class $\alpha \in H^m(T, F)$ there exists a closed subset Z of codimension $i + m$ such that α is zero on $T - Z$. Together with the Leray spectral sequence associated with p this implies the statement of the lemma.

Recall that we let T^n denote the n-sphere $\mathbf{A}^n/\mathbf{A}^n - \{0\}$. The following proposition is a key step in the proof of Theorem 2.11.

Proposition **2.7.** — *There exists a vector bundle* V_d *of dimension* n_d *on* $\mathbf{P}^d$ *and a morphism* $f : T^{d+n_d} \to Th_{\mathbf{P}^d}(V_d)$ *in* $H_\bullet$ *such that:*

1. *if* $T_{\mathbf{P}^d}$ *is the tangent bundle of* $\mathbf{P}^d$ *then* $V_d + T_{\mathbf{P}^d} = \mathscr{O}^{n_d+d}$ *in* $K_0(\mathbf{P}^d)$
2. *for a rational point* x *of* $\mathbf{P}^d$ *one has* $f^*(t_V \wedge a_x) = t$ *where* t *is the canonical generator of* $H^{2(d+n_d),d+n_d}(T^{d+n_d})$.

Proof. — Let $W = \Omega \oplus (\Omega \otimes T_{\mathbf{P}})$ where $T_{\mathbf{P}}$ is the tangent bundle on $\mathbf{P}^d$ and Ω is its dual. The dimension of W is $n = d^2 + d$.

Lemma **2.8.** — $W + T_{\mathbf{P}} = \mathscr{O}^{d^2+2d}$

Proof. — We have to show that $T_{\mathbf{P}} + \Omega + \Omega \otimes T_{\mathbf{P}} = \mathscr{O}^{d^2+2d}$. Consider the standard exact sequence

$$0 \to \Omega \to \mathscr{O}(-1)^{d+1} \to \mathscr{O} \to 0$$

(see [9, Th. 8.13]) and its dual

$$0 \to \mathscr{O} \to \mathscr{O}(1)^{d+1} \to T_{\mathbf{P}} \to 0.$$

The first sequence implies that $\Omega + \mathcal{O} = (d+1)\mathcal{O}(-1)$, hence

$$\mathrm{T}_{\mathbf{P}} + \Omega \otimes \mathrm{T}_{\mathbf{P}} = \mathrm{T}_{\mathbf{P}} \otimes (\mathcal{O} + \Omega) = (d+1)\mathrm{T}_{\mathbf{P}}(-1).$$

Form the second sequence $\mathrm{T}_{\mathbf{P}}(-1) = (d+1)\mathcal{O} - \mathcal{O}(-1)$, hence

$$\begin{aligned}\Omega + \mathrm{T}_{\mathbf{P}} + \Omega \otimes \mathrm{T}_{\mathbf{P}} &= \Omega + (d+1)\mathrm{T}_{\mathbf{P}}(-1) = \\ &= \Omega + (d+1)^2\mathcal{O} - (d+1)\mathcal{O}(-1) = (d^2+2d)\mathcal{O}.\end{aligned}$$

Consider the incidence hyperplane H in $\mathbf{P}^d \times \mathbf{P}^d$ where the first projective space is thought of as the projective space of a vector space $\mathcal{O}^{d+1}$ and the second one as the projective space of the dual $(\mathcal{O}^{d+1})^*$. The complement

$$\widetilde{\mathbf{P}}^d = \mathbf{P}^d \times \mathbf{P}^d - \mathrm{H}$$

considered as a scheme over $\mathbf{P}^d$ by means of the projection to the first factor $p\colon \widetilde{\mathbf{P}}^d \to \mathbf{P}^d$ is an affine bundle over $\mathbf{P}^d$. On the other hand, the Segre embedding

$$i_{d,d} : \mathbf{P}^d \times \mathbf{P}^d \to \mathbf{P}^{d^2+2d}$$

gives H as the divisor at infinity for the appropriate choice of an intersecting hyperplane H_∞. Therefore, $\widetilde{\mathbf{P}}^d$ is an affine variety. This construction is known as the Jouanolou trick (see [10]).

Consider the fiber product:

(**8**)
$$\begin{array}{ccc}\widetilde{\mathbf{P}}^d \times \mathbf{P}^d & \xrightarrow{h} & \mathbf{P}^d \times \mathbf{P}^d \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle pr} \\ \widetilde{\mathbf{P}}^d & \xrightarrow{p} & \mathbf{P}^d.\end{array}$$

The open embedding $\widetilde{\mathbf{P}}^d \to \mathbf{P}^d \times \mathbf{P}^d$ defines a section s of v. Let N be the normal bundle to the Segre embedding $i_{d,d}$ and let E be the normal bundle to s.

Lemma **2.9.** — *There is an integer m and an isomorphism of vector bundles*

(**9**)
$$\mathrm{E} \oplus \mathrm{N} \oplus \mathcal{O}^m \cong p^*(\mathrm{W}) \oplus \mathcal{O}^m$$

on $\widetilde{\mathbf{P}}^d$.

Proof. — Since $\widetilde{\mathbf{P}}^d$ is affine, two vector bundles give the same class in K_0 if and only if they become isomorphic after the addition of $\mathcal{O}^m$ for some $m \geq 0$. Therefore, to prove the lemma it is sufficient to show that one has $\mathrm{E} + \mathrm{N} + p^*(\mathrm{T}_{\mathbf{P}^d}) = \mathcal{O}^{d^2+2d}$ in

$K_0(\widetilde{\mathbf{P}}^d)$. Let $p' : \widetilde{\mathbf{P}}^d \to \mathbf{P}^d$ be the second of the two projections. One can easily see from the definition of E that $E = (p')^*(T_{\mathbf{P}^d})$. Hence $E + p^*(T_{\mathbf{P}^d})$ is the restriction to $\widetilde{\mathbf{P}}^d$ of the tangent bundle on $\mathbf{P}^d \times \mathbf{P}^d$. On the other hand the short exact sequence which defines N shows that over $\widetilde{\mathbf{P}}^d$ we have

$$T_{\mathbf{P}^d \times \mathbf{P}^d} + N = \mathcal{O}^{d^2+d}$$

which finishes the proof of the lemma.

Lemma **2.10.** — *There exists a map* $g : T^{d^2+2d} \to Th_{\widetilde{\mathbf{P}}^d}(E \oplus N)$ *such that for a rational point* x *of* $\mathbf{P}^d$ *one has* $g^*(t_{E\oplus N} \wedge p^*(a_x)) = t$.

Proof. — Consider the pointed sheaf

$$F = \mathbf{P}^{d^2+2d}/\big(H_\infty \cup \big(\mathbf{P}^{d^2+2d} - i_{d,d}(\mathbf{P}^d \times \mathbf{P}^d)\big)\big).$$

There is an obvious map η from $T^{d^2+2d} \cong \mathbf{P}^{d^2+2d}/H_\infty$ to F. We claim that F is isomorphic in $H_\bullet$ to $Th_{\widetilde{\mathbf{P}}^n}(E \oplus N)$ and that the composition of η with this isomorphism satisfies the condition of the lemma.

The same reasoning as in the proof of the homotopy purity theorem in [23, Theorem 2.23], shows that in $H_\bullet$ one has

$$F \cong Th_{\mathbf{P}^d \times \mathbf{P}^d}(N)/Th_H(N).$$

Consider again the pull-back square (8). The map

$$Th_{\widetilde{\mathbf{P}}^d \times \mathbf{P}^d}(h^*(N))/Th_{h^{-1}(H)}(h^*(N)) \to Th_{\mathbf{P}^d \times \mathbf{P}^d}(N)/Th_H(N)$$

is clearly an $\mathbf{A}^1$-weak equivalence. On the other hand one verifies easily that $h^{-1}(H)$ is contained in $\widetilde{\mathbf{P}}^d \times \mathbf{P}^d - s(\widetilde{\mathbf{P}}^d)$ and that this embedding is an $\mathbf{A}^1$-weak equivalence. Hence we have a weak equivalence from

$$Th_{\widetilde{\mathbf{P}}^d \times \mathbf{P}^d}(h^*(N))/Th_{h^{-1}(H)}(h^*(N))$$

to

$$Th_{\widetilde{\mathbf{P}}^d \times \mathbf{P}^d}(h^*(N))/Th_{\widetilde{\mathbf{P}}^d \times \mathbf{P}^d - s(\widetilde{\mathbf{P}}^d)}(h^*(N)).$$

The latter quotient is isomorphic to $h^*(N)/(h^*(N) - s(\widetilde{\mathbf{P}}^d))$ and since the normal bundle to s in h^*N is $N \oplus E$ we conclude by [23, Theorem 2.23] that it is weakly equivalent to $Th_{\widetilde{\mathbf{P}}^d}(E \oplus N)$. This finishes the construction of the map $g : T^{d^2+2d} \to Th_{\widetilde{\mathbf{P}}^d}(E \oplus N)$. It remains to check that for a rational point x of $\mathbf{P}^d$ we have $g^*(t_{E\oplus N} \wedge p^*(a_x)) = t$.

Consider the following diagram

$$\begin{array}{ccc}
\mathbf{P}_+^{d^2+2d} & \longrightarrow & \mathbf{P}^{d^2+2d}/\mathrm{H} \\
\Big\downarrow & & \Big\downarrow \\
\mathbf{P}^{d^2+2d}/\big(\mathbf{P}^{d^2+2d} - i_{d,d}(\mathbf{P}^d \times \mathbf{P}^d)\big) & \longrightarrow & \mathrm{F} \\
{\scriptstyle \rho_{i_{d,d}}}\Big\downarrow & & \Big\downarrow \\
\mathrm{Th}_{\mathbf{P}^d\times\mathbf{P}^d}(\mathrm{N}) & \longrightarrow & \mathrm{Th}_{\mathbf{P}^d\times\mathbf{P}^d}(\mathrm{N})/\mathrm{Th}_{\mathrm{H}}(\mathrm{N}) \\
{\scriptstyle \tilde{h}}\Big\uparrow & & \Big\uparrow \\
\mathrm{Th}_{\widetilde{\mathbf{P}}^d\times\mathbf{P}^d}(\mathrm{N}) & \longrightarrow & \mathrm{Th}_{\widetilde{\mathbf{P}}^d\times\mathbf{P}^d}(\mathrm{N})/\mathrm{Th}_{h^{-1}(\mathrm{H})}(\mathrm{N}) \\
\Big\downarrow & & \Big\downarrow \\
h^*(\mathrm{N})/(h^*(\mathrm{N}) - s(\widetilde{\mathbf{P}}^d)) & \xrightarrow{=} & h^*(\mathrm{N})/(h^*(\mathrm{N}) - s(\widetilde{\mathbf{P}}^d)) \\
{\scriptstyle \rho_s}\Big\downarrow & & \Big\downarrow \\
\mathrm{Th}_{\widetilde{\mathbf{P}}^d}(\mathrm{E}\oplus\mathrm{N}) & \xrightarrow{=} & \mathrm{Th}_{\widetilde{\mathbf{P}}^d}(\mathrm{E}\oplus\mathrm{N})
\end{array}$$

where the arrows going up are weak equivalences and g is the morphism defined by the right vertical side of the diagram. One verifies immediately that it commutes. The upper horizontal arrow gives a monomorphism on motivic cohomology and (by [42, Lemma 4.2]) the pull-back of t along this arrow coincides with a_y for any rational point y of $\mathbf{P}^{d^2+2d}$. On the other hand applying several times Lemma 2.4 we conclude that the pull-back of $p^*(a_x)$ along the left vertical side of the diagram is $a_{j(x)}$ where j is the composition $\mathbf{P}^d \xrightarrow{\Delta} \mathbf{P}^d \times \mathbf{P}^d \xrightarrow{i_{d,d}} \mathbf{P}^{d^2+2d}$. Lemma is proved.

We are now ready to finish the proof of Proposition 2.7. We take $\mathrm{V}_d = \mathrm{W} \oplus \mathcal{O}^m$ and $n = d^2+d+m$. The map $p : \widetilde{\mathbf{P}}^d \to \mathbf{P}^d$ defines a map of Thom spaces $\mathrm{Th}_{\widetilde{\mathbf{P}}^n}(p^*(\mathrm{V}_d)) \to \mathrm{Th}_{\mathbf{P}^n}(\mathrm{V}_d)$. The isomorphism (9) defines an isomorphism

$$\Sigma_{\mathrm{T}}^m \mathrm{Th}_{\widetilde{\mathbf{P}}^n}(\mathrm{E}\oplus\mathrm{N}) \cong \mathrm{Th}_{\widetilde{\mathbf{P}}^n}(p^*(\mathrm{V}_d)).$$

Composing these maps with the m-fold suspension of g we get a map

$$f : \mathrm{T}^{d^2+2d+m} \to \mathrm{Th}_{\mathbf{P}^d}(\mathrm{V}_d).$$

For a_x we have

$$\begin{aligned} f^*(t_{\mathrm{V},d} \wedge a_x) &= (\Sigma_{\mathrm{T}}^m g)^*\big(t_{\mathrm{E}\oplus\mathrm{N}\oplus\mathcal{O}^m} \wedge p^*(a_x)\big) = \\ &= g^*\big(t_{\mathrm{E}\oplus\mathrm{N}} \wedge p^*(a_x)\big) = t \end{aligned}$$

which shows that f satisfies the second condition of the proposition.

Theorem **2.11.** — *Let* X *be a smooth projective variety of pure dimension* d *over a field* k. *Then there exists an integer* n *and a vector bundle* V *on* X *of dimension* n *such that:*

1. *if* T_X *is the tangent bundle of* X *then* $V + T_X = \mathcal{O}^{n+d}$ *in* $K_0(X)$
2. *There exists a morphism in* $H_\bullet$ *of the form* $f_V : T^{n+d} \to Th_X(V)$ *such that the map* $H^{2d,d}(X) \to \mathbf{Z}$ *defined by* f_V *and the Thom isomorphism coincides with the degree map.*

Proof. — Let $i : X \to \mathbf{P}^m$ be a closed embedding and N the normal bundle to i. Let further N' be the normal bundle to X in V_m. We claim that the composition

$$T^{n_m+m} \xrightarrow{f} Th_{\mathbf{P}^m}(V_m) \xrightarrow{\pi} V_m/(V_m - i(X)) \xrightarrow{\rho} Th_X(N') \tag{10}$$

where the first arrow is the morphism of Proposition 2.7, the second arrow is the projection and the third is the purity map for the pair $i(X) \subset V_m$, satisfies the conditions of the theorem.

To verify the first condition observe that $N' = N \oplus i^*(V_m)$ and $T_X = i^*(T_{\mathbf{P}^m}) - N$. Therefore,

$$T_X + N' = T_X + N + i^*(V_m) = \mathcal{O}^{n_m+m}.$$

To verify that (10) induces the degree map on $H^{2d,d}$ we may pass to the algebraic closure of the base field. Over an algebraically closed field $H^{2d,d}$ is generated by classes of rational points and it is enough to check that for a point x of X one has

$$f^*\pi^*\rho^*(t_{N'} \wedge a_x) = t \tag{11}$$

where $t_{N'}$ is the Thom class of N' and t is the canonical generator of

$$H^{2(n_m+m), n_m+m}(T^{n_m+m}) = \mathbf{Z}.$$

By Lemma 2.4 we have $\rho^*(t_{N'} \wedge a_x) = a_{si(x)}$ where $s : \mathbf{P}^d \to Th(V_m)$ is the zero section. By Lemma 2.2 we have $\pi^*(a_{si(x)}) = t_N \wedge a_{i(x)}$ and finally $f^*(a_{i(x)}) = t$ by Proposition 2.7.

3. The motivic analog of Margolis homology

In this section we introduce the motivic version of Margolis homology. The definition of topological Margolis homology[1] is based on the fact that the Steenrod algebra contains some very special elements Q_i called Milnor's primitives. These elements generate an exterior subalgebra in the Steenrod algebra and in particular $Q_i^2 = 0$. Hence, one may consider the cohomology of a space or a spectrum as a complex

[1] which appeared in [16] and which I learned about from [26].

with the differential given by $\widetilde{Q}_i$. The homology of this complex are known in topology as Margolis homology $\widetilde{MH}^*_i$. Spaces or spectra whose Margolis homology vanish for $i \leq n$ play an important role in the proofs of the amazing recent results on the structure of the stable homotopy category.

Since we only know how to construct reduced power operations in the motivic cohomology with $\mathbf{Z}/l$ coefficients for $l \neq char(k)$

everywhere in this section l is a prime not equal to the characteristic of the base field.

Recall that we defined in [42, §13] operations Q_i in the motivic cohomology with $\mathbf{Z}/l$-coefficients of the form

$$Q_i : \widetilde{H}^{p,q}(-, \mathbf{Z}/l) \to \widetilde{H}^{p+2l^i-1,q+l^i-1}(-, \mathbf{Z}/l).$$

We proved in [42, Proposition 13.4] that operations Q_i have the property $Q_i^2 = 0$. For any pointed simplicial sheaf $\mathscr{X}$ the cohomology of the complex

$$\widetilde{H}^{p-2l^i+1,q-l^i+1}(\mathscr{X}, \mathbf{Z}/l) \to \widetilde{H}^{p,q}(\mathscr{X}, \mathbf{Z}/l) \to \widetilde{H}^{p+2l^i-1,q+l^i-1}(\mathscr{X}, \mathbf{Z}/l)$$

at the term $\widetilde{H}^{p,q}$ is called the i-th motivic Margolis cohomology of $\mathscr{X}$ of bidegree (p, q) and we denote it by $\widetilde{MH}^{p,q}_i(\mathscr{X})$. By [42, Lemma 13.5] Q_0 is the Bockstein homomorphism and we get the following important sufficient condition for the vanishing of $\widetilde{MH}^{*,*}_0(\mathscr{X})$.

Lemma **3.1.** — *Let $\mathscr{X}$ be a pointed simplicial sheaf such that $l\widetilde{H}^{*,*}(\mathscr{X}, \mathbf{Z}_{(l)}) = 0$. Then $\widetilde{MH}^{*,*}_0(\mathscr{X}) = 0$.*

Proof. — Follows easily from the long exact sequences in the motivic cohomology defined by the short exact sequence $0 \to \mathbf{Z} \xrightarrow{l} \mathbf{Z} \to \mathbf{Z}/l \to 0$.

For a smooth variety X over k we denote by $s_d(T_X)$ the d-th Milnor class of X i.e. the characteristic class of the tangent bundle T_X which corresponds to the Newton polynomial in the Chern classes of degree d (see [42, Corollary 14.3] for a more careful definition).

Theorem **3.2.** — *Let* Y *be a smooth projective variety over k such that there exists a map* X → Y *where* X *is a smooth projective variety of dimension $l^m - 1$ satisfying*

$$deg(s_{l^m-1}(X)) \neq 0 (mod\ l^2).$$

Then $\widetilde{MH}^{p,q}_m(\widetilde{C}(Y), \mathbf{Z}/l) = 0$ *for all p and q (see Appendix B for the definition of* $\widetilde{C}(Y)$*).*

Proof. — For $m = 0$ our condition means that Y has a rational point over a separable extension of k of degree n where $n \neq 0 (mod\ l^2)$. By Lemma 9.3 this implies that the group $\widetilde{H}^{*,*}(\widetilde{C}(Y), \mathbf{Z}_{(l)})$ has exponent l and by Lemma 3.1 we conclude that $\widetilde{MH}_0^{*,*}(\widetilde{C}(Y)) = 0$.

Assume that $m > 0$. Set $d = l^m - 1$. Let V be a vector bundle on X and $f_V : T^{d+n} \to Th_X(V)$ a map satisfying the conclusion of Theorem 2.11. Consider the cofibration sequence

$$T^{d+n} \xrightarrow{f_V} Th_X(V) \to cone \xrightarrow{\psi} \Sigma_s^1 T^{n+d}$$

corresponding to f_V. The long exact sequence of motivic cohomology corresponding to this cofibration sequence shows that there exists a unique class

$$\alpha \in \tilde{H}^{2n,n}(cone)$$

whose restriction to $Th_X(V)$ is the Thom class t_V. Multiplication with this class gives us a map

$$\widetilde{H}^{p,q}(\widetilde{C}(Y)) \to \widetilde{H}^{p+2n,q+n}(cone \wedge \widetilde{C}(Y)). \tag{12}$$

Consider the map $cone \wedge \widetilde{C}(Y) \to (\Sigma_s^1 T^{n+d}) \wedge \widetilde{C}(Y)$. We claim that this map is a weak equivalence. Indeed, it is a part of a cofibration sequence and to verify that it is a weak equivalence it is enough to check that $Th_X(V) \wedge \widetilde{C}(Y)$ is contractible. This follows from the cofibration sequence

$$(V - \{0\})_+ \wedge \widetilde{C}(Y) \to V_+ \wedge \widetilde{C}(Y) \to Th_X(V) \wedge \widetilde{C}(Y)$$

and Lemma 9.2. Since $cone \wedge \widetilde{C}(Y) \to (\Sigma_s^1 T^{n+d}) \wedge \widetilde{C}(Y)$ is a weak equivalence, (12) defines a map

$$\phi : \widetilde{H}^{p,q}(\widetilde{C}(Y)) \to \widetilde{H}^{p-2d-1,q-d}(\widetilde{C}(Y)).$$

We claim that ϕ is something like a contracting homotopy for the complex $(\widetilde{H}_n^{*,*}(\widetilde{C}(Y)), Q_m)$. More precisely we have the following lemma which clearly imply the statement of the theorem.

Lemma **3.3.** — *There exists $c \in (\mathbf{Z}/l)^*$ such that for any $x \in \widetilde{H}^{p,q}(\widetilde{C}(Y), \mathbf{Z}/l)$ one has*

$$cx = \phi Q_m(x) - Q_m \phi(x).$$

Proof. — Let $\gamma \in \widetilde{H}^{2n+2d+1,n+d}(cone)$ be the pull-back of the tautological motivic cohomology class on $\Sigma_s^1 T^{n+d}$ with respect to ψ. Since the map

$$cone \wedge \widetilde{C}(Y) \to (\Sigma_s^1 T^{n+d}) \wedge \widetilde{C}(Y)$$

is a weak equivalence it is sufficient to verify that there exists $c \in (\mathbf{Z}/l)^*$ such that

$$c\gamma \wedge x = \alpha \wedge Q_m(x) - Q_m(\alpha \wedge x).$$

Let us show that for $i < m$ we have $Q_i(\alpha) = 0$. Indeed, the restriction of $Q_i(\alpha)$ to $Th_X(V)$ is zero by [42, Theorem 14.2(1)] for any i. On the other hand for $i < m$ the restriction map is a mono since there are is motivic cohomology of negative weight. This fact together with [42, Proposition 13.4] implies that we have

$$Q_m(\alpha \wedge x) = \alpha \wedge Q_m(x) + Q_m(\alpha) \wedge x$$

and therefore

$$Q_m(\alpha) \wedge x = \alpha \wedge Q_m(x) - Q_m(\alpha \wedge x).$$

We now have two possibilities. If Y has a point over an extension of degree prime to l then $\widetilde{H}^{p,q}(\widetilde{C}(X)) = 0$ and the statement of our lemma obviously holds. Assume that Y has no points over extensions of degree prime to l. Let us show that under this assumption $Q_m(\alpha) = c\gamma$ for $c \in (\mathbf{Z}/l)^*$. Note first that since Y has no points over extensions of degree prime to l, X does not have points over such extensions either. Since $Q_m(\alpha)$ restricts to zero on $Th_X(V)$ by [42, Theorem 14.2(1)] it is sufficient to show that $Q_m(\alpha) \neq 0$. By [42, Proposition 13.6] we have $Q_m = \beta q_m \pm q_m\beta$ where β is the Bockstein homomorphism and q_m is another cohomological operation. Since α is a reduction of an integral class we have $Q_m(\alpha) = \beta q_m(\alpha)$ and to show that it is non-zero we have to check that $q_m(\alpha)$ can not be lifted to cohomology with $\mathbf{Z}/l^2$-coefficients. If it could there would be a lifting y of $q_m(t_V)$ to the cohomology with $\mathbf{Z}/l^2$-coefficients such that $f_V^*(y) = 0$. Our condition that X has no points over extensions of degree prime to l implies, by Lemma 3.4, that the value of $f_V^*(y)$ does not depend on the choice of y. On the other hand by [42, Corollary 14.3] we know that $q_m(t_V)$ is the reduction of the integral class $s_d(V)$. It remains to show that this class is non-zero modulo l^2. By Theorem 2.11(1) we have $V + T_X = \mathcal{O}^N$ where T_X is the tangent bundle on X. Since s_d is an additive class we conclude that $s_d(V)$ equals to $-s_d(X)$ which is non-zero modulo l^2 by our assumption on X.

Lemma **3.4.** — *Let* X *be a smooth projective variety of pure dimension d and n an integer prime to char*(k). *Then the map*

$$deg_n : H^{2d,d}(X, \mathbf{Z}) \xrightarrow{deg} \mathbf{Z} \longrightarrow \mathbf{Z}/n$$

is surjective if and only if X *has a rational point over an extension of k of degree prime to n.*

Proof. — The transfer argument shows that $coker(deg_n)$ does not change under the base field extensions of degree prime to n. Hence, we may pass to the maximal purely inseparable extension and assume that k is perfect. Under this assumption any zero cycle is supported in a smooth subscheme and the reasoning of the proof of Proposition 2.5 shows that any class in $H^{2d,d}$ is the class of a zero cycle. This implies the statement of the lemma.

Remark **3.5.** — The conclusion of Lemma 3.4 holds without the assumption that n is prime to $char(k)$.

The following result provides us with a class of varieties X satisfying the condition of Theorem 3.2.

Proposition **3.6.** — *Let* X *be a smooth hypersurface of degree* d *in* $\mathbf{P}^n$. *Then*

$$deg(s_{n-1}(X)) := deg(s_{n-1}(T_X)) = d(n+1-d^{n-1}).$$

In particular, if X *is a smooth hypersurface of degree* l *in* $\mathbf{P}^{l^m}$ *then*

$$deg(s_{l^m-1}(X)) = l(mod\ l^2).$$

Proof. — For the hypersurface given by a generic section of a vector bundle L the normal bundle is canonically isomorphic to L. In particular we have a short exact sequence of the form

$$0 \to T_X \to i^*(T_{\mathbf{P}^n}) \to i^*(\mathcal{O}(d)) \to 0.$$

The tangent bundle on $\mathbf{P}^n$ fits into an exact sequence of the form

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T_{\mathbf{P}^n} \to 0.$$

Therefore, in $K_0(X)$ we have

$$T_X = i^*(T_{\mathbf{P}^n}) - i^*(\mathcal{O}(d)) = (n+1)i^*(\mathcal{O}(1)) - \mathcal{O} - i^*(\mathcal{O}(d)).$$

Since s_{n-1} is an additive characteristic class which on line bundles is given by $s_{n-1}(L) = e(L)^{n-1}$ we get for $n > 1$

$$s_{n-1}(T_X) = (n+1-d^{n-1})i^*((-\sigma)^{n-1})$$

where $\sigma = e(\mathcal{O}(-1))$. By Lemma 3.7 we conclude that

$$deg(s_{n-1}(X)) := deg(s_{n-1}(T_X)) = d(n+1-d^{n-1}).$$

Lemma **3.7.** — *Let* $i : X \to \mathbf{P}^n$ *be a smooth hypersurface of degree* d. *Then* $deg(i^*(e(\mathcal{O}(1))^{n-1})) = d$.

Proof. — It is easy to see that the Euler class $e(\mathcal{O}(1))$ coincides with the class a_H where H is a hyperplane in $\mathbf{P}^n$. Together with Lemma 2.3 it shows that $e(\mathcal{O}(1))^{n-1}$

coincides with a_L where L is a line in $\mathbf{P}^n$. Choosing L which intersects $i(X)$ transversally we conclude by Lemma 2.1 that $i^*(e(\mathcal{O}(1))^{n-1}) = a_{i^{-1}(L)}$. Therefore, the degree of this class is the degree of the zero cycle $i^{-1}(L)$ is d.

Combining Theorem 3.2 and Proposition 3.6 we get the following result which is the only result of this section used for the proof of Theorem 7.4.

Corollary **3.8.** — *Let Q be a smooth quadric in* $\mathbf{P}^{2^n}$. *Then*

$$\widetilde{\mathrm{MH}}_i^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Q}), \mathbf{Z}/2) = 0$$

for all $i \leq n$.

4. Norm quadrics and their motives

The goal of this section is to prove Theorems 4.4 and 4.9. Everywhere in this section (except for Lemmas 4.5, 4.7 and 4.11) k is a field of characteristic $\neq 2$.

For elements $a_1, \ldots, a_n$ in k^* let $\langle a_1, \ldots, a_n\rangle$ be the quadratic form $\sum a_i x_i^2$. One defines the Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ as the tensor product

$$\langle\langle a_1, \ldots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \ldots \otimes \langle 1, -a_n\rangle. \tag{13}$$

Denote by $\mathrm{Q}_{\underline{a}} = \mathrm{Q}_{a_1,\ldots,a_n}$ the projective quadric of dimension $2^{n-1} - 1$ given by the equation $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle = a_n t^2$. For $n = 2$ the rational points of the affine part of this quadric ($t \neq 0$) can be identified with non-zero elements x of $\mathrm{E} = k(\sqrt{a_1})$ such that $\mathrm{N}_{\mathrm{E}/k}(x) = a_2$. Because of this interpretation the quadric given by (13) is called the norm quadric associated with the sequence $\underline{a} = (a_1, \ldots, a_n)$.

The following result is well known but we decided to include the proof since it is crucial for our main theorem.

Proposition **4.1.** — *The symbol* $\{a_1, \ldots, a_n\}$ *is divisible by* 2 *in* $\mathrm{K}_n^{\mathrm{M}}(k(\mathrm{Q}_{\underline{a}}))$.

Proof. — We are going to show that if $\mathrm{Q}_{\underline{a}}$ has a rational point over k then $\underline{a}$ is divisible by 2 in $\mathrm{K}_n^{\mathrm{M}}(k)$. Since any variety has a point over its function field this implies that $\underline{a}$ is divisible by 2 in the generic point of $\mathrm{Q}_{\underline{a}}$. Let $\mathrm{P}_{\underline{a}}$ denote the quadric given by the equation $\langle\langle a_1, \ldots, a_n\rangle\rangle = 0$.

Lemma **4.2.** — *For any* $\underline{a} = (a_1, \ldots, a_n)$ *the following two conditions are equivalent*

1. $\mathrm{Q}_{\underline{a}}$ *has a k-rational point*
2. $\mathrm{P}_{\underline{a}}$ *has a k-rational point*

Proof. — The first condition implies the second one because the form

$$\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \oplus \langle -a_n\rangle$$

is a subform in $\langle\langle a_1, \ldots, a_n\rangle\rangle$ and therefore $Q_{\underline{a}}$ is a subvariety in $P_{\underline{a}}$. Assume that the second condition holds. By [14, Corollary 1.6] it implies that the form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is hyperbolic. Hence, for any rational point p of $P_{\underline{a}}$ there exists a linear subspace H of dimension $dim(P_{\underline{a}})/2 = 2^{n-1} - 1$ which lies on $P_{\underline{a}}$ and passes through p. The quadric $Q_{\underline{a}}$ is a section of $P_{\underline{a}}$ by a linear subspace L of codimension $2^{n-1} - 1$ in $\mathbf{P}^{2^n-1}$. The intersection of H and L is a rational point on $Q_{\underline{a}}$.

To prove Proposition 4.1 we proceed by induction on n. Consider first the case $n = 2$. Then $Q_{\underline{a}}$ is given by the equation $x^2 - a_1 y^2 = a_2 z^2$. We may assume that it has a point of the form $(x_0, y_0, 1)$ (otherwise a_1 is a square root in k and the statement is obvious). Then a_2 is the norm of the element $x_0 + \sqrt{a_1}\, y_0$ from $k(\sqrt{a_1})$ and thus the symbol $\{a_1, a_2\}$ is divisible by 2.

Suppose that the proposition is proved for sequences $(a_1, \ldots, a_i)$ of length smaller than n. The quadric $Q_{\underline{a}}$ is given by the equation

$$\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle = a_n t^2.$$

The form $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ is of the form $\langle 1\rangle \oplus \mu_{\underline{a}}$. By induction we may assume that our point $q \in Q_{\underline{a}}(k)$ belongs to the affine part $t \neq 0$. Consider the plane L generated by points $(1, 0, \ldots, 0)$ and q. The restriction of the quadratic form $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ to L is of the form $\langle\langle b\rangle\rangle$ for some b (the idea is that L is a "subfield" in the vector space where $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ lives). Consider the field extension $k(\sqrt{b})$. The form $\langle\langle b\rangle\rangle$ and therefore the form $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ represents zero over $k(\sqrt{b})$ and thus by Lemma 4.2 and the inductive assumption $(a_1, \ldots, a_{n-1}) = 0$ in $K^M_{n-1}(k(\sqrt{b}))/2$. On the other hand by the construction $\langle\langle b\rangle\rangle$ represents a_n and therefore we have $a_n \in \operatorname{Im} N_{k(\sqrt{b})/k} \subset k^*$ which proves the proposition.

Denote by $\mathscr{C}$ the category of Chow motives over k and let $\mathbf{Z}\{n\}$ be the n-th Tate (or, rather, Lefshetz) motive in this category. We will use the following important result.

Theorem **4.3** *(Markus Rost).* — *There exists a direct summand* $M_{\underline{a}}$ *of* $Q_{\underline{a}}$ *in* $\mathscr{C}$ *together with two morphisms*

$$\psi^* : \mathbf{Z}\{2^{n-1} - 1\} \to M_{\underline{a}}$$
$$\psi_* : M_{\underline{a}} \to \mathbf{Z}$$

such that

1. *the composition* $Q_{\underline{a}} \to M_{\underline{a}} \xrightarrow{\psi_*} \mathbf{Z}$ *is the morphism defined by the projection* $Q_{\underline{a}} \to \operatorname{Spec}(k)$

2. *for any field* F *over* k *where* $Q_{\underline{a}}$ *has a point the pull-back of the sequence*

$$\mathbf{Z}\{2^{n-1}-1\} \to M_{\underline{a}} \to \mathbf{Z}$$

to F *is split-exact.*

Proof. — See [29], [30], [12, Prop. 5.2].

The Friedlander–Lawson moving lemma for families of cycles shows that for any k there is a functor from the category of Chow motives over k to $DM_-^{eff}(k)$. Therefore, the Rost motive $M_{\underline{a}}$ can be also considered in DM_-^{eff} where it is a direct summand of the motive $M(Q_{\underline{a}})$ of the norm quadric.

Let $\mathcal{X}_{\underline{a}}$ denote the simplicial scheme $\check{C}(Q_{\underline{a}})$ (see Appendix B). The following result is formulated in terms of the triangulated category of mixed motives $DM_-^{eff} = DM_-^{eff}(k)$ introduced in [39]. The motive $M(\mathcal{X}_a)$ of $\mathcal{X}_{\underline{a}}$ is the class in DM_-^{eff} of the complex obtained from $\mathcal{X}_{\underline{a}}$ by applying the functor $\mathbf{Z}_{tr}(-)$ termwise and then taking the alternating sum of boundary maps.

Theorem **4.4.** — *Let* k *be a perfect field. Then, there exists a distinguished triangle in* DM_-^{eff} *of the form*

$$M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-1]. \tag{14}$$

We start with the following lemmas.

Lemma **4.5.** — *Let* k *be a perfect field,* M *an object of* $DM_-^{eff}(k)$ *and* X *a smooth variety over* k *such that for any generic point* η *of* X *the pull-back of* M *to the residue field* k_η *is zero. Then one has*

$$\mathrm{Hom}_{\mathrm{DM}}(M(X), M) = 0$$
$$M \otimes M(X) = 0.$$

Proof. — Since k is perfect, the group $\mathrm{Hom}(M(X), M)$ is the hypercohomology group of X with coefficients in a complex of sheaves with transfers with homotopy invariant cohomology sheaves K which represents M. Our condition on X and M implies easily that the cohomology sheaves $h_i(K)$ of K vanish on smooth schemes over the generic points of X. Since the $h_i(K)$ are homotopy invariant sheaves with transfers we conclude by [38, Corollary 4.19, p. 116] that they vanish on all schemes etale over X. Therefore, $\mathrm{Hom}(M(X), M) = 0$.

To prove that $M \otimes M(X) = 0$ it is sufficient to show that the class of objects N such that

$$\mathrm{Hom}(N, M \otimes M(X)[m]) = 0$$

for all m, contains $M\otimes M(X)$. Since this class is a localizing subcategory[2], it is sufficient to show that for any smooth Y over k and any m one has

$$\mathrm{Hom}(M(Y \times X), M \otimes M(X)[m]) = 0. \tag{15}$$

If X and M satisfy the conditions of the lemma then for any Y and any N, $Y \times X$ and $M \otimes N$ satify these conditions. Therefore, (15) follows from the first assertion of the lemma.

Lemma **4.6.** — *Let k be a perfect field. Then the sequence of*

$$M(Q_{\underline{a}})(2^{n-1} - 1)[2^n - 2] \xrightarrow{\mathrm{Id}\otimes\psi^*} M(Q_{\underline{a}}) \otimes M_{\underline{a}} \xrightarrow{\mathrm{Id}\otimes\psi_*} M(Q_{\underline{a}}) \tag{16}$$

is split-exact.

Proof. — Observe first that $\mathrm{Id} \otimes \psi_*$ is a split epimorphism. Indeed, Theorem 4.3(1) implies that the morphism

$$M(Q_{\underline{a}}) \to M(Q_{\underline{a}}) \otimes M(Q_{\underline{a}}) \to M(Q_{\underline{a}}) \otimes M_{\underline{a}}$$

where the first arrow is defined by the diagonal and the second by the projection $M(Q_{\underline{a}}) \to M_{\underline{a}}$ is a section of $\mathrm{Id}\otimes\psi_*$. It remains to show that (16) extends to a distinguished triangle. Let *cone* be a cone of the morphism $\psi^* : \mathbf{Z}(2^{n-1} - 1)[2^n - 2] \to M_{\underline{a}}$. Since there is no motivic cohomology of negative weight, the morphism $\psi_* : M_{\underline{a}} \to \mathbf{Z}$ factors through a morphism $\phi : cone \to \mathbf{Z}$. Let $cone'$ be the a cone of ϕ. Standard properties of triangles in triangulated categories imply that the sequence (16) extends to a distinguished triangle if and only if $cone' \otimes M(Q_{\underline{a}}) = 0$. This follows from Theorem 4.3(2) and Lemma 4.5.

Lemma **4.7.** — *Let* X *be a smooth scheme over* k *and* M *an object of the localizing subcategory generated by* M(X)*. Then one has*

1. *the morphism* $M \otimes M(\check{C}(X)) \to M$ *is an isomorphism*
2. *the homomorphism* $\mathrm{Hom}(M, M(\check{C}(X))) \to \mathrm{Hom}(M, \mathbf{Z})$ *is an isomorphism.*

Proof. — It is clearly sufficient to prove the lemma for $M = M(X)$. In this case the first statement follows immediately from Lemma 9.2 and the fact that M takes simplicial weak equivalences to isomorphisms. Let $M(\widetilde{C}(X))$ be the cone of the morphism $M(\check{C}(X)) \to \mathbf{Z}$. To prove the second statement we have to show that any morphism $f : M(X) \to M(\widetilde{C}(X))$ is zero. The morphism f can be written as the composition

$$M(X) \xrightarrow{M(\Delta)} M(X) \otimes M(X) \xrightarrow{f\otimes \mathrm{Id}} M(\widetilde{C}(X)) \otimes M(X) \to M(\widetilde{C}(X))$$

[2] A subcategory in a triangulated category is called localizing if it is closed under triangles, direct sums and direct summands.

where the last arrow is defined by the morphism $M(X) \to \mathbf{Z}$. This composition is zero because the first part of the lemma implies that

$$M(\widetilde{C}(X)) \otimes M(X) = 0.$$

Proof of Theorem 4.4. — The morphism $\psi_* : M_{\underline{a}} \to \mathbf{Z}$ has a canonical lifting to a morphism $\tilde{\psi}_* : M_{\underline{a}} \to M(\mathscr{X}_{\underline{a}})$ by Lemma 4.7(2). Together with the composition

$$\tilde{\psi}^* : M(\mathscr{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to \mathbf{Z}(2^{n-1}-1)[2^n-2] \xrightarrow{\psi^*} M_{\underline{a}}$$

this lifting gives us a sequence of morphisms

$$M(\mathscr{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \xrightarrow{\tilde{\psi}^*} M_{\underline{a}} \xrightarrow{\tilde{\psi}_*} M(\mathscr{X}_{\underline{a}}).$$

The composition $\tilde{\psi}_* \circ \tilde{\psi}^*$ is zero by Lemma 4.7(2) and the fact that

$$\mathrm{Hom}(\mathbf{Z}(2^{n-1}-1)[2^n-2], \mathbf{Z}) = 0.$$

Let *cone* be a cone of $\tilde{\psi}^*$. The morphism $\tilde{\psi}_*$ factors through a morphism $\phi : \mathit{cone} \to M(\mathscr{X}_{\underline{a}})$ and we have to show that ϕ is an isomorphism. The category $\mathscr{C}$ of objects N such that $\phi \otimes \mathrm{Id}_N$ is an isomorphism is a localizing subcategory. By Lemma 4.7(1), the morphism $M(\mathscr{X}_{\underline{a}}) \otimes M(Q^i_{\underline{a}}) \to M(Q^i_{\underline{a}})$ is an isomorphism for any $i > 0$ and we conclude by Lemma 4.6 that $\mathscr{C}$ contains objects $M(Q^i_{\underline{a}})$. Since $\mathscr{C}$ is a localizing subcategory and $M(\mathscr{X}_{\underline{a}})$ can be obtained from the $M(Q^i_{\underline{a}})$ by taking cones and (infinite) direct sums this implies that $\mathscr{C}$ contains $M(\mathscr{X}_{\underline{a}})$.

On the other hand we have a commutative diagram

$$\begin{array}{ccc} \mathit{cone} \otimes M(\mathscr{X}_{\underline{a}}) & \to & M(\mathscr{X}_{\underline{a}}) \otimes M(\mathscr{X}_{\underline{a}}) \\ \downarrow & & \downarrow \\ \mathit{cone} & \to & M(\mathscr{X}_{\underline{a}}) \end{array}$$

with both vertical arrows are isomorphisms by Lemma 4.7(1). This finishes the proof of Theorem 4.4.

Remark **4.8.** — It is essential to use the category DM^{eff}_{-} in Theorem 4.4 because the triangle (14) can not be lifted to the motivic homotopy category (stable or unstable). One can see this for $n = 2$ using the fact that the map $\mathscr{X}_{\underline{a}} \to \mathrm{Spec}(k)$ defines an isomorphism on algebraic K-theory but $K^{*,*}(Q_{\underline{a}}) \neq K^{*,*}(k) \oplus K^{*,*}(k)$.

Theorem **4.9.** — $H^{2^n-1,2^{n-1}}(\mathscr{X}_{\underline{a}}, \mathbf{Z}_{(2)}) = 0.$

Proof. — Since $char(k) \neq 2$ any purely inseparable extension of k is of odd degree and the transfer argument shows that it is sufficient to consider the case of a perfect k. Denote by $\underline{\mathrm{K}}_n^{\mathrm{M}}$ the sheaf on Sm/k such that for a connected smooth scheme X over k the group $\underline{\mathrm{K}}_n^{\mathrm{M}}(\mathrm{X})$ is the subgroup in the n-th Milnor K-group of the function field of X which consists of elements u such that all residues of u in points of codimension 1 are zero. The proof is based on the following result.

Theorem **4.10** *(Markus Rost).* — *The natural homomorphism*

$$\mathrm{H}^{2^{n-1}-1}\left(\mathrm{Q}_{\underline{a}}, \underline{\mathrm{K}}^{\mathrm{M}}_{2^{n-1}}\right) \to k^* \qquad (\mathbf{17})$$

is a monomorphism.

Proof. — See [28] or [11].

The following lemma shows that the cohomology group on the left hand side of (17) can be replaced by a motivic cohomology group.

Lemma **4.11.** — *Let* X *be a smooth scheme over a field k. Then for any p, q there is a canonical homomorphism*

$$\mathrm{H}^{p,q}(\mathrm{X}, \mathbf{Z}) \to \mathrm{H}^{p-q}\left(\mathrm{X}, \underline{\mathrm{K}}_q^{\mathrm{M}}\right)$$

which is an isomorphism if $p \geq q + dim(\mathrm{X})$.

Proof. — Considering X as a limit of smooth schemes over the subfield of constants of k we may assume that k is perfect. Let $h^i = \underline{\mathrm{H}}^i(\mathbf{Z}(q))$ denote the cohomology sheaves of the complex $\mathbf{Z}(q)$. Since $h^i = 0$ for $i > q$ the standard spectral sequence which goes from cohomology with coefficients in h^i and converges to the hypercohomology with coefficients in $\mathbf{Z}(q)$ shows that there is a canonical homomorphism

$$\mathrm{H}^{p,q}(\mathrm{X}, \mathbf{Z}) \to \mathrm{H}^{p-q}(\mathrm{X}, h^q). \qquad (\mathbf{18})$$

The same spectral sequence implies that the kernel and cokernel of this homomorphism are bulit out of groups $\mathrm{H}^{p-i}(\mathrm{X}, h^i)$ and $\mathrm{H}^{p-i+1}(\mathrm{X}, h^i)$ respectively, where $i < q$. Since $p \geq q + dim(\mathrm{X})$ we get $p - i > p - q \geq dim(\mathrm{X})$ and the cohomological dimension theorem for the Nisnevich topology implies that these groups are zero.

It remains to show that $h^q = \underline{\mathrm{K}}_q^{\mathrm{M}}$. Since h^q is a homotopy invariant sheaf with transfers for any smooth connected X the restriction homomorphism

$$\mathrm{H}^0(\mathrm{X}, h^q) \to \mathrm{H}^0(\mathrm{Spec}(k(\mathrm{X})), h^q)$$

is a monomorphism ([38, Corollary 4.18, p. 116]). It was shown in [33, Prop. 4.1] that for any field k one has canonical isomorphisms $\mathrm{H}^{q,q}(k, \mathbf{Z}) = \mathrm{K}_q^{\mathrm{M}}(k)$. In particular for any X the group $\mathrm{H}^0(\mathrm{X}, h^q)$ is a subgroup in $\mathrm{K}_q^{\mathrm{M}}(k(\mathrm{X}))$ and one verifies easily that it coincides with the subgroup $\underline{\mathrm{K}}_q^{\mathrm{M}}(\mathrm{X})$ of elements with zero residues.

By Theorem 4.4 and the suspension isomorphism (see [42, Theorem 2.4]) we have an exact sequence

$$\text{(19)} \qquad H^{0,1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}) \to H^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}}, \mathbf{Z}) \to H^{2^n-1,2^{n-1}}(M_{\underline{a}}, \mathbf{Z}) \to H^{1,1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}).$$

Since the dimension of $Q_{\underline{a}}$ equals $2^{n-1} - 1$, the left hand side group in (17) is isomorphic to the group $H^{2^n-1,2^{n-1}}(Q_{\underline{a}}, \mathbf{Z})$ by Lemma 4.11 and we obtain a natural monomorphism

$$H^{2^n-1,2^{n-1}}(Q_{\underline{a}}, \mathbf{Z}) \to k^*.$$

Let $\overline{M}_{\underline{a}}$ be the pull-back of $M_{\underline{a}}$ to the algebraic closure of k. Since $M_{\underline{a}}$ is a direct summand of $M(Q_{\underline{a}})$ we conclude that the map

$$H^{2^n-1,2^{n-1}}(M_{\underline{a}}, \mathbf{Z}) \to H^{2^n-1,2^{n-1}}(\overline{M}_{\underline{a}}, \mathbf{Z})$$

is injective. Since

$$H^{2^n-1,2^{n-1}}(\mathcal{X}_{\underline{a}} \times \mathrm{Spec}(\bar{k}), \mathbf{Z}) = H^{2^n-1,2^{n-1}}(\mathrm{Spec}(\bar{k}), \mathbf{Z}) = 0$$

we conclude that the second arrow in (19) is zero. On the other hand since $\mathbf{Z}(1) = \mathbf{G}_m[-1]$ we have

$$H^{0,1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}) = H^{-1}(\mathcal{X}_{\underline{a}}, \mathbf{G}_m) = 0.$$

Theorem 4.9 is proved.

5. Computations with Galois cohomology

In this section we are only concerned with classical objects – Milnor K-theory and etale cohomology. More general motivic cohomology does not appear here. The only result of this section which we will directly use below is Theorem 5.9. One may observe that in the case of **Z**/2-coefficients it can be proved in a much easier way, but we decided to include the case of general l for possible future use.

Definition **5.1.** — *We say that* $\mathrm{BK}(w, l)$ *holds if for any field k of characteristic $\neq l$ and any $q \leq w$ one has:*

1. *the norm residue homomorphism* $K_q^M(k)/l \to H_{et}^q\big(k, \boldsymbol{\mu}_l^{\otimes q}\big)$ *is an isomorphism*
2. *for any cyclic extension E/k of degree l the sequence*

$$K_q^M(E) \xrightarrow{1-\sigma} K_q^M(E) \xrightarrow{N_{E/k}} K_q^M(k)$$

where σ is a generator of $\mathrm{Gal}(E/k)$ *is exact.*

Proposition **5.2.** — *Let k be a field of characteristic $\neq l$ which has no extensions of degree prime to l. Assume that* $\mathrm{BK}(w, l)$ *holds. Then for any cyclic extension* E/k *of k of degree l there is an exact sequence of the form*

$$\mathrm{H}^w_{et}(\mathrm{E}, \mathbf{Z}/l) \xrightarrow{\mathrm{N}_{\mathrm{E}/k}} \mathrm{H}^w_{et}(k, \mathbf{Z}/l) \xrightarrow{-\wedge[a]} \mathrm{H}^{w+1}_{et}(k, \mathbf{Z}/l) \longrightarrow \mathrm{H}^{w+1}_{et}(\mathrm{E}, \mathbf{Z}/l)$$

where $[a] \in \mathrm{H}^1_{et}(k, \mathbf{Z}/l)$ is the class which corresponds to E/k.

Proof. — In order to prove the proposition we will need to do some preliminary computations. Fix an algebraic closure $\bar{k}$ of k. Since k has no extensions of degree prime to l there exists a primitive root of unit $\xi \in k$ of degree l. Let $\mathrm{E} \subset \bar{k}$ be a cyclic extension of k of degree l. We have $\mathrm{E} = k(b)$ where $b^l = a$ for an element a in k^*. Denote by σ_ξ the generator of the Galois group $\mathrm{G}_b = \mathrm{Gal}(\mathrm{E}/k)$ which acts on b by multiplication by ξ and by $[a]_\xi$ the class in $\mathrm{H}^1_{et}(k, \mathbf{Z}/l)$ which corresponds to the homomorphism $\mathrm{Gal}(\bar{k}/k) \to \mathrm{G} \to \mathbf{Z}/l$ which takes σ_ξ to the canonical generator of $\mathbf{Z}/l$ (one can easily see that this class is determined by a and ξ and does not depend on b).

Let $p : \mathrm{Spec}(\mathrm{E}) \to \mathrm{Spec}(k)$ be the projection. Consider the etale sheaf $\mathrm{F} = p_*(\mathbf{Z}/l)$. The group G acts on F in the natural way. Denote by F_i the kernel of the homomorphism $(1-\sigma)^i : \mathrm{F} \to \mathrm{F}$. One can verify easily that $\mathrm{F}_i = \mathrm{Im}(1-\sigma)^{l-i}$ and that as a $\mathbf{Z}/l[\mathrm{Gal}(\bar{k}/k)]$-module F_i has dimension i. In particular we have $\mathrm{F} = \mathrm{F}_l$. Note that the extension

$$0 \to \mathbf{Z}/l \to \mathrm{F}_2 \to \mathbf{Z}/l \to 0$$

represents the element $[a]_\xi$ in $\mathrm{H}^1_{et}(k, \mathbf{Z}/l) = \mathrm{Ext}^1_{\mathbf{Z}/l[\mathrm{Gal}(\bar{k}/k)]}(\mathbf{Z}/l, \mathbf{Z}/l)$. Let α_i be the element in $\mathrm{H}^2(k, \mathbf{Z}/l) = \mathrm{Ext}^2_{\mathbf{Z}/l[\mathrm{Gal}(\bar{k}/k)]}(\mathbf{Z}/l, \mathbf{Z}/l)$ defined by the exact sequence

$$0 \longrightarrow \mathbf{Z}/l \longrightarrow \mathrm{F}_i \xrightarrow{u_i} \mathrm{F}_i \longrightarrow \mathbf{Z}/l \longrightarrow 0$$

where $u_i = 1 - \sigma$ and $\mathrm{Im}(u_i) = \mathrm{F}_{i-1}$.

Lemma **5.3.** — *One has $\alpha_l = c([a]_\xi \wedge [\xi]_\xi)$ where c is an invertible element of $\mathbf{Z}/l$ and $\alpha_i = 0$ for $i < l$.*

Proof. — The fact that $\alpha_i = 0$ for $i < l$ follows from the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}/l & \to & \mathrm{F}_{i+1} & \to & \mathrm{F}_{i+1} & \to & \mathbf{Z}/l & \to 0 \\ & {\scriptstyle 0}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & \\ 0 \to & \mathbf{Z}/l & \to & \mathrm{F}_i & \to & \mathrm{F}_i & \to & \mathbf{Z}/l & \to 0. \end{array}$$

To compute α_l note first that since the action of $\mathrm{Gal}(\bar{k}/k)$ on $F = F_l$ factors through $G = \mathrm{Gal}(E/k)$ it comes from a well defined element in $H^2(G, \mathbf{Z}/l)$. This element is not zero for trivial reasons. On the other hand the group

$$H^2(G, \mathbf{Z}/l) = H^2(\mathbf{Z}/l, \mathbf{Z}/l) = \mathbf{Z}/l$$

is generated by the element $\beta(\gamma)$ where γ is the canonical generator of $H^1(G, \mathbf{Z}/l)$ and β is the Bockstein homomorphism. Thus we conclude that up to multiplication by an invertible element of $\mathbf{Z}/l$ our class α_l equals $\beta([a]_\xi)$. It remains to show that $\beta([a]_\xi) = c[a]_\xi \wedge [\xi]_\xi$ which follows by simple explicit computations from the fact that $[a]_\xi$ has a lifting to an element of $H^1_{et}(\mathrm{Spec}(k), \mu_{l^2})$.

Lemma **5.4.** — *Assume that* BK(w, l) *holds. Then for all fields k of characteristic $\neq l$, all $q \leq w$ and all $i = 1, \ldots, l-1$ one has:*

1. *The sequence* $H^q_{et}(k, \mathbf{Z}/l) \oplus H^q_{et}(k, F_{i+1}) \to H^q_{et}(k, F_{i+1}) \to H^q_{et}(k, \mathbf{Z}/l)$ *where the first homomorphism is given on the second summand by* $1 - \sigma$ *is exact.*
2. *The homomorphisms* $\nu_{q,i} : H^q_{et}(k, \mathbf{Z}/l) \oplus H^q_{et}(k, F_{i+1}) \to H^q_{et}(k, F_i)$ *given by the canonical morphisms* $\mathbf{Z}/l \to F_i$, $F_{i+1} \to F_i$ *are surjective.*

Proof. — We proceed by induction on i. Consider first the case $i = l-1$. The first statement follows immediately the assumption that BK(w, l) holds.

Let us prove the second one. The image of $H^q_{et}(k, F_{l-1})$ in

$$H^q_{et}(k, F_l) = H^q_{et}(E, \mathbf{Z}/l)$$

coincides with the kernel of the norm homomorphism

$$H^q_{et}(E, \mathbf{Z}/l) \to H^q_{et}(k, \mathbf{Z}/l).$$

The first statement implies then that $H^q_{et}(k, \mathbf{Z}/l) \oplus H^q_{et}(k, F_l)$ maps surjectively to this image. It is therefore sufficient to show that an element $\gamma \in H^q_{et}(k, F_{l-1})$ which goes to zero in $H^q_{et}(k, F_l)$ belongs to the image of $\nu_{q,l-1}$. Any such element is a composition of a cohomology class in $H^{q-1}_{et}(k, \mathbf{Z}/l)$ with the canonical extension

$$0 \to F_{l-1} \to F_l \to \mathbf{Z}/l \to 0.$$

Thus we may assume that $q = 1$ and γ is the element which corresponds to this extension. Let δ be the image of $c[\xi]_\xi$ (where c is as in Lemma 5.3) under the homomorphism $H^1_{et}(k, \mathbf{Z}/l) \to H^1_{et}(k, F_{l-1})$. The composition

$$H^1_{et}(k, \mathbf{Z}/l) \to H^1_{et}(k, F_{l-1}) \to H^2_{et}(k, \mathbf{Z}/l)$$

where the latter homomorphism corresponds to the extension

$$0 \to \mathbf{Z}/l \to F_l \to F_{l-1} \to 0$$

equals to multiplication by $[a]_\xi$. We now conclude by Lemma 5.3 that the image of $\gamma - \delta$ in $H^2(k, \mathbf{Z}/l)$ is zero. Then it lifts to $H^1_{et}(k, F_l)$ which proves our Lemma in the case $i = l - 1$.

Suppose that the lemma is proved for all $i > j$. Let us show that it holds for $i = j$. The first statement follows immediately from the inductive assumption and the commutativity of the diagram

$$\begin{array}{ccc} F_{j+2} & \xrightarrow{1-\sigma} & F_{j+2} \\ \downarrow & & \downarrow \\ F_{j+1} & \xrightarrow{1-\sigma} & F_{j+1}. \end{array}$$

The proof of the second one is now similar to the case $i = l - 1$ with a simplification due to the fact that $\alpha_i = 0$ for $i < l$ (Lemma 5.3).

The statement of the proposition follows immediately from Lemma 5.4.

Remark **5.5.** — For $l = 2$ Proposition 5.2 is a trivial corollary of the exactness of the sequence $0 \to \mathbf{Z}/2 \to F_2 \to \mathbf{Z}/2 \to 0$. In particular it holds without the $BK(w, l)$ assumption and not only in the context of Galois cohomology but for cohomology of any (pro-)finite group. For $l > 2$ this is not true anymore which one can see by considering cohomology of $\mathbf{Z}/l$.

Lemma **5.6.** — *Assume that* $BK(w, l)$ *holds and let* k *be a field of characteristic* $\neq l$ *which has no extensions of degree prime to* l. *Let further* E/k *be a cyclic extension of degree* l *such that the norm homomorphism* $K^M_w(E) \to K^M_w(k)$ *is surjective. Then the sequence*

$$K^M_{w+1}(E) \xrightarrow{1-\sigma} K^M_{w+1}(E) \xrightarrow{N_{E/k}} K^M_{w+1}(k)$$

where σ *is a generator of* $Gal(E/k)$ *is exact.*

Proof. — It is essentially a version of the proof given in [31] for $w = 2$ and in [20] for $w = 3$. Let us define a homomorphism

$$\phi : K^M_{w+1}(k) \to K^M_{w+1}(E)/(Im(1 - \sigma))$$

as follows. Let a be an element in $K^M_{w+1}(k)$ of the form $(a_0, ..., a_w)$ and let b be an element in $K^M_w(E)$ such that

$$N_{E/k}(b) = (a_0, ..., a_{w-1}).$$

We set $\phi(a) = b \wedge a_n$. Since $BK(w, l)$ holds the element $\phi(a)$ does not depend on the choice of b and one can easily see that ϕ is a homomorphism from $(k^*)^{\otimes(w+1)}$ to $K^M_{w+1}(E)/(Im(1 - \sigma))$. To show that it is a homomorphism from $K^M_w(k)$ it is sufficient to verify that ϕ takes an element of the form $(a_0, ..., a_w)$ such that say $a_0 + a_w = 1$ to zero. Let b be a preimage of $(a_0, \dots, a_{w-1})$ in $K^M_w(k)$. We have to show that $(b, a_w) \in (1-\sigma)K^M_{w+1}(E)$. Assume first that a_0 is not in $(k^*)^l$ and let c be an element in $\bar{k}^*$ such that $c^l = a_0$. Let further $F = k(c)$. Then by $BK(w, l)$ one has

$$b \wedge a_w = b \wedge (1 - a_0) = N_{EF/E}(b_{EL} \wedge (1 - c)) =$$
$$= N_{EF/E}((b - (c, a_1, \dots, a_w - 1)) \wedge (1 - c)) \in (1 - \sigma)K^M_{w+1}(E)$$

since $N_{EF/F}(b-(c, a_1, \dots, a_w-1)) = 0$. The proof for the case when $a_0 \in (k^*)^l$ is similar.

Clearly ϕ is a section for the obvious morphism

$$K^M_{w+1}(E)/(\mathrm{Im}(1-\sigma)) \to K^M_{w+1}(k).$$

It remains to show that it is surjective. It follows immediately from the fact that under our assumption on k the group $K^M_{w+1}(E)$ is generated by symbols of the form $\{b, a_1, \dots, a_w\}$ where $b \in E^*$ and $a_1, \dots, a_w \in k^*$ (see [1]).

Lemma **5.7.** — *Assume that* $BK(w, l)$ *holds and let* k *be a field of characteristic* $\neq l$ *which has no extensions of degree prime to* l. *Then the following two conditions are equivalent*

1. $K^M_{w+1}(k) = lK^M_{w+1}(k)$
2. for any cyclic extension E/k *the norm homomorphism*

$$K^M_w(E) \to K^M_w(k)$$

is surjective.

Proof. — The $2 \Rightarrow 1$ part follows from the projection formula. Since $BK(w, l)$ holds we conclude that there is a commutative square with surjective horizontal arrows of the form

$$\begin{array}{ccc} K^M_w(E) & \to & H^w(E, \mathbf{Z}/l) \\ \downarrow & & \downarrow \\ K^M_w(k) & \to & H^w(k, \mathbf{Z}/l) \end{array}$$

and thus the cokernel of the left vertical arrow is the same as the cokernel of the right one. By Proposition 5.2 it gives us an exact sequence

$$K^M_w(E) \to K^M_w(k) \to H^{w+1}(k, \mathbf{Z}/l)$$

and since the last arrow clearly factors through $K^M_{w+1}(k)/l$ it is zero. The lemma is proved.

Lemma **5.8.** — *Assume that* $BK(w, l)$ *holds and let* k *be a field of characteristic not equal to* l *which has no extensions of degree prime to* l. *Assume further that* $K^M_{w+1}(k) = lK^M_{w+1}(k)$. *Then for any finite extension* E/k *one has* $K^M_{w+1}(E) = lK^M_{w+1}(E)$.

Proof. — This proof is a variant of the proof given in [31] for $w = 1$. Since k has no extensions of degree prime to l it is separable and its Galois group is an l-group. Therefore it is sufficient to prove the lemma in the case of a cyclic extension E/k of degree l. By Proposition 5.2 and Lemma 5.6 we have an exact sequence

$$K^M_{w+1}(E) \xrightarrow{1-\sigma} K^M_{w+1}(E) \xrightarrow{N_{E/k}} K^M_{w+1}(k).$$

Let α be an element in $K^M_{w+1}(E)$ and let $\beta \in K^M_{w+1}(k)$ be an element such that $N_{E/k}(\alpha) = l\beta$. Then $N_{E/k}(\alpha - \beta_E) = 0$ and we conclude that the endomorphism

$$1 - \sigma : K^M_{w+1}(E)/l \to K^M_{w+1}(E)/l$$

is surjective. Since $(1-\sigma)^l = 0$ this implies that $K^M_{w+1}(E)/l = 0$.

Theorem **5.9.** — *Assume that* $BK(w, l)$ *holds and let k be a field of characteristic not equal to l which has no extensions of degree prime to l such that* $K^M_{w+1}(k) = lK^M_{w+1}(k)$. *Then* $H^{w+1}_{et}(k, \mathbf{Z}/l) = 0$.

Proof. — Let α be an element of $H^{w+1}_{et}(k, \mathbf{Z}/l)$. We have to show that $\alpha = 0$. By Lemma 5.8 and obvious induction we may assume that α vanishes on a cyclic extension of k. Then by Proposition 5.2 α is of the form $\alpha_0 \wedge a$ where $a \in H^1_{et}(k, \mathbf{Z}/l)$ is the element which represents our cyclic extension. Thus since $BK(w, l)$ holds it belongs to the image of the homomorphism $K^M_{w+1}(k)/l \to H^{w+1}_{et}(k, \mathbf{Z}/l)$ and therefore is zero.

6. Beilinson–Lichtenbaum conjectures

For a smooth variety X and an abelian group A define the *Lichtenbaum motivic cohomology groups* of X with coefficients in A as the hypercohomology groups

$$H^{p,q}_L(X, A) := \mathbf{H}^p_{et}(X, A \otimes \mathbf{Z}(q)).$$

For n prime to the characteristic of the base field, Lichtenbaum motivic cohomology groups with $\mathbf{Z}/n$ coefficients are closely related to the "usual" etale cohomology.

Theorem **6.1.** — *Let k be a field and n be an integer prime to characteristic of k. Denote by* $\boldsymbol{\mu}_n$ *the etale sheaf of n-th roots of unit on* Sm/k *and let* $\boldsymbol{\mu}_n^{\otimes q}$ *be the n-th tensor power of* $\boldsymbol{\mu}_n$ *in the category of* $\mathbf{Z}/n$*-modules. Then there is a canonical isomorphism* $H^{p,q}_L(-, \mathbf{Z}/n) = H^p_{et}(-, \boldsymbol{\mu}_n^{\otimes q})$.

Proof. — We have to show that the complex $\mathbf{Z}/n(q)$ is canonically quasi-isomorphic in the etale topology to the sheaf $\boldsymbol{\mu}_n^{\otimes q}$. In the category of complexes of sheaves with transfers of $\mathbf{Z}/n$-modules in the etale topology with homotopy invariant cohomology sheaves $DM^{eff}_-(k, \mathbf{Z}/n, et)$ we have

$$\mathbf{Z}/n(q) = (\mathbf{Z}/n(1))^{\otimes q}.$$

Since $\mathbf{Z}(1) = \mathbf{G}_m[-1]$ in the etale topology we have $\mathbf{Z}/n(1) = \mu_n$. The sheaf $\boldsymbol{\mu}_n$ is a locally constant sheaf of exponent prime to $char(k)$. Hence all the tensor powers of this sheaf are strictly homotopy invariant which implies that the n-th tensor power of $\mathbf{Z}/n(1)$ in DM_{-}^{eff} coincide with the n-th tensor power in the derived category of sheaves with transfers i.e. with $\boldsymbol{\mu}_n^{\otimes q}$.

Remark **6.2.** — A very detailed proof of this theorem which uses only the most basic facts about motivic cohomology can be found in [17]. There is also a different proof using Bloch's complexes in [8].

The following fundamental conjecture is due to Alexander Beilinson and Stephen Lichtenbaum (see [2], [15]).

Conjecture **6.3.** — *Let k be a field. Then for any $n \geq 0$ one has*

$$H_L^{n+1,n}(\mathrm{Spec}(k), \mathbf{Z}) = 0.$$

For $n = 0$ we have $\mathbf{Z}(0) = \mathbf{Z}$ and this conjecture follows from the simple fact that $H^1(k, \mathbf{Z}) = 0$. For $n = 1$ we have $\mathbf{Z}(1) = \mathbf{G}_m[-1]$ and the conjecture is equivalent to the cohomological form of the Hilbert 90 Theorem. Because of this fact Conjecture 6.3 is called the generalized Hilbert 90 Conjecture.

The standard proofs of Conjecture 6.3 for $n = 0$ and $n = 1$ work integrally. In the proofs of all the known cases of Conjecture 6.3 for $n > 1$ one considers the vanishing of the groups $H_L^{n+1,n}(\mathrm{Spec}(k), \mathbf{Z}_{(l)}) = 0$ for different primes l separately.

Definition **6.4.** — *We say that* $H90(n, l)$ *holds if for any k one has*

$$H_L^{n+1,n}(\mathrm{Spec}(k), \mathbf{Z}_{(l)}) = 0.$$

The following result is proved in [7, Theorem 8.6].[3]

Theorem **6.5** *(Geisser–Levine).* — *Let* S *be the local ring of a smooth scheme over a field of characteristic $p > 0$. Then for any $n \geq 0$ one has*

$$H_L^{n+1,n}(S, \mathbf{Z}_{(p)}) = 0.$$

Let $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$ be the obvious morphism of sites. Consider the complex $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ of Nisnevich sheaves with transfers on Sm/k. We have

$$H_L^{p,q}(X, \mathbf{Z}) = \mathbf{H}_{Nis}^p(X, \mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))).$$

[3] The authors of [7] define motivic cohomology through the higher Chow groups. Their definition is equivalent to the one used here by the comparison theorem of [32], [6] and [41].

Let $\mathrm{L}(q)$ be the canonical truncation of the complex $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ at level $q+1$ i.e. $\mathrm{L}(q)$ is the subcomplex of sheaves in $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ whose cohomology sheaves $\underline{\mathrm{H}}^i(\mathrm{L}(q))$ are the same as for $\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ for $i \leq q+1$ and zero for $i > q+1$. Since $\underline{\mathrm{H}}^i(\mathbf{Z}(q)) = 0$ for $i > q$ the canonical morphism $\mathbf{Z}(q) \to \mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))$ factors through $\mathrm{L}(q)$. Let $\mathrm{K}(q)$ be the complex of sheaves with transfers on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ defined by the distinguished triangle

$$\mathbf{Z}(q) \to \mathrm{L}(q) \to \mathrm{K}(q) \to \mathbf{Z}(q)[1]. \tag{20}$$

Theorem **6.6.** — *Assume that* H90(w, l) *holds. Then for any k such that char$(k) \neq l$ the complex* $\mathrm{K}(w) \otimes \mathbf{Z}_{(l)}$ *is quasi-isomorphic to zero.*

Proof. — We will use the following two lemmas.

Lemma **6.7.** — *Let k be a field of characteristic not equal to l. Then for any $q \geq 0$ and any $\mathbf{Z}_{(l)}$-module* A*, the complex* $\mathbf{R}\pi_*(\pi^*(\mathrm{A} \otimes \mathbf{Z}(q)))$ *has homotopy invariant cohomology sheaves.*

Proof. — Since the sheaf associated with a homotopy invariant presheaf with transfers is homotopy invariant ([38, Propositions 4.26, p. 118] and [38, Propositions 5.5, p. 128]) it is sufficient to show that the functors $\mathrm{H}^{p,q}_{\mathrm{L}}(-, \mathrm{A})$ are homotopy invariant i.e. that for any smooth U the homomorphism

$$\mathrm{H}^{p,q}_{\mathrm{L}}(\mathrm{U} \times \mathbf{A}^1, \mathrm{A}) \to \mathrm{H}^{p,q}_{\mathrm{L}}(\mathrm{U}, \mathrm{A})$$

given by the restriction to $\mathrm{U} \times \{0\}$ is an isomorphism. Consider the universal coefficients long exact sequence relating Lichtenbaum motivic cohomology with coefficients in A to Lichtenbaum motivic cohomology with coefficients in $\mathrm{A}\otimes\mathbf{Q}$ and $\mathrm{A}\otimes\mathbf{Q}/\mathbf{Z}_{(l)}$. The cohomology with $\mathrm{A} \otimes \mathbf{Q}$-coefficients is homotopy invariant by Lemma 6.8. The cohomology with $\mathrm{A} \otimes \mathbf{Q}/\mathbf{Z}_{(l)}$-coefficients is isomorphic to the etale cohomology by Theorem 6.1 and therefore homotopy invariant as well. Our claim follows now from the five lemma.

Lemma **6.8.** — *The canonical homomorphisms*

$$\mathrm{H}^{p,q}(-, \mathbf{Q}) \to \mathrm{H}^{p,q}_{\mathrm{L}}(-, \mathbf{Q})$$

are isomorphisms.

Proof. — It is sufficient to show that for any $i, j \in \mathbf{Z}$ and any smooth scheme X one has

$$\mathrm{H}^i_{et}(\mathrm{X}, \underline{\mathrm{H}}^j(\mathbf{Q}(q))_{et}) = \mathrm{H}^i_{\mathrm{Nis}}(\mathrm{X}, \underline{\mathrm{H}}^j(\mathbf{Q}(q))).$$

Since the $\underline{\mathrm{H}}^j(\mathbf{Q}(q))$ are sheaves with transfers it is a particular case of [38, Propositions 5.24, 5.27, p. 135].

Lemma 6.7 implies that $\underline{H}^{n+1}(K(w)\otimes \mathbf{Z}_{(l)})$ is a homotopy invariant sheaf with transfers and by the assumption that H90(w,l) holds we know that it vanishes over fields. By [38, Corollary 4.18, p. 116] we conclude that $\underline{H}^{n+1}(K(w)\otimes \mathbf{Z}_{(l)})=0$.

In order to show that $K(w)\otimes \mathbf{Z}_{(l)}$ is quasi-isomorphic to zero it remains to verify that for any smooth scheme X over k and any $p\leq w$ the homomorphism $H^{p,w}(X,\mathbf{Z}_{(l)})\to H_L^{p,w}(X,\mathbf{Z}_{(l)})$ is an isomorphism. Lemma 6.8 and the universal coefficients long exact sequence imply that it is sufficient to verify that the homomorphisms

$$H^{p,w}(X,\mathbf{Q}/\mathbf{Z}_{(l)})\to H_L^{p,w}(X,\mathbf{Q}/\mathbf{Z}_{(l)}) \tag{21}$$

are isomorphisms for $p\leq w$. The diagram

$$\begin{array}{ccc}
H_B^{p,p}(\mathrm{Spec}(F),\mathbf{Q}) & \longrightarrow & H_L^{p,p}(\mathrm{Spec}(F),\mathbf{Q})\\
\downarrow & & \downarrow\\
H_B^{p,p}(\mathrm{Spec}(F),\mathbf{Q}/\mathbf{Z}_{(l)}) & \longrightarrow & H_L^{p,p}(\mathrm{Spec}(F),\mathbf{Q}/\mathbf{Z}_{(l)})\\
\downarrow & & \downarrow\\
H_B^{p+1,p}(\mathrm{Spec}(F),\mathbf{Z}_{(l)}) & \longrightarrow & H_L^{p+1,p}(\mathrm{Spec}(F),\mathbf{Z}_{(l)})
\end{array}$$

implies by Lemma 6.8 and our assumption that the map (21) is surjective for $X=\mathrm{Spec}(F)$ where F is a field and $p=w$. By the analog of [8, Theorem 1.1] for $\mathbf{Q}/\mathbf{Z}_{(l)}$-coefficients and the comparison between the higher Chow groups and the etale cohomology we conclude that (21) is an isomorphism.

Corollary **6.9.** — *Assume that* H90(w,l) *holds. Then for any field k and any smooth simplicial scheme $\mathcal{X}$ over k one has*

1. the homomorphisms

$$H^{p,q}(\mathcal{X},\mathbf{Z}_{(l)})\to H_L^{p,q}(\mathcal{X},\mathbf{Z}_{(l)})$$

are isomorphisms for $p-1\leq q\leq w$ and monomorphisms for $p=q+2$ and $q\leq w$

2. the homomorphisms

$$H^{p,q}(\mathcal{X},\mathbf{Z}/l^n)\to H_L^{p,q}(\mathcal{X},\mathbf{Z}/l^n)$$

are isomorphisms for $p\leq q\leq w$ and monomorphisms for $p=q+1$ and $q\leq w$.

Corollary **6.10.** — *Assume that* H90(w,l) *holds. Then for any k of characteristic not equal to l and any $q\leq w$ the norm residue map*

$$K_q^M(k)/l\to H_{et}^q\big(\mathrm{Spec}(k),\boldsymbol{\mu}_l^{\otimes q}\big) \tag{22}$$

is an isomorphism.

Proof. — By Corollary 6.9(2) the homomorphisms

$$H^{p,q}(X, \mathbf{Z}/l) \to H^{p,q}_{L}(X, \mathbf{Z}/l) \tag{23}$$

are isomorphisms for $p \leq q \leq w$. By Theorem 6.1, for $p = q$ and $X = \mathrm{Spec}(k)$ the homomorphism (23) is isomorphic to the homomorphism (22).

Lemma **6.11.** — *Assume that* H90(w, l) *holds. Then for any field k, any $q \leq w$ and any cyclic extension E/k of degree l the sequence*

$$K^M_q(E) \xrightarrow{1-\sigma} K^M_q(E) \xrightarrow{N_{E/k}} K^M_q(k)$$

(where σ is a generator of $\mathrm{Gal}(E/k)$*) is exact.*

Proof. — One verifies easily that this complex becomes exact after tensoring with $\mathbf{Z}[1/l]$. It remains to show that it becomes exact after tensoring with $\mathbf{Z}_{(l)}$. Recall that for a smooth scheme X over k we let $\mathbf{Z}_{tr}(X)$ denote the free sheaf with transfers generated by X. Consider the complex of presheaves with transfers of the form

$$0 \to \mathbf{Z}_{tr}(k) \to \mathbf{Z}_{tr}(E) \xrightarrow{1-\sigma} \mathbf{Z}_{tr}(E) \to \mathbf{Z}_{tr}(k) \to 0$$

where the second arrow is the transfer map. Denote this complex with the last $\mathbf{Z}_{tr}(k)$ placed in degree zero by $\underline{K}$. One can easily see that it is exact in the etale topology and therefore

$$\mathrm{Hom}_D\big(\underline{K}, \mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))[*]\big) = \mathrm{Hom}_{D_{et}}(\underline{K}, \mathbf{Z}(q)[*]) = 0$$

where D_{et} is the category of etale sheaves with transfers. Since

$$\underline{H}^{q+1}\big(\mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))\big) = 0,$$

the map

$$\mathrm{Hom}_D(\underline{K}, L(q)[q+2]) \to \mathrm{Hom}_D\big(\underline{K}, \mathbf{R}\pi_*(\pi^*(\mathbf{Z}(q)))[q+2]\big)$$

is injective and we conclude by Theorem 6.6 that

$$\mathrm{Hom}_D(\underline{K}, \mathbf{Z}_{(l)}(q)[q+2]) = 0. \tag{24}$$

We have

$$H^{p,q}(X, \mathbf{Z}) = \mathrm{Hom}_D(\mathbf{Z}_{tr}(X), \mathbf{Z}(q)[p])$$

and in particular for a separable extension F of k we have

$$\mathrm{Hom}_{DM}(\mathbf{Z}_{tr}(\mathrm{Spec}(F)), \mathbf{Z}(q)[p]) = \begin{cases} K^M_q(F) & \text{for } p = q \\ 0 & \text{for } p > q. \end{cases}$$

Our result follows now from (24) and the standard spectral sequence which computes morphisms in a triangulated category from a complex in terms of morphisms from its terms.

Let $\mathrm{K}(w)$ be the complex of sheaves on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$ defined in (20).

Lemma **6.12.** — *Let k be a field of characteristic not equal to l and assume that* $\mathrm{H90}(w-1, l)$ *holds. Then for any smooth scheme* X *the map*

$$\mathbf{H}^*(\mathrm{X}, \mathrm{K}(w) \otimes \mathbf{Z}_{(l)}) \to \mathbf{H}^*(\mathrm{X} \times (\mathbf{A}^1 - \{0\}), \mathrm{K}(w) \otimes \mathbf{Z}_{(l)})$$

defined by the projection $\mathbf{A}^1 - \{0\} \to \mathrm{Spec}(k)$*, is an isomorphism.*

Proof. — Recall from [38] that for a functor F from schemes to abelian groups we denote by F_{-1} the functor $\mathrm{X} \mapsto \mathit{coker}(\mathrm{F}(\mathrm{X}) \to \mathrm{F}(\mathrm{X} \times (\mathbf{A}^1 - \{0\}))$ where the map is defined by the projection. To prove the lemma we have to show that

$$\mathbf{H}^*(\mathrm{X}, \mathrm{K}(w) \otimes \mathbf{Z}_{(l)})_{-1} = 0.$$

The long exact sequence defined by the distinguished triangle (20) shows that this is equivalent to checking that the standard map

$$\mathrm{H}^{*,w}(\mathrm{X}, \mathbf{Z}_{(l)})_{-1} = \mathbf{H}^*(\mathrm{X}, \mathbf{Z}_{(l)}(w))_{-1} \to \mathbf{H}^*_{\mathrm{L}}(\mathrm{X}, \mathbf{Z}_{(l)}(w))_{-1}$$

is an isomorphism. For a complex of sheaves with transfers K there is a complex K_{-1} (defined up to a canonical quasi-isomorphism) such that $\mathbf{H}^*(-, \mathrm{K})_{-1} = \mathbf{H}^*(-, \mathrm{K}_{-1})$. By [38, Proposition 4.34, p. 124], if K is a complex of sheaves with transfers with homotopy invariant cohomology sheaves $\underline{\mathrm{H}}^i$ then

$$\underline{\mathrm{H}}^i(\mathrm{K}_{-1}) = (\underline{\mathrm{H}}^i(\mathrm{K}))_{-1}.$$

Therefore, it is sufficient to check that the maps

$$\underline{\mathrm{H}}^i(\mathbf{Z}_{(l)}(w))_{-1} \to \underline{\mathrm{H}}^i(\mathrm{K}(w) \otimes \mathbf{Z}_{(l)})_{-1}$$

are isomorphisms. Since both sides are zero for $i > w + 1$ and since $\underline{\mathrm{H}}^i(\mathrm{K})$ are the sheaves associated with the presheaves $\mathrm{X} \mapsto \mathbf{H}^i(\mathrm{X}, \mathrm{K})$ it remains to check that the maps

$$\mathrm{H}^{i,w}(\mathrm{X}, \mathbf{Z}_{(l)})_{-1} \to \mathbf{H}^i(\mathrm{X}, \mathrm{L}(w) \otimes \mathbf{Z}_{(l)})_{-1}$$

are isomorphisms for $i \leq w + 1$. In this range the map

$$\mathbf{H}^i(\mathrm{X}, \mathrm{L}(w) \otimes \mathbf{Z}_{(l)}) \to \mathbf{H}^i(\mathrm{X}, \mathbf{R}\pi_*(\pi^*(\mathbf{Z}_l(w)))) = \mathrm{H}^{i,w}_{\mathrm{L}}(\mathrm{X}, \mathbf{Z}_{(l)})$$

is an isomorphism and therefore it remains to check that the map

$$\mathrm{H}^{i,w}(\mathrm{X}, \mathbf{Z}_{(l)})_{-1} \to \mathrm{H}^{i,w}_{\mathrm{L}}(\mathrm{X}, \mathbf{Z}_{(l)})_{-1}$$

is an isomorphism for $i \leq w+1$. Consider the commutative diagram

$$\begin{array}{ccc} H^{i-1,w-1}(X, \mathbf{Z}_{(l)}) & \longrightarrow & H_L^{i-1,w-1}(X, \mathbf{Z}_{(l)}) \\ \downarrow & & \downarrow \\ H^{i,w}(X, \mathbf{Z}_{(l)})_{-1} & \longrightarrow & H_L^{i,w}(X, \mathbf{Z}_{(l)})_{-1} \end{array}$$

where the vertical arrows are defined by the multiplication with the canonical class $\eta \in H^{1,1}(\mathbf{A}^1 - \{0\}, \mathbf{Z})$. The upper horizontal arrow is an isomorphism by our assumption that $H90(w-1, l)$ holds and Corollary 6.9(1). The left hand side vertical arrow is an isomorphism by the supension theorem [42, Theorem 2.4]. It remains to check that the right hand side vertical arrow is an isomorphism. This we can verify separately for rational coefficients and $\mathbf{Q}/\mathbf{Z}_{(l)}$-coefficients. In the former case the result follows from Lemma 6.8 and again [42, Theorem 2.4]. In the later case it follows from Theorem 6.1 and the corresponding result for the etale cohomology.

Lemma **6.13.** — *Assume that* $H90(w-1, l)$ *holds and let k be a field of characteristic not equal to l,* X *a smooth scheme over k and* U *a dense open subscheme in* X. *Then the map*

$$\mathbf{H}^*(X, K(w) \otimes \mathbf{Z}_{(l)}) \to \mathbf{H}^*(U, K(w) \otimes \mathbf{Z}_{(l)}) \tag{25}$$

is an isomorphism.

Proof. — Considering X to be a limit of smooth schemes (possibly of greater dimension) over the subfield of constants in k we may assume that k is perfect. By obvious induction it is sufficient to show that the statement of the lemma holds for $U = X - Z$ where Z is a smooth closed subscheme in X. Moreover, one can easily see that it is sufficient to prove that for any point z of Z there exists a neighborhood V of z in X such that

$$\mathbf{H}^*(V, K(w) \otimes \mathbf{Z}_{(l)}) \to \mathbf{H}^*(V - V \cap Z, K(w) \otimes \mathbf{Z}_{(l)})$$

is an isomorphism. The homotopy purity theorem [23, Theorem 2.23, p. 115] implies that this homomorphism is a part of a long exact sequence with the third term being $\mathbf{H}^*(Th(N), K(w) \otimes \mathbf{Z}_{(l)})$ where Th(N) is the Thom space of the normal bundle to $Z \cap V$ in V. Taking V sufficiently small we may assume that this bundle is trivial. It remain to show that $\mathbf{H}^*(Z_+ \wedge T^c, K(w) \otimes \mathbf{Z}_{(l)}) = 0$ for any $c > 0$. Since $T = \mathbf{A}^1/(\mathbf{A}^1 - \{0\})$ this follows easily from Lemma 6.12.

7. Main theorem

In this section we prove Theorem 7.4 which shows that $H90(w, 2)$ holds for all w. We also prove several corollaries of this theorem including the Milnor Conjecture (Corollary 7.5). We start with the following result which is the basis for the inductive step in the proof of 7.4. Recall that we let $\mathcal{X}_{\underline{a}}$ denote the simplicial scheme $\check{C}(Q_{\underline{a}})$ where $\underline{a} = (a_1, ..., a_w)$ is a sequence of elements of k^* and $Q_{\underline{a}}$ is the norm quadric defined by $\underline{a}$.

Proposition **7.1.** — *Assume that char*$(k) \neq 2$ *and that* H90$(w-1, 2)$ *holds. Then one has*

$$H^{w+1,w}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}) = 0.$$

Proof. — Consider the pointed simplicial scheme $\tilde{\mathcal{X}}_{\underline{a}} = \widetilde{C}(Q_{\underline{a}})$ defined by the cofibration sequence

$$(\mathcal{X}_{\underline{a}})_+ \to S^0 \to \tilde{\mathcal{X}}_{\underline{a}} \to \Sigma^1_s((\mathcal{X}_{\underline{a}})_+).$$

Since $H^{p,q}(\mathrm{Spec}(k), \mathbf{Z}) = 0$ for $p > q$ the homomorphisms

$$H^{p-1,q}(\mathcal{X}_{\underline{a}}, \mathbf{Z}) \to \widetilde{H}^{p,q}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z})$$

defined by the third arrow of this sequence are isomorphisms for $p - 1 > q$. Thus it is sufficient to verify that $\tilde{H}^{w+2,w}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}_{(2)}) = 0$. Since there exists an extension of degree two E/k such that $Q_{\underline{a}}(\mathrm{Spec}(E)) \neq \emptyset$ Lemma 9.3 implies that all the motivic cohomology groups of $\tilde{\mathcal{X}}_{\underline{a}}$ have exponent at most 2. Thus it is sufficient to show that the image of $\tilde{H}^{w+2,w}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}_{(2)})$ in $\tilde{H}^{w+2,w}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/2)$ is zero. Let u be an element of this image. Consider the composition of cohomological operations $Q_{w-2}Q_{w-3}...Q_1$. It maps u to an element of

$$\tilde{H}^{2^w,2^{w-1}}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/2) = H^{2^w-1,2^{w-1}}(\mathcal{X}_{\underline{a}}, \mathbf{Z}/2).$$

We are going to show that this element is zero and that the composition $Q_{w-2}Q_{w-3}$ $...Q_1$ is a monomorphism.

Lemma **7.2.** — *Let a be a class in* $H^{*,*}(\mathcal{X}, \mathbf{Z}/l)$ *which is the image of an integral class. Then* $Q_i(a)$ *is the image of an integral class for any i.*

Proof. — By [42, Proposition 13.6] we have $Q_i = \beta q_i \pm q_i \beta$ where β is the Bockstein homomorphism and q_i is another cohomological operation. Since a is the image of an integral class we have $\beta(a) = 0$ and $Q_i(a) = \beta q_i(a)$. On the other hand, the Bockstein homomorphism β can be written as the composition

$$\widetilde{H}^{*,*}(-, \mathbf{Z}/l) \to \widetilde{H}^{*+1,*}(-, \mathbf{Z}) \to \widetilde{H}^{*+1,*}(-, \mathbf{Z}/l)$$

where the first map is the connecting homomorphism for the exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/l \to 0$ and the second map is the reduction modulo l. Therefore, any element of the form $\beta(x)$ is the image of an integral class.

Lemma 7.2 implies that $Q_{w-2}Q_{w-3}...Q_1(u)$ belongs to the image of the corresponding integral motivic cohomology group. Therefore by Theorem 4.9 it is zero.

It remains to verify that the composition $Q_{w-2}Q_{w-3}...Q_1$ is a monomorphism. Counting dimensions we see that we have to show that the operation Q_i acts monomorphically on the group $\tilde{H}^{w-i+2^i-1,w-i+2^{i-1}-1}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/2)$ for $i = 1, ..., w-2$. By Corollary 3.8, the motivic Margolis homology of $\tilde{\mathcal{X}}_{\underline{a}}$ are zero for Q_i with $i \leq w-1$. Therefore, the kernel of Q_i on this group is covered by the image of $\tilde{H}^{w-i,w-i}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/2)$. The later group is zero by the inductive assumption that $H90(w-1, 2)$ holds, Lemma 7.3 and Corollary 6.9(2).

Lemma **7.3.** — *Let* X *be a non empty smooth scheme over* k. *Then the homomorphisms*

$$H_L^{p,q}(\mathrm{Spec}(k), \mathbf{Z}) \to H_L^{p,q}(\check{C}(X), \mathbf{Z}) \tag{26}$$

defined by the morphism $\check{C}(X) \to \mathrm{Spec}(k)$ *are isomorphisms for all* $p, q \in \mathbf{Z}$.

Proof. — By definition, (see Appendix A) we can rewrite the homomorphism (26) as the homomorphism

$$\mathrm{Hom}_{D_{et}}(\mathbf{Z}, \mathbf{Z}(q)[p]) \to \mathrm{Hom}_{D_{et}}(\mathbf{Z}(\check{C}(X)), \mathbf{Z}(q)[p])$$

where the morphisms are in the derived category of sheaves of abelian groups in the etale topology on Sm/k. We claim that the morphism of complexes $\mathbf{Z}(\check{C}(X)) \to \mathbf{Z}$ is a quasi-isomorphism. To check this statement we have to verify that for any strictly henselian local scheme S the map of complexes of abelian groups $\mathbf{Z}(\check{C}(X))(S) \to \mathbf{Z}$ is a quasi-isomorphism. Since X is non-empty there exists a morphism $S \to X$ and therefore, the simplicial set $\check{C}(X)(S)$ is contractible (see the proof of Lemma 9.2). Therefore we have

$$\mathbf{Z}(\check{C}(X))(S) = \mathbf{Z}(\check{C}(X)(S)) \cong \mathbf{Z}.$$

Theorem **7.4.** — *For any field* k *and any* $w \geq 0$ *one has*

$$H_L^{w+1,w}(\mathrm{Spec}(k), \mathbf{Z}_{(2)}) = 0.$$

Proof. — For $char(k) = 2$ the theorem is proved in [8, Theorem 8.6]. Assume that $char(k) \neq 2$. By induction on w we may assume that $H_L^{q+1,q}(k, \mathbf{Z}_{(2)}) = 0$ for all $q < w$.

Let k be a field which has no extensions of degree prime to 2 and such that $K_w^M(k)$ is 2-divisible. By Lemma 6.11 our inductive assumption implies that $BK(w-1, 2)$ holds. By Theorem 5.9 we conclude that

$$H_{et}^{w}(\mathrm{Spec}(k), \mathbf{Z}/2) = 0.$$

Together with Theorem 6.1 this shows that the group $H_L^{w+1,w}(k, \mathbf{Z}_{(2)})$ is torsion free. On the other hand $H_L^{w+1,w}(k, \mathbf{Q}) = 0$ by Lemma 6.8 and we conclude that $H_L^{w+1,w}(\mathrm{Spec}(k), \mathbf{Z}_{(2)}) = 0$.

Consider now an arbitrary field k of odd characteristic. For a finite extension E/k of degree prime to 2 the homomorphism

$$H_L^{w+1,w}(\mathrm{Spec}(k), \mathbf{Z}_{(2)}) \to H_L^{w+1,w}(\mathrm{Spec}(E), \mathbf{Z}_{(2)})$$

is a monomorphism by the transfer argument. Thus in order to prove $H90(w, 2)$ it is sufficient to show that for any element $\underline{a} \in K_w^M(k)$ there exists an extension $K_{\underline{a}}/k$ such that $\underline{a}$ is divisible by 2 in $K_w^M(K_{\underline{a}})$ and the homomorphism

$$H_L^{w+1,w}(\mathrm{Spec}(k), \mathbf{Z}_{(2)}) \to H_L^{w+1,w}(\mathrm{Spec}(K_{\underline{a}}), \mathbf{Z}_{(2)}) \tag{27}$$

is a monomorphism. Since any element in $K_w^M(k)$ is a sum of symbols it is sufficient to construct $K_{\underline{a}}$ for $\underline{a}$ of the form $(a_1, ..., a_w)$.

Let us show that the function field $K = k(Q_{\underline{a}})$ of the norm quadric has the required properties. The fact that $\underline{a}$ becomes divisible by 2 in the Milnor K-theory of K is proved in Proposition 4.1. It remains to show that the map (27) is injective.

Let u be an element in the kernel of (27). We have homomorphisms

$$H_L^{w+1,w}(\mathrm{Spec}(k), \mathbf{Z}_{(2)}) \to H_L^{w+1,w}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}) \to \mathbf{H}_{\mathrm{Nis}}^{w+1}(\mathcal{X}_{\underline{a}}, K(w)). \tag{28}$$

The first of these homomorphisms is an isomorphism by Lemma 7.3. The second one is a monomorphism by the long exact sequence relating the Beilinson and the Lichtenbaum motivic cohomology and Proposition 7.1. Let u' be the image of u under the composition (28). We want to show that $u' = 0$. The projection $Q_{\underline{a}} \to \mathrm{Spec}(k)$ factors through the canonical morphism $Q_{\underline{a}} \to \mathcal{X}_{\underline{a}}$. Since u becomes zero at the generic point of $Q_{\underline{a}}$ there exists a nonempty open subscheme U of $Q_{\underline{a}}$ such that the restriction of u and therefore of u' to U is zero. By Lemma 6.13 we conclude that the restriction of u' to $Q_{\underline{a}}$ is zero.

The canonical morphism $M(Q_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}})$ factors through the morphism $M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}})$ which is a part of the distinguished triangle of Theorem 4.4. Since $M(Q_{\underline{a}}) \to M_{\underline{a}}$ has a section, our class u' becomes zero on $M_{\underline{a}}$ and by (14) we conclude that it belongs to the image of the group $\mathrm{Hom}(M(\mathcal{X}_{\underline{a}})(2^{w-1} - 1)[2^w - 1], K(w)[w + 1])$. Since $w > 1$ this group is zero by Lemma 6.12. Theorem 7.4 is proved.

Corollary **7.5.** — *Let k be a field of characteristic not equal to* 2. *Then the norm residue homomorphisms $K_w^M(k)/2 \to H_{et}^w(k, \mathbf{Z}/2)$ are isomorphisms for all $w \geq 0$.*

Proof. — This follows from Theorem 7.4 and Corollary 6.10.

The following corollary refers to the continuous Galois cohomology introduced in [34].

Corollary **7.6.** — *For any field k of characteristic not equal to 2 and any $q \geq 0$ the 2-adic cohomology group $H^{q+1}_{et}(k, \mathbf{Z}_2(q))$ is torsion free.*

Proof. — Since all primes but 2 are invertible in $\mathbf{Z}_2$ it is sufficient to show that our group has no 2-torsion. The long exact sequence defined by the short exact sequence of continuous Galois modules

$$0 \to \mathbf{Z}_2(q) \to \mathbf{Z}_2(q) \to \mathbf{Z}/2(q) \to 0$$

implies that it is sufficient to show that the homomorphism $H^q(k, \mathbf{Z}_2(q)) \to H^q(k, \mathbf{Z}/2(q))$ is surjective. Since there is a surjective homomorphism from $H^q(k, \mathbf{Z}_2(q))$ to $lim_n H^q(k, \mathbf{Z}/2^n(q))$ (see e.g. [34]) it is sufficient to show that all the maps $H^q(k, \mathbf{Z}/2^n(q)) \to H^q(k, \mathbf{Z}/2(q))$ are surjective. This follows from Corollary 7.5 and the fact that the norm residue homomorphism factors through $H^q(k, \mathbf{Z}/2^n(q))$.

The following nice corollary of Theorem 7.4 is due to S. Bloch.

Corollary **7.7.** — *Let $\alpha \in H^i(X, \mathbf{Z})$ be a 2-torsion element in the integral cohomology of a complex algebraic variety X. Then there exists a divisor Z on X such that the restriction of α to $X - Z$ is zero.*

Proof. — Since $2\alpha = 0$, α is the image of an element α' in $H^{i-1}(X, \mathbf{Z}/2)$ with respect to the Bockstein homomorphism $\beta : H^{i-1}(X, \mathbf{Z}/2) \to H^i(X, \mathbf{Z})$. By Corollary 7.5 there exists a dense open subset $U = X - Z$ of X such that α' on U is in the image of the canonical map $(\mathcal{O}^*(U))^{\otimes(i-1)} \to H^{i-1}(U, \mathbf{Z}/2)$. Since this map factors through the integral cohomology group we have $\beta(\alpha') = 0$ on U.

Denote by $B/n(q)$ the canonical truncation at the cohomological level q of the complex of sheaves $\mathbf{R}\pi_*(\pi^*(\boldsymbol{\mu}_n^{\otimes q}))$ where $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$ is the usual morphism of sites.

Theorem **7.8.** — *For a smooth variety X and any $n > 0$ there are canonical isomorphisms*

$$H^{p,q}(X, \mathbf{Z}/2^n) = \mathbf{H}^p_{Nis}(X, B/2^n(q)).$$

In particular, for any X as above $H^{p,q}(X, \mathbf{Z}/2^n) = 0$ for $p < 0$.

Proof. — By Theorem 6.1 we have a quasi-isomorphism

$$\mathbf{R}\pi_*(\pi^*(\mathbf{Z}/2^n(q))) \to \mathbf{R}\pi_*\left(\pi^*\left(\boldsymbol{\mu}_{2^n}^{\otimes q}\right)\right)$$

which defines a morphism

$$\mathbf{Z}/2^n(q) \to \mathbf{R}\pi_*\left(\pi^*\left(\boldsymbol{\mu}_{2^n}^{\otimes q}\right)\right).$$

Since the complex $\mathbf{Z}/2^n(q)$ has no cohomology in dimension $> q$ this morphism factors through a morphism

$$(\mathbf{29}) \qquad \mathbf{Z}/2^n(q) \to \mathrm{B}/2^n(q).$$

To prove the theorem it is sufficient to show that (29) is a quasi-isomorphism. The cohomology sheaves on both sides are zero in dimension $> q$. For $i \leq q$ the i-th cohomology sheaves are the sheaves associated with the presheaves $\mathrm{H}^{i,q}(-, \mathbf{Z}/2^n)$ and $\mathrm{H}^i_{et}\big(-, \boldsymbol{\mu}_{2^n}^{\otimes q}\big)$ respectively. The morphism from the first presheaf to the second is an isomorphism by Corollary 6.9(2). Therefore the morphism of associated sheaves is an isomorphism as well.

In [4] Spencer Bloch and Stephen Lichtenbaum constructed the *motivic spectral sequence* which starts from Higher Chow groups of a field and converges to its algebraic K-theory. Their construction was reformulated in much more natural terms and extended to all smooth varieties over fields in [6]. Combining the motivic spectral sequence with Theorem 7.8 and using [41] to identify motivic cohomology of [6] with the motivic cohomology of this paper we obtain the following results.

Theorem **7.9.** — *Let k be a field of characteristic $\neq 2$. Then for all $n > 0$ there exists a natural spectral sequence with the* E_2*-term of the form*

$$\mathrm{E}_2^{p,q} = \begin{cases} \mathrm{H}^{p+q}_{et}\big(\mathrm{Spec}(k), \boldsymbol{\mu}_{2^n}^{\otimes q}\big) & \text{for } q \geq 0,\ p \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

which converges to $\mathrm{K}_{q-p}(\mathrm{Spec}(k), \mathbf{Z}/2^n)$ *(K-theory with* $\mathbf{Z}/2^n$*-coefficients).*

Theorem **7.10.** — *Let* X *be a complex algebraic variety of dimension d. Then the canonical homomorphisms* $\mathrm{K}_i^{alg}(\mathrm{X}, \mathbf{Z}/2^n) \to \mathrm{K}^{-i}(\mathrm{X}(\mathbf{C}), \mathbf{Z}/2^n)$ *are isomorphisms for $i \geq d-1$ and monomorphisms for $i = d-2$.*

8. Appendix A. Hypercohomology of pointed simplicial sheaves

We recall here some basic notions related to the hypercohomology of simplicial sheaves. Let T be a site with final object pt. For the purpose of the present paper one may assume that T is the category of smooth schemes over a field k with the Nisnevich or the etale topology and $pt = \mathrm{Spec}(k)$. Denote by AbShv(T) the category of sheaves of abelian groups on T. For an object X of T let $\mathbf{Z}(\mathrm{X})$ be the sheaf characterized by the property that

$$\text{(30)} \qquad \mathrm{Hom}(\mathbf{Z}(X), F) = F(X)$$

for any sheaf of abelian groups F. Note that $\mathbf{Z}(pt)$ is the constant sheaf $\mathbf{Z}$.

For a pointed simplicial object $\mathcal{X}$ we let $\mathbf{Z}(\mathcal{X})$ denote the simplical sheaf of abelian groups obtained from $\mathcal{X}$ by applying the functor $\mathbf{Z}(-)$ termwise. We further let $\mathbf{Z}(\mathcal{X})_*$ denote the complex obtained from $\mathbf{Z}(\mathcal{X})$ by taking the alternating sums of boundary maps. For a complex of sheaves of abelian groups K we define the hypercohomology of $\mathcal{X}$ with coefficients in K by the formula

$$\mathbf{H}^n(\mathcal{X}, K) := \mathrm{Hom}_D(\mathbf{Z}(\mathcal{X}), K)$$

where $D = D(AbShv(T))$ is the derived category of complexes of sheaves of abelian groups on T. The formula (30) together with the standard method of computing cohomology by means of injective resolutions (or more general fibrant models) implies that for $\mathcal{X} = X$ an object of T our definition agrees with the usual one.

Let now Y be a sheaf of sets on T. Then we can define $\mathbf{Z}(Y)$ as the free sheaf of abelian groups generated by Y such that for every sheaf of abelian groups F one has $\mathrm{Hom}(\mathbf{Z}(Y), F) = \mathrm{Hom}(Y, F)$ where on the left hand side we have morphisms of sheaves of abelian groups and on the right hand side morphisms of sheaves of sets. The Yoneda Lemma shows that for an object X, $\mathbf{Z}(X) = \mathbf{Z}(h_X)$ where h_X is the sheaf of sets represented by X. Therefore, we may immediately extend our definition of hypercohomology groups to simplicial sheaves in a way which agrees with the definition for simplicial objects in T on representable sheaves. Starting from this point we consider objects of T to be a particular type of sheaves of sets.

If $\mathcal{X}$ is a pointed simplicial sheaf the distinguished point defines a homomorphism $\mathbf{H}^*(\mathcal{X}, K) \to \mathbf{H}^*(pt, K)$ which has a canonical section defined by the projection $\mathcal{X} \to pt$. We define reduced hypercohomology of $\mathcal{X}$ by the formula:

$$\widetilde{H}^*(\mathcal{X}, K) = \mathit{ker}(\mathbf{H}^*(\mathcal{X}, K) \to \mathbf{H}^*(pt, K)).$$

Alternatively, we can define $\widetilde{\mathbf{Z}}(\mathcal{X})$ setting

$$\widetilde{\mathbf{Z}}(\mathcal{X}) := \mathit{coker}(\mathbf{Z} \to \mathbf{Z}(\mathcal{X}))$$

where the morphism is defined by the distinguished point. Then

$$\widetilde{H}^n(\mathcal{X}, K) = \mathrm{Hom}_D(\widetilde{\mathbf{Z}}(X), K[n]).$$

If $\mathcal{X}_+$ denotes the simplicial sheaf $\mathcal{X} \coprod pt$ pointed by the canonical embedding $pt \to \mathcal{X} \coprod \mathrm{Spec}(k)$, then

$$H^{*,*}(\mathcal{X}, -) = \tilde{H}^{*,*}(\mathcal{X}_+, -).$$

There is a standard spectral sequence with the E_1-term consisting of the hypercohomology of the terms $\mathcal{X}_i$ of $\mathcal{X}$ with coefficients in K which tries to converge to hypercohomology of $\mathcal{X}$. However, this spectral sequence is of limited use since $\mathbf{Z}(\mathcal{X})_*$ is unbounded on the left and the convergence properties of this spectral sequence are uncertain. The proof of the following proposition contains the trick which allows one get around this problem.

Proposition **8.1.** — *Let $n \geq 0$ be an integer and $\mathcal{X}$ and* K *be such that*

$$\mathbf{H}^m(\mathcal{X}_j, \mathrm{K}) = 0$$

for all $j \geq 0$ and $m \leq n$. Then $\mathbf{H}^m(\mathcal{X}, \mathrm{K}) = 0$ for $m \leq n$.

Proof. — Let $s_{\leq i}\mathbf{Z}(\mathcal{X})_*$ be the subcomplex of $\mathbf{Z}(\mathcal{X})_*$ such that

$$s_{\leq i}\mathbf{Z}(\mathcal{X})_j = \begin{cases} \mathbf{Z}(\mathcal{X})_j & \text{for } j \leq i \\ 0 & \text{for } j > i. \end{cases}$$

Since the complexes $s_{\leq i}\mathbf{Z}(\mathcal{X})_*$ are bounded the usual argument shows that $\mathrm{Hom}_{\mathrm{D}}(s_{\leq i}\mathbf{Z}(\mathcal{X})_*, \mathrm{K}[m]) = 0$ for $m \leq n$ and all i. On the other hand one has an exact sequence of complexes

$$0 \to \oplus_i s_{\leq i}\mathbf{Z}(\mathcal{X})_* \to \oplus_i s_{\leq i}\mathbf{Z}(\mathcal{X})_* \to \mathbf{Z}(\mathcal{X})_* \to 0$$

which expresses the fact that $\mathbf{Z}(\mathcal{X})_* = \mathit{colim}_i s_{\leq i}\mathbf{Z}(\mathcal{X})_*$. This short exact sequence defines a long exact sequence of groups of morphisms in the derived category which shows that $\mathbf{H}^m(\mathcal{X}, \mathrm{K}) = 0$ for $m \leq n$.

For a morphism of pointed simplicial sheaves $f : \mathcal{X} \to \mathcal{X}'$ let $\mathit{cone}(f)$ denote the pointed simplicial sheaf such that for a smooth scheme U one has

$$\mathit{cone}(f) : \mathrm{U} \mapsto \mathit{cone}(\mathcal{X}(\mathrm{U}) \to \mathcal{X}'(\mathrm{U})).$$

Lemma **8.2.** — *The sequence*

$$\mathcal{X} \xrightarrow{f} \mathcal{X}' \to \mathit{cone}(f)$$

defines a long exact sequence of hypercohomology groups of the form

$$\ldots \to \widetilde{\mathrm{H}}^n(\mathit{cone}(f), \mathrm{K}) \to \widetilde{\mathrm{H}}^n(\mathcal{X}, \mathrm{K}) \to \widetilde{\mathrm{H}}^n(\mathcal{X}', \mathrm{K}) \to$$
$$\to \widetilde{\mathrm{H}}^{n+1}(\mathit{cone}(f), \mathrm{K}) \to \ldots .$$

9. Appendix B. Čech simplicial schemes

We define here for any smooth scheme X a simplicial scheme $\check{\mathrm{C}}(\mathrm{X})$ such that the map $\pi : \check{\mathrm{C}}(\mathrm{X}) \to \mathrm{Spec}(k)$ is a weak equivalence if and only if X has a rational point. This map defines an isomorphism in motivic cohomology with $\mathbf{Z}_{(n)}$-coefficients if and only if X has a 0-cycle of degree prime to n. In particular, the motivic cohomology of $\widetilde{\mathrm{C}}(\mathrm{X}) := cone(\pi)$ with $\mathbf{Z}/n$-coefficients provide obstructions to the existence of 0-cycles on X of degree prime to n.

Definition **9.1.** — *For a variety* X *over* k, $\check{\mathrm{C}}(\mathrm{X})$ *is the simplicial scheme with terms* $\check{\mathrm{C}}(\mathrm{X})_n = \mathrm{X}^{n+1}$ *and face and degeneracy morphisms given by partial projections and diagonals respectively.*

Lemma **9.2.** — *Let* X, Y *be smooth schemes over* k *such that*

$$\mathrm{Hom}(\mathrm{X}, \mathrm{Y}) \neq \emptyset.$$

Then the projection $\check{\mathrm{C}}(\mathrm{Y}) \times \mathrm{X} \to \mathrm{X}$ *is a simplicial weak equivalence.*

Proof. — Let U be a smooth scheme over k. Then the simplicial set $\check{\mathrm{C}}(\mathrm{Y})(\mathrm{U})$ is the simplex generated by the set $\mathrm{Hom}(\mathrm{U}, \mathrm{Y})$ which is contractible if and only if $\mathrm{Hom}(\mathrm{U}, \mathrm{Y}) \neq \emptyset$. This implies immediately that for any U the map of simplicial sets

$$(\check{\mathrm{C}}(\mathrm{Y}) \times \mathrm{X})(\mathrm{U}) \to \mathrm{X}(\mathrm{U}) = \mathrm{Hom}(\mathrm{U}, \mathrm{X})$$

is a weak equivalence.

Let $\widetilde{\mathrm{C}}(\mathrm{X})$ denote the unreduced suspension of $\check{\mathrm{C}}(\mathrm{X})$ i.e. the cone of of the morphism $\check{\mathrm{C}}(\mathrm{X})_+ \to \mathrm{Spec}(k)_+$.

Lemma **9.3.** — *Let* Y *be a smooth scheme which has a rational point over an extension of* k *of degree* n. *Then* $n\widetilde{\mathrm{H}}^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Y}), \mathbf{Z}) = 0$.

Proof. — Let E be an extension of k and $\mathrm{Y}_{\mathrm{E}} = \mathrm{Y} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathrm{E})$ considered as a smooth scheme over E. Then there are homomorphisms

$$\widetilde{\mathrm{H}}^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Y}), \mathbf{Z}) \to \widetilde{\mathrm{H}}^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Y}_{\mathrm{E}}), \mathbf{Z})$$

and

$$\widetilde{\mathrm{H}}^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Y}_{\mathrm{E}}), \mathbf{Z}) \to \widetilde{\mathrm{H}}^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Y}), \mathbf{Z})$$

and the composition of the first one with the second is multiplication by $deg(\mathrm{E}/k)$. Let E be an extension of degree n such that $\mathrm{Y}(\mathrm{E}) \neq \emptyset$. Since $\mathrm{Y}(\mathrm{E}) \neq \emptyset$ the pointed sheaf $\widetilde{\mathrm{C}}(\mathrm{Y}_{\mathrm{E}})$ is contractible and therefore,

$$\widetilde{\mathrm{H}}^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Y}_{\mathrm{E}}), \mathbf{Z}) = 0.$$

We conclude that $n\widetilde{\mathrm{H}}^{*,*}(\widetilde{\mathrm{C}}(\mathrm{Y}), \mathbf{Z}) = 0$.

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V. V.
School of Mathematics,
Institute for Advanced Study,
Princeton, NJ 08540, USA,
vladimir@ias.edu

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