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Motives over simplicial schemes

by

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Abstract

This paper was written as a part of [8] and is intended primarily to provide the definitions and results concerning motives over simplicial schemes, which are used in the proof of the Bloch-Kato conjecture.

Key Words: relative motives, motivic cohomology, embedded simplicial schemes

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Contents

1. Introduction

For the purpose of this paper *a scheme* means a possibly infinite disjoint union of separated noetherian schemes of finite dimension. A *smooth* scheme over a scheme

S is a disjoint union of smooth separated schemes of finite type over S. A *smooth simplicial scheme* $\mathcal{X}$ over S is a simplicial scheme such that all terms of $\mathcal{X}$ are smooth schemes over S and all morphisms are over S.

If $\mathcal{X}$ is a smooth simplicial scheme over a field k then the complex of presheaves with transfers defined by the simplicial presheaf with transfers $\mathbf{Z}_{tr}(\mathcal{X})$ gives an object $M(\mathcal{X})$ in the triangulated category of motives $DM_{-}^{eff}(k)$ over k. The motivic cohomology of this object is called the motivic cohomology of $\mathcal{X}$ and we denote its cohomology groups by

$$H^{p,q}(\mathcal{X},A) := Hom_{DM}(M(\mathcal{X}),A(q)[p])$$

where A is an abelian group of coefficients.

The main goal of this paper is to define for any smooth simplicial scheme $\mathcal{X}$ over a perfect field k, a tensor triangulated category $DM_{-}^{eff}(\mathcal{X})$ such that

$$H^{p,q}(\mathcal{X},A) = Hom_{DM^{eff}(\mathcal{X})}(\mathbf{Z},A(p)[q]). \tag{1.1}$$

For completeness we make our construction of $DM_{-}^{eff}(\mathcal{X})$ for the case of a general simplicial scheme and in particular we provide a definition of "motivic cohomology" of simplicial schemes based on (1.1). If the terms of $\mathcal{X}$ are not regular, there are examples which show that the motivic cohomology defined by (1.1) does not satisfy the suspension isomorphism with respect to the T-suspension (which implies that they do not satisfy the projective bundle formula and do not have the Gysin long exact sequence). Therefore in the general case we have to distinguish the "effective" motivic cohomology groups given by (1.1) and the stable motivic cohomology groups given by

$$H^{p,q}_{stable}(\mathcal{X},A) := \lim_{n} Hom_{DM^{eff}(\mathcal{X})}(\mathbf{Z}(n),A(n+q)[p]). \tag{1.2}$$

The stable motivic cohomology groups should also have a description as morphisms between Tate objects in the properly defined T-stable version of DM and should have many good properties including the long exact sequence for blow-ups, which the unstable groups in the non-regular case do not have.

If the terms of $\mathcal{X}$ are regular schemes of equal characteristic then the cancellation theorem over perfect fields implies that this problem does not arise and the stable groups are same as the effective ones (see Corollary 5.5). Since in applications to the Bloch-Kato conjecture we need only the case of smooth schemes over a perfect field, we shall not consider stable motivic cohomology in this paper.

Note also that while we use schemes which are smooth over a base as the basic building blocks of motives over a base, one can also consider all (separated) schemes instead, as done in [7] and [4]. As far as the constructions of this paper are concerned, this make no difference except that the resulting motivic category gets bigger.

2. Presheaves with transfers

Let $\mathcal{X}$ be a simplicial scheme with terms X_i, $i \geq 0$. For a morphism $\phi : [j] \to [i]$ in Δ we let $\mathcal{X}_\phi$ denote the corresponding morphism $\mathcal{X}_i \to \mathcal{X}_j$. Denote by $Sm/\mathcal{X}$ the category defined as follows:

1. An object of $Sm/\mathcal{X}$ is a pair of the form (Y,i) where i is a non-negative integer and $Y \to \mathcal{X}_i$ is a smooth scheme over $\mathcal{X}_i$.
2. A morphism from (Y,i) to (Z,j) is a pair (u,ϕ) where $\phi : [j] \to [i]$ is a morphism in Δ and $u : Y \to Z$ is a morphism of schemes such that the square

$$\begin{array}{ccc} Y & \xrightarrow{u} & Z \\ \downarrow & & \downarrow \\ \mathcal{X}_i & \xrightarrow{\mathcal{X}_\phi} & \mathcal{X}_j \end{array} \tag{2.1}$$

commutes.

A presheaf of sets on $Sm/\mathcal{X}$ is a contravariant functor from $Sm/\mathcal{X}$ to sets. Each presheaf F on $Sm/\mathcal{X}$ defines in the obvious way a famlily of presheaves F_i on $Sm/\mathcal{X}_i$ together with natural transformations $F_\phi : \mathcal{X}_\phi^*(F_j) \to F_i$ for all morphisms $\phi : [j] \to [i]$ in Δ.

One can easily see that this construction provides a bijection between presheaves on $Sm/\mathcal{X}$ and families (F_i, F_ϕ) such that $F_{Id} = Id$ and the obvious compatibility condition holds for composable pairs of morphisms in Δ. Under this bijection the presheaf $h_{(Y,i)}$ represented by (Y,i) has as its j-th component the presheaf

$$(h_{(Y,i)})_j = \coprod_\phi h_{Y \times_\phi \mathcal{X}_j}$$

where ϕ runs through the morphisms $[i] \to [j]$ in Δ.

Our first goal is to develop an analog of this picture where the presheaves of sets are replaced with presheaves with transfers. Let us recall first the basic notions for presheaves with transfers over usual schemes. For a scheme X denote by $SmCor(X)$ the category whose objects are smooth schemes over X and whose morphisms are finite correspondences over X (in the case of a non-smooth $\mathcal{X}$ see [7] for a detailed definition of finite correspondences and their compositions). Note that we allow schemes which are infinite disjoint unions of smooth schemes of finite type to be objects of $SmCor(X)$. In particular our $SmCor(X)$ has infinite direct sums. A presheaf with transfers on Sm/X is an additive contravariant functor from $SmCor(X)$ to abelian groups which takes infinite direct sums to products.

Presheaves with transfers form an abelian category $PST(X)$. The forgetful functor from presheaves with transfers to presheaves of sets has a left adjoint which we denote by $\mathbf{Z}_{tr}(-)$. If Y is a smooth scheme over X and h_Y is the presheaf of sets represented by Y then $\mathbf{Z}_{tr}(h_Y)$ coincides with the presheaf with transfers represented by Y on $SmCor(X)$ and we denote this object by $\mathbf{Z}_{tr}(Y)$. It will be convenient for us to identify $SmCor(X)$ with its image in $PST(X)$ and to denote the object of $SmCor(X)$ corresponding to a smooth scheme Y over X by $\mathbf{Z}_{tr}(Y)$.

A morphism of schemes $f : Y \to X$ defines the pull-back functor

$$\mathbf{Z}_{tr}(U) \mapsto \mathbf{Z}_{tr}(U \times_X Y)$$

from $SmCor(X)$ to $SmCor(Y)$ and therefore a pair of adjoint functors f_*, f^* between the corresponding categories of presheaves with transfers. Since f_* commutes with the forgetful functor, we conclude by adjunction that for a presheaf of sets F over X one has

$$\mathbf{Z}_{tr}(f^*(F)) = f^*(\mathbf{Z}_{tr}(F)). \tag{2.2}$$

It is not necessarily true that pull-back functors on presheaves of sets and on presheaves with transfers commute with the forgetful functor.

Definition 2.1 Let $\mathcal{X}$ be a simplicial scheme. A presheaf with transfers on $\mathcal{X}$ is the following collection of data:

1. For each $i \geq 0$, a presheaf with transfers F_i on $Sm/\mathcal{X}_i$.

2. For each morphism $\phi : [j] \to [i]$ in the simplicial category Δ, a morphism of presheaves with transfers

$$F_\phi : \mathcal{X}_\phi^*(F_j) \to F_i.$$

These data should satisfy the condition that $F_{id} = Id$ and for a composable pair of morphisms $\phi : [j] \to [i]$, $\psi : [k] \to [j]$ in Δ, the obvious diagram of morphisms of presheaves commutes.

We let $PST(\mathcal{X})$ denote the category of presheaves with transfers on $\mathcal{X}$. This is an abelian category with kernels and cokernels computed termwise.

Example 2.2 Let X be a scheme and $\mathcal{X}$ a simplicial scheme such that $\mathcal{X}_i = X$ for all i and all the structure morphisms are identities. Then a presheaf with transfers over $\mathcal{X}$ is the same as a cosimplicial object in the category of presheaves with transfers over X.

Let $F = (F_i, F_\phi)$ be a presheaf of sets on $Sm/\mathcal{X}$. In view of (2.2), the collection of presheaves with transfers $\mathbf{Z}_{tr}(F_i)$ has the natural structure of a presheaf with transfers on $Sm/\mathcal{X}$, which we denote by $\mathbf{Z}_{tr}(F)$. One observes easily that $F \mapsto \mathbf{Z}_{tr}(F)$ is left adjoint to the corresponding forgetful functor. If (Y,i) is an object of $Sm/\mathcal{X}$ and $h_{(Y,i)}$ is the corresponding representable presheaf of sets, we let $\mathbf{Z}_{tr}(Y,i)$ denote the presheaf with transfers $\mathbf{Z}_{tr}(h_{(Y,i)})$. For any presheaf with transfers F, we have

$$Hom(\mathbf{Z}_{tr}(Y,i), F) = F_i(Y). \tag{2.3}$$

By construction, the i-th component of $\mathbf{Z}_{tr}(Z,j)$ is

$$\mathbf{Z}_{tr}((h_{(Z,j)})_i) = \mathbf{Z}_{tr}(\coprod_{\phi} h_{Z\times_\phi \mathcal{X}_i}) = \oplus_\phi \mathbf{Z}_{tr}(Z \times_\phi \mathcal{X}_i) \tag{2.4}$$

where ϕ runs through all morphisms $[j] \to [i]$ in Δ. Together with (2.3) this shows that

$$Hom(\mathbf{Z}_{tr}(Y,i), \mathbf{Z}_{tr}(Z,j)) = \oplus_\phi Hom_{SmCor(\mathcal{X}_i)}(Y, Z \times_\phi \mathcal{X}_i).$$

Denote by $SmCor(\mathcal{X})$ the full subcategory in $PST(\mathcal{X})$ generated by direct sums of objects of the form $\mathbf{Z}_{tr}(Y_i)$. The following lemma is an immediate corollary of (2.3).

Lemma 2.3 *The category $PST(\mathcal{X})$ is naturally equivalent to the category of additive contravariant functors from $SmCor(\mathcal{X})$ to the category of abelian groups, which commute with $\oplus$.*

Lemma 2.3 implies in particular that we can apply in the context of $PST(\mathcal{X})$ the usual construction of the canonical left resolution of a functor by direct sums of representable functors. It provides us with a functor $Lres$ from $PST(\mathcal{X})$ to complexes over $SmCor(\mathcal{X})$ together with a familiy of natural quasi-isomorphisms

$$Lres(F) \to F.$$

We let

$$D(\mathcal{X}) := D_-(PST(\mathcal{X}))$$

denote the derived category of complexes bounded from above over $PST(\mathcal{X})$. In view of Lemma 2.3, it can be identified with the homotopy category of complexes bounded from above over $SmCor(\mathcal{X})$ by means of the functor

$$K \mapsto Tot(Lres(K)) \tag{2.5}$$

which we also denote by $Lres$.

For a morphism of simplicial schemes $f_\bullet : \mathcal{X} \to \mathcal{Y}$, the direct and inverse image functors $f_i^*, f_{i,*}$ define in the obvious way functors

$$f_\bullet^* : PST(\mathcal{Y}) \to PST(\mathcal{X})$$

$$f_{\bullet,*} : PST(\mathcal{X}) \to PST(\mathcal{Y})$$

and the adjunction morphisms $Id \to f_{i,*} f_i^*$, $f_i^* f_{*,i} \to Id$ define morphisms

$$Id \to f_{\bullet,*} f_\bullet^*$$

$$f_{\bullet,*} f_\bullet^* \to Id$$

which automatically satisfy the adjunction axioms and therefore make $f_{\bullet,*}$ into a right adjoint to $f_\bullet^*$.

The functor $f_\bullet^*$ takes $\mathbf{Z}_{tr}(Z,i)$ to $\mathbf{Z}_{tr}(Z \times_{\mathcal{Y}_i} \mathcal{X}_i, i)$ and commutes with direct sums. Therefore it restricts to a functor

$$f_\bullet^{-1} : SmCor(\mathcal{Y}) \to SmCor(\mathcal{X}).$$

Using the equivalence of Lemma 2.3, we can now recover the functors $f_\bullet^*$ and $f_{*,\bullet}$ as the direct and inverse image functors defined by $f_\bullet^{-1}$.

The functors $f_{\bullet,*}$ are clearly exact and therefore define functors on the corresponding derived categories. The functors $f_\bullet^*$ for non-smooth f are in general only right exact but not left exact. To define the corresponding left adjoints one sets

$$Lf_\bullet^*(K) = f_\bullet^*(Lres(K))$$

where $Lres$ is defined on complexes by (2.5). The corresponding functor on the derived categories, which we continue to denote by $Lf_\bullet^*$, is then a left adjoint to $f_{\bullet,*}$.

A group of functors relates the presheaves with transfers over $\mathcal{X}$ with the presheaves with transfers over the terms of $\mathcal{X}$. For any $i \geq 0$, let

$$r_i : SmCor(\mathcal{X}_i) \to SmCor(\mathcal{X})$$

be the functor which takes a smooth scheme Y over $\mathcal{X}_i$ to $\mathbf{Z}_{tr}(Y,i)$. This functor defines in the usual way a pair of adjoints

$$r_{i,\#} : PST(\mathcal{X}_i) \to PST(\mathcal{X})$$

and

$$r_i^* : PST(\mathcal{X}) \to PST(\mathcal{X}_i)$$

where r_i^* is the right adjoint and $r_{i,\#}$ the left adjoint. Equation (2.3) implies that for a presheaf with transfers F on $\mathcal{X}$, $r_i^*(F)$ is the i-th component of F. To compute $r_{i,\#}$, note that

$$r_{i,\#}(\mathbf{Z}_{tr}(Y)) = \mathbf{Z}_{tr}(Y,i) \tag{2.6}$$

and $r_{i,\#}$ is right exact. Therefore for a presheaf with transfers F over $\mathcal{X}_i$, one has

$$r_{i,\#}(F) = h_0(r_{i,\#}(Lres(F))) \tag{2.7}$$

where $Lres$ is the canonical left resolution by representable presheaves with transfers and the right hand side of (2.7) is defined by (2.6).

The functors r_i^* are exact and therefore define functors between the corresponding derived categories which we again denote by r_i^*. We do not know if the functors $r_{i,\#}$ are exact, but in any event one can define the left derived functor

$$Lr_{i,\#} := r_{i,\#} \circ Lres.$$

This functor respects quasi-isomorphisms and the corresponding functor between the derived categories, which we continue to denote by $Lr_{i,\#}$, is the left adjoint to r_i^*.

Lemma 2.4 *The family of functors*

$$r_i^* : D(\mathcal{X}) \to D(\mathcal{X}_i)$$

is conservative i.e. if $r_i^*(K) \cong 0$ *for all* i *then* $K \cong 0$.

Proof: Let K be an object such that $r_i^*(K) \cong 0$ for all i. Then by adjunction

$$Hom(\mathbf{Z}_{tr}(Y,i), K[n]) = Hom(Lr_{i,\#}(\mathbf{Z}_{tr}(Y)), K[n]) =$$

$$= Hom(\mathbf{Z}_{tr}(Y), r_i^*(K)[n]) = 0.$$

Since objects of the form $\mathbf{Z}_{tr}(Y,i)$ generate $D(\mathcal{X})$, we conclude that $K \cong 0$. □

Consider the composition

$$r_i^* r_{j,\#} : PST(\mathcal{X}_j) \to PST(\mathcal{X}_i).$$

By (2.4) it takes $\mathbf{Z}_{tr}(Y)$ to $\oplus_\phi \mathbf{Z}_{tr}(Y \times_\phi \mathcal{X}_i)$ where ϕ runs through the morphisms $[j] \to [i]$ in Δ. Therefore we have

$$r_i^* Lr_{j,\#} = \oplus_\phi L\mathcal{X}_\phi^* \tag{2.8}$$

and passing to $h_0(-)$ we get

$$r_i^* r_{j,\#} = \oplus_\phi \mathcal{X}_\phi^*.$$

Remark 2.5 The functors r_i behave as if the terms $\mathcal{X}_i$ formed a covering of the simplicial scheme $\mathcal{X}$ where the r_i^* were the inverse image functors for this covering and the $r_{i,\#}$ were the functors which in the case of an open covering $j_i : U_i \to X$ would be denoted by $(j_i)_!$.

The functors r_i^* commute in the obvious sense with the functors f^* for morphisms $f : \mathcal{X} \to \mathcal{Y}$ of simplicial schemes.

Let now $\mathcal{X}$ be a simplicial scheme over a scheme S. We have a functor

$$c^* : PST(S) \to PST(\mathcal{X})$$

which sends a presheaf with transfers F over S to the collection

$$c^*(F) = ((\mathcal{X}_i \to S)^*(F))_{i \geq 0}$$

with the obvious structure morphisms. This functor is clearly right exact and using the representable resolution $Lres$ over S we may define a functor Lc^* from complexes over $PST(S)$ to complexes over $PST(\mathcal{X})$. Then Lc^* respects quasi-isomorphisms and therefore defines a triangulated functor

$$Lc^* : D(S) \to D(\mathcal{X}).$$

The functors c^* are compatible with the pull-back functors f^* such that for $f : \mathcal{X} \to \mathcal{Y}$, we have a natural isomorphism

$$c^* = f^* c^*$$

and for the functors on the derived categories, we have natural isomorphisms

$$Lc^* = Lf^* Lc^*.$$

They are also compatible with the functors r_i^* such that one has

$$r_i^* c^* = p_i^*$$

and

$$r_i^* Lc^* = Lp_i^*$$

where p_i is the morphism $\mathcal{X}_i \to S$.

If $\mathcal{X}$ is a smooth simplicial scheme over S then the functor c^* has a left adjoint $c_\#$ which takes $\mathbf{Z}_{tr}(Y,i)$ to the presheaf with transfers $\mathbf{Z}_{tr}(Y/S)$ on $SmCor(S)$. In particular in this case c^* is exact. The functor $c_\#$ being a left adjoint is right exact and we use representable resolutions to define the left derived functor

$$Lc_\# := c_\# \circ Lres.$$

The functor $Lc_\#$ respects quasi-isomorphisms and the corresponding functor on the derived categories is a left adjoint to $c^* = Lc^*$.

Functors $c_\#$ are compatible with the functors $r_{i,\#}$ such that one has

$$r_{i,\#}c_\# = p_{i,\#}$$

where p_i is the smooth morphism $\mathcal{X}_i \to S$, and on the level of the derived categories one has

$$Lr_{i,\#}Lc_\# = Lp_{i,\#}.$$

3. Tensor structure

Recall that for a scheme X one uses the fiber product of smooth schemes over X and the corresponding external product of finite correspondences to define the tensor structure on $SmCor(X)$ (see e.g. [7]). One then defines a tensor structure on $PST(X)$ setting

$$F \otimes G := h_0(Lres(F) \otimes Lres(G))$$

where the tensor product on the right is defined by the tensor product on $SmCor(X)$. If $f : X' \to X$ is a morphism of schemes then there are natural isomorphisms

$$f^*(F \otimes G) = f^*(F) \otimes f^*(G) \tag{3.1}$$

which are compatible on representable presheaves with transfers with the isomorphisms

$$(Y \times_X X') \times_{X'} (Z \times_X X') = (Y \times_X Z) \times_X X'.$$

Let now $\mathcal{X}$ be a simplicial scheme. For presheaves with transfers F, G over $\mathcal{X}$ the collection of presheaves with transfers $F_i \otimes G_i$ over $\mathcal{X}_i$ has the natural structure of a presheaf with transfers over $\mathcal{X}$ defined by the isomorphisms (3.1). This structure is natural in F and G and one can easily see that the pairing

$$(F,G) \mapsto F \otimes G$$

extends to a tensor structure on presheaves with transfers over $\mathcal{X}$. The unit of this tensor structure is the constant presheaf with transfers $\mathbf{Z}$ which has as its components the constant presheaves with transfers over $\mathcal{X}_i$. The following lemma is straightforward.

Lemma 3.1 *Let F, G be presheaves of sets over $\mathcal{X}$. Then there is a natural isomorphism*

$$\mathbf{Z}_{tr}(F \times G) = \mathbf{Z}_{tr}(F) \otimes \mathbf{Z}_{tr}(G).$$

A major difference between the categories of presheaves with transfers over a scheme and over a simplicial scheme lies in the fact that the tensor structure on $PST(\mathcal{X})$ does not come from a tensor structure on $SmCor(\mathcal{X})$. In particular, for a general $\mathcal{X}$, $\mathbf{Z}$ is not representable and the tensor product of two representable presheaves with transfers is not representable.

Let us say that a presheaf with transfers F is *admissible*, if its components F_i are direct sums of representable presheaves with transfers over $\mathcal{X}_i$. The class of admissible presheaves contains $\mathbf{Z}$ and is closed under tensor products. The following straightforward lemma implies that any representable presheaf with transfers is admissible and in particular that $Lres$ provides a resolution by admissible presheaves.

Lemma 3.2 *A presheaf with transfers of the form $\mathbf{Z}_{tr}(Y,i)$ is admissible.*

Proof: Follows immediately from (2.4). □

Lemma 3.3 *Let K, K', L be complexes of admissible presheaves with transfers and $K \to K'$ be a quasi-isomorphism. Then $K \otimes L \to K' \otimes L$ is a quasi-isomorphism.*

Proof: The analog of this proposition for presheaves with transfers over each $\mathcal{X}_i$ holds since free presheaves with transfers are projective objects in $PST(\mathcal{X}_i)$. Since both quasi-isomorphisms and tensor products in $PST(\mathcal{X})$ are defined term-wise, the proposition follows. □

In view of Lemmas 3.2 and 3.3 the functor

$$K \overset{L}{\otimes} L := Lres(K) \otimes Lres(L)$$

respects quasi-isomorphisms in K and L and therefore defines a functor on the derived categories, which we also denote by $\overset{L}{\otimes}$.

To see that this functor is a part of a good tensor triangulated structure on $D(\mathcal{X})$, we may use the following equivalent definition. Let A be the additive category of admissible presheaves with transfers over $\mathcal{X}$ and $H_-(A)$ the homotopy category of complexes bounded from above over A. The tensor product of presheaves with transfers makes A into a tensor additive category and we may consider the corresponding structure of the tensor triangulated category on $H_-(A)$. Observe now that the natural functor

$$H_-(A) \to D(\mathcal{X})$$

is the localization with respect to the class of quasi-isomorphisms and that the tensor product $\overset{L}{\otimes}$ on $D(\mathcal{X})$ is the localization of the tensor product on $H_-(A)$. Since a tensor trinagulated structure localizes well we conclude that $D(\mathcal{X})$ is a tensor trinagulated category with respect to $\overset{L}{\otimes}$. To formulate this statement more precisely we will use the axioms connecting tensor and triangulated structures, which were introduced in [2]. Since we often work with categories which do not have internal Hom-objects we will write (TC2a) for that part of the axiom (TC2) of [2, Def. 4.1, p. 47], which refers to the tensor product, and (TC2b) for that part of the axiom which refers to the internal Hom-objects. As we show in the appendix, axiom (TC2b) follows from the axioms (TC1), (TC2a) and (TC3) whenever the internal Hom-objects exist.

Proposition 3.4 *The category $D(\mathcal{X})$ is symmetric monoidal with respect to the tensor product introduced above and this symmetric monoidal structure satisfies axioms (TC1), (TC2a) and (TC3) with respect to the standard triangulated structure.*

The interaction between the tensor structure and the standard functors introduced above are given by the following lemmas.

Lemma 3.5 *For a morphism of simplicial schemes $f : \mathcal{X} \to \mathcal{Y}$ one has canonical isomorphisms in $PST(\mathcal{X})$ of the form*

$$f^*(F \otimes G) = f^*(F) \otimes f^*(G) \tag{3.2}$$

and canonical isomorphisms in $D(\mathcal{X})$ of the form

$$Lf^*(K \overset{L}{\otimes} L) = Lf^*(K) \overset{L}{\otimes} Lf^*(L). \tag{3.3}$$

Proof: The first statement follows immediately from (3.1). The second follows from the first and the fact that f^* takes admissible objects to admissible objects. □

Lemma 3.6 *For a simplicial scheme $\mathcal{X}$ one has canonical isomorphisms in $PST(\mathcal{X}_i)$ of the form*

$$r_i^*(F \otimes G) = r_i^*(F) \otimes r_i^*(G)$$

and canonical isomorphisms in $D(\mathcal{X}_i)$ of the form

$$Lr_i^*(K \overset{L}{\otimes} L) = Lr_i^*(K) \overset{L}{\otimes} Lr_i^*(L).$$

Lemma 3.7 *For a simplicial scheme $\mathcal{X}$ over a scheme S one has canonical isomorphisms in $PST(\mathcal{X})$ of the form*

$$c^*(F \otimes G) = c^*(F) \otimes c^*(G)$$

and canonical isomorphisms in $D(\mathcal{X})$ of the form

$$Lc^*(K \overset{L}{\otimes} L) = Lc^*(K) \overset{L}{\otimes} Lc^*(L).$$

Proof: The first statement follows immediately from (3.1). The second from the first and the fact that f^* takes representable presheaves with transfers over S to admissible presheaves with transfers over $\mathcal{X}$. $\square$

Lemma 3.8 *For a simplicial scheme $\mathcal{X}$ over a scheme S such that all $\mathcal{X}_i$ are smooth over S, one has canonical isomorphisms in $PST(\mathcal{X})$ of the form*

$$c_\#(F \otimes c^*(G)) = c_\#(F) \otimes G$$

and canonical isomorphisms in $D(\mathcal{X})$ of the form

$$Lc_\#(F \overset{L}{\otimes} Lc^*(G)) = Lc_\#(F) \overset{L}{\otimes} G.$$

Proof: Since (3.2) holds and $c_\#$ is the left adjoint to c^* there is a natural map

$$c_\#(F \otimes c^*(G)) \to c_\#(F) \otimes G.$$

Since all the functors here are right exact and every presheaf with transfers is the colimit of a diagram of representable presheaves with transfers, it is sufficient to check that this map is an isomorphism for representable F and G. This follows immediately from the isomorphisms

$$\mathbf{Z}_{tr}(Y,i) \otimes c^*(\mathbf{Z}_{tr}(Z)) = \mathbf{Z}_{tr}(Y \times_S Z, i) \tag{3.4}$$

and

$$c_\#(\mathbf{Z}_{tr}(Y,i)) = \mathbf{Z}_{tr}(Y/S).$$

The isomorphism (3.4) implies also that for a representable G and a representable F, $F \otimes c^*(G)$ is representable. Therefore the first statement of the lemma implies the second. $\square$

To compute $Lc_\#$ on the constant sheaf, we need the following result.

Lemma 3.9 *Consider the simplicial object $L\mathbf{Z}_\bullet$ in $SmCor(\mathcal{X})$ with terms*

$$L\mathbf{Z}_i = \mathbf{Z}_{tr}(\mathcal{X}_i, i)$$

and the obvious structure morphisms. Let $L\mathbf{Z}_*$ be the corresponding complex. Then there is a natural quasi-isomorphism

$$L\mathbf{Z}_* \to \mathbf{Z}.$$

Proof: We have to show that for any (Y, j) the simplicial abelian group $L\mathbf{Z}_\bullet(Y, j)$ is a resolution for the abelian group

$$\mathbf{Z}(Y) = H^0(Y,\mathbf{Z}).$$

Indeed one verifies easily that

$$L\mathbf{Z}_\bullet(Y, j) = \mathbf{Z}(\Delta^j) \otimes \mathbf{Z}(Y)$$

and since Δ^j is contractible, the projection $\Delta^j \to pt$ defines a natural quasi-isomorphism $L\mathbf{Z}_\bullet(Y, j) \to \mathbf{Z}(Y)$. □

If $\mathcal{X}$ is such that all its terms are disjoint unions of smooth schemes over S then we may consider the complex $\mathbf{Z}_{tr}(\mathcal{X})_*$ defined by the simplicial object represented by $\mathcal{X}$ in the derived categories of presheaves with transfers over S. Note that

$$\mathbf{Z}_{tr}(\mathcal{X}) = c_\#(L\mathbf{Z}_\bullet)$$

and therefore Lemmas 3.8 and 3.9 imply the following formula.

Proposition 3.10 *For a complex of presheaves with transfers K over S one has*

$$Lc_\# Lc^*(K) \cong \mathbf{Z}_{tr}(\mathcal{X})_* \overset{L}{\otimes} K.$$

Remark 3.11 It is easy to see that the functors Lf^* can be computed using more general admissible resolutions instead of the representable resolutions. But we cannot use admissible resolutions to compute $Lc_\#$, since for the (admissible) constant presheaf with transfers

$$\mathbf{Z} = Lc^*(\mathbf{Z})$$

we have by 3.10:

$$c_\#(\mathbf{Z}) = h_0(Lc_\#(\mathbf{Z})) = h_0(\mathbf{Z}_{tr}(\mathcal{X})_*) \neq \mathbf{Z}_{tr}(\mathcal{X}_*) = Lc_\#(\mathbf{Z}).$$

Remark 3.12 It would be interesting to find a nice explicit description of the complex $\mathbf{Z}_{tr}(\mathcal{X}_i, i) \otimes \mathbf{Z}_{tr}(\mathcal{X}_j, j)$ or, equivalently, a nice simplicial resolution of $h_{(\mathcal{X}_i, i)} \times h_{(\mathcal{X}_j, j)}$ by representable presheaves (of sets).

4. Relative motives

For a scheme X, let $W^{el}(X)$ be the class of complexes over $PST(X)$ defined as follows:

1. For any pull-back square
$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & Y \end{array} \tag{4.1}$$
in Sm/X such that p is etale, j an open embedding and $p^{-1}(Y\backslash U) \to Y\backslash U$ is an isomorphism, the corresponding Mayer-Vietoris complex
$$\mathbf{Z}_{tr}(W) \to \mathbf{Z}_{tr}(U) \oplus \mathbf{Z}_{tr}(V) \to \mathbf{Z}_{tr}(Y)$$
is in $W^{el}(X)$.

2. For any Y in Sm/X, the complex $\mathbf{Z}_{tr}(Y \times \mathbf{A}^1) \to \mathbf{Z}_{tr}(Y)$ is in $W^{el}(X)$.

Let further $W(X)$ be the smallest class in $D(X)$ which contains $W^{el}(X)$ and is closed under triangles, direct sums and direct summands. One says that a morphism in $D(X)$ is an $\mathbf{A}^1$*-equivalence*, if its cone lies in $W(X)$ and defines the triangulated category $DM_{-}^{eff}(X)$ of (effective, connective) motives over X as the localization of $D(X)$ with respect to $\mathbf{A}^1$-equivalences.

For a simplicial $\mathcal{X}$, consider
$$W_i^{el}(\mathcal{X}) := r_{i,\#}(W^{el}(\mathcal{X}_i))$$
as classes of complexes in $PST(\mathcal{X})$. Let $W(\mathcal{X})$ be the smallest class in $D(\mathcal{X})$ which contains all $W_i^{el}(\mathcal{X})$ and is closed under triangles, direct sums and direct summands.

Definition 4.1 A morphism u in $D(\mathcal{X})$ is called an $\mathbf{A}^1$*-equivalence*, if its cone lies in $W(\mathcal{X})$.

Definition 4.2 Let $\mathcal{X}$ be a simplicial scheme. The triangulated category $DM_{-}^{eff}(\mathcal{X})$ of (effective, connective) motives over $\mathcal{X}$ is the localization of $D(\mathcal{X})$ with respect to $\mathbf{A}^1$-equivalences.

Lemma 4.3 *1. For any morphism f of simplicial schemes the functor Lf^* takes $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences,*

2. for any simplicial scheme the functors r_i^* take $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences,

3. for any simplicial scheme the functors $Lr_{i,\#}$ take $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences,

4. for any simplicial scheme over S the functor Lc^* takes $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences,

5. *for any smooth simplicial scheme over S the functor $Lc_\#$ takes $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences.*

Proof: It follows immediately from the definitions that the functors Lf^* and $Lc_\#$ and $r_{i,\#}$ take $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences.

The functor r_i^* takes $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences by (2.8).

To see that Lc^* takes $\mathbf{A}^1$-equivalences to $\mathbf{A}^1$-equivalences consider a complex L over S which consists of representable presheaves with transfers. Let further L_i be the pull-back of L to $\mathcal{X}_i$ which we consider as a complex of representable presheaves with transfers over $\mathcal{X}$. One has

$$Lc^*(L) = \mathbf{Z} \otimes Lc^*(L) = L\mathbf{Z}_* \otimes Lc^*(L)$$

where $L\mathbf{Z}_*$ is the complex of Lemma 3.9. By (3.4) we conclude that $Lc^*(L)$ is quasi-isomorphic to the total complex of a bicomplex with terms of the form $r_{i,\#}(L_i)$. Since for $L \in W^{el}(S)$ we have $L_i \in W^{el}(\mathcal{X}_i)$ for all i, this implies that Lc^* takes $W(S)$ to $W(\mathcal{X})$. □

We keep the notations Lf^*, r_i^*, $Lr_{i,\#}$, Lc^* and $Lc_\#$ for the functors between the categories DM^{eff} which are defined by Lf^*, Lc^* and $Lc_\#$ respectively. Note (cf. [1, Prop. 2.6.2]) that these functors have the same adjunction properties as the original functors.

Lemma 4.4 *The family of functors*

$$r_i^* : DM_-^{eff}(\mathcal{X}) \to DM_-^{eff}(\mathcal{X}_i)$$

is conservative i.e. if $r_i^*(K) \cong 0$ for all i then $K \cong 0$.

Proof: Same as in Lemma 2.4. □

Proposition 4.5 *The tensor product $\overset{L}{\otimes}$ respects $\mathbf{A}^1$-equivalences.*

Proof: It is enough to show that for $K \in r_{i,\#}(W^{el}(\mathcal{X}_i))$ and any L, the object $K \overset{L}{\otimes} L$ is zero in $DM_-^{eff}(\mathcal{X})$. By Lemma 4.4 it is sufficient to show that $r_j^*(K) \cong 0$ for all j. This follows immediately from Lemma 3.6 and (2.8). □

By Proposition 4.5, the tensor structure on $D(\mathcal{X})$ defines a tensor structure on $DM_-^{eff}(\mathcal{X})$. Since any distinguished triangle in $DM_-^{eff}(\mathcal{X})$ is, by definition, isomorphic to the image of a distinguished triangle in $D(\mathcal{X})$, Proposition 3.4 implies immediately the following result.

Proposition 4.6 *The axioms (TC1)-(TC3) of [2] hold for $DM_-^{eff}(\mathcal{X})$.*

Proposition 4.7 *The category $DM_-^{eff}(\mathcal{X})$ is Karoubian, i.e. projectors in this category have kernels and images.*

Proof: For any K in $DM_{-}^{eff}(\mathcal{X})$ the countable direct sum $\oplus_{i=1}^{\infty}K$ exists in $DM_{-}^{eff}(\mathcal{X})$ for obvious reasons. This implies the statement of the proposition in view of the following easy generalization of [3, Prop.1.6.8 p.65]. □

Lemma 4.8 *Let D be a triangulated category such that for any object K in D the countable direct sum $\oplus_{i=1}^{\infty}K$ exists. Then D is Karoubian.*

Proof: Same as the proof of [3, Prop.1.6.8 p.65]. □

5. Relative Tate motives

For any S we may define the elementary *Tate objects* $\mathbf{Z}(p)[q]$ in $DM_{-}^{eff}(S)$ in the same way they were defined in [9, p.192] for $S = Spec(k)$. For $\mathcal{X}$ over S we define the *Tate objects* $\mathbf{Z}(p)[q]$ in $DM_{-}^{eff}(\mathcal{X})$ as $Lc^*(\mathbf{Z}(p)[q])$. Note that this definition does not depend on S - one may always consider $\mathcal{X}$ as a simplicial scheme over $Spec(\mathbf{Z})$ and lift the Tate objects from $Spec(\mathbf{Z})$. We denote by $DT(\mathcal{X})$ (resp. $\overline{DT}(\mathcal{X})$) the thick (resp. localizing) subcategory in $DM_{-}^{eff}(\mathcal{X})$ generated by Tate objects, i.e. the smallest subcategory which is closed under shifts, triangles, direct summands (resp. and direct sums) and contains $\mathbf{Z}(i)$ for all $i \geq 0$. We will call these categories the category of (effective) Tate objects over $\mathcal{X}$ and the category of effective Tate objects of finite type over $\mathcal{X}$, respectively. When $\mathcal{X}$ is clear from the context, we will write DT and $\overline{DT}$ instead of $DT(\mathcal{X})$ and $\overline{DT}(\mathcal{X})$, respectively, and since we never work with non-effective objects in this paper we will often omit the word "effective" .

Both subcategories $DT(\mathcal{X})$ and $\overline{DT}(\mathcal{X})$ are clearly closed under tensor product and Proposition 4.6 implies that they are tensor triangulated categories satisfying May's axiom $TC3$.

Remark 5.1 The category $DT(\mathcal{X})$ does not coincide in general with the triangulated subcategory generated in $DM_{-}^{eff}(\mathcal{X})$ by Tate objects. Consider for example the case when $\mathcal{X} = \mathcal{X}_1 \amalg \mathcal{X}_2$ and both $\mathcal{X}_1$ and $\mathcal{X}_2$ are non-empty. Then the constant presheaf with transfers $\mathbf{Z}$ is a direct sum of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ where $\mathbf{Z}_i$ is the constant presheaf with transfers on $\mathcal{X}_i$. One can easily show that the $\mathbf{Z}_i$'s are not in the triangulated subcategory generated by Tate objects.

However, one can show that the problem demonstrated by this example is the only possible one - if $H^0(\mathcal{X},\mathbf{Z})$ is $\mathbf{Z}$ then the triangulated subcategory in $DM_{-}^{eff}(\mathcal{X})$ generated by Tate objects is closed under directs summands and therefore coincides with $DT(\mathcal{X})$.

For M in DM_{-}^{eff}, we denote as usual by $H_{*,*}(M)$ the groups

$$H_{p,q}(M) = \begin{cases} Hom(\mathbf{Z}, M(-q)[-p]) & \text{for } q \leq 0 \\ Hom(\mathbf{Z}(q)[p], M) & \text{for } q \geq 0 \end{cases}$$

and by $H^{*,*}(M)$ the groups

$$H^{p,q}(M) = \begin{cases} Hom(M,\mathbf{Z}(q)[p]) & \text{for } q \geq 0 \\ 0 & \text{for } q < 0. \end{cases}$$

Lemma 5.2 *Let $f : M \to M'$ be a morphism in DT which defines isomorphisms on the groups $H_{p,q}(-)$ for $q \geq 0$. Then f is an isomorphism.*

Proof: For a given f, the class of all N such that the maps

$$Hom(N[p],M) \to Hom(N[p],M')$$

are isomorphisms for all p is a thick subcategory of DT. Our condition means that it contains all $\mathbf{Z}(q)$. Therefore it coincides with the whole of DT and we conclude that f is an isomorphism by the Yoneda Lemma. □

Let $\mathcal{X}$ be a smooth simplicial scheme over S. For such a $\mathcal{X}$ we define $M(\mathcal{X})$ as the object in $DM_{-}^{eff}(S)$ given by the complex $\mathbf{Z}_{tr}(\mathcal{X})$ associated with the simplicial object $\mathcal{X}$ in $SmCor(S)$. Note that this definition is compatible with the definition of motives of smooth simplicial schemes given in [6].

Proposition 5.3 *For $\mathcal{X}$ as above, there are natural isomorphisms*

$$Hom_{DM(\mathcal{X})}(\mathbf{Z}(q')[p'],\mathbf{Z}(q)[p]) = Hom_{DM(S)}(M(\mathcal{X})(q')[p'],\mathbf{Z}(q)[p])).$$

Proof: We have by adjunction

$$Hom_{DM(\mathcal{X})}(\mathbf{Z}(p')[q'],\mathbf{Z}(q)[p]) = Hom_{DM(\mathcal{X})}(c^*\mathbf{Z}(q')[p'],c^*\mathbf{Z}(q)[p])) =$$

$$= Hom_{DM(S)}(Lc_{\#}c^*\mathbf{Z}(q')[p'],\mathbf{Z}(q)[p])$$

and Proposition 3.10 implies that for any M in $DM_{-}^{eff}(S)$, one has

$$Lc_{\#}c^*(M) = M(\mathcal{X}) \otimes M.$$

□

Corollary 5.4 *For $\mathcal{X}$ as above and any $i > 0$, one has*

$$Hom_{DM(\mathcal{X})}(\mathbf{Z},\mathbf{Z}[-i]) = 0.$$

Combining Proposition 5.3 with the Cancellation Theorem [7] we get the following result.

Corollary 5.5 *Let now $\mathcal{X}$ be a smooth simplicial scheme over a perfect field k. Then one has*

$$Hom_{DM(\mathcal{X})}(\mathbf{Z}(q')[p'],\mathbf{Z}(q)[p]) =$$
$$= \begin{cases} 0 & \text{for } q < q' \\ Hom_{DM(\mathcal{X})}(\mathbf{Z},\mathbf{Z}(q-q')[p-p']) & \text{for } q \geq q'. \end{cases}$$

Remark 5.6 Using the fact that a regular scheme of equal characteristic is the inverse limit of a system of smooth schemes over a perfect field, it is easy to generalize Corollary 5.5 to smooth simplicial schemes over regular schemes of equal characteristic. We expect this hold for all regular simplicial schemes but not for general (simplicial) schemes.

From now on we assume that $S = Spec(k)$ where k is a perfect field and $\mathcal{X}$ is a smooth simplicial scheme over S.

Lemma 5.7 *For any X, Y in $DT(\mathcal{X})$ there exists an internal Hom-object (Z,e) from X to Y.*

Proof: Consider first the case when $X = \mathbf{Z}(i)$ and $Y = \mathbf{Z}(j)$. Corollary 5.5 implies immediately that (0,0) is an internal Hom-object from $\mathbf{Z}(i)$ to $\mathbf{Z}(j)$ for $j < i$. The same corollary shows that $(\mathbf{Z}(j-i),e)$, where e is the isomorphism $\mathbf{Z}(j-i) \otimes \mathbf{Z}(i) \to \mathbf{Z}(j)$, is an internal Hom-object from $\mathbf{Z}(i)$ to $\mathbf{Z}(j)$ for $j \geq i$. The fact that (Z,e) exists for arbitrary X and Y follows now from Theorem 8.3 and the obvious argument for direct summands. □

From now on we choose a specification of internal Hom-objects in $DT(\mathcal{X})$ (see Appendix) such that for $i \geq j$ one has $\underline{Hom}(\mathbf{Z}(j),\mathbf{Z}(i)) = \mathbf{Z}(i-j)$.

Let $DT_{\geq n}$ (resp. $DT_{<n}$) be the thick subcategory in $DT(\mathcal{X})$ generated by Tate objects $\mathbf{Z}(i)$ for $i \geq n$ (resp. $i < n$). The subcategories $DT_{\geq n}$ form a decreasing filtration

$$\dots \subset DT_{\geq 1} \subset DT_{\geq 0} = DT(\mathcal{X})$$

and we have

$$\cap_n DT_{\geq n} = 0.$$

Similarly the subcategories $DT_{<n}$ form an increasing filtration

$$0 = DT_{<0} \subset DT_{<1} \subset \dots \subset DT(\mathcal{X})$$

and we have

$$\cup_n DT_{<n} = DT(\mathcal{X}).$$

We call these filtrations *the slice filtrations* on DT since they are similar to the slice filtration on the motivic stable homotopy category. Since we consider here only Tate motives the slice filtration coincides (up to numbering) with the weight filtration, but for more general motives they are different.

Lemma 5.8 *Let M be such that $H_{*,i}(M) = 0$ for all $i \geq n$. Then M lies in $DT_{<n}$.*

Proof: Set

$$\Psi(M) = \underline{Hom}(\underline{Hom}(M, \mathbf{Z}(n-1)), \mathbf{Z}(n-1)).$$

The adjoint to the morphism

$$ev \circ \sigma : M \otimes \underline{Hom}(M, \mathbf{Z}(n-1)) \to \mathbf{Z}(n-1),$$

where σ is the permutation of multiples, is a morphism

$$\psi : M \to \Psi(M)$$

which is natural in M. Using Proposition 8.5 and Corollary 5.5 one verifies immediately that $\Psi(M)$ lies in $DT_{<n}$ for $M = \mathbf{Z}(q)[p]$, $q \geq 0$ and therefore, by Proposition 8.5, $\Psi(M)$ lies in $DT_{<n}$ for all M. It remains to check that for M satisfying the condition of the lemma, ψ is an isomorphism. Consider the maps

$$H_{*,i}(M) \to H_{*,i}(\Psi(M)).$$

For $i < n$ and $M = \mathbf{Z}(q)[p]$, these maps are isomorphisms by Corollary 5.5. Together with Proposition 8.5 and the five lemma, we conclude that they are isomorphisms for $i < n$ and all M. On the other hand $H_{*,i}(\Psi(M)) = 0$ for $i \geq n$ and any M and therefore under the conditions of the lemma, ψ is an isomorphism by Lemma 5.2. □

Lemma 5.9 *For any M in DT and any n, there exists a distinguished triangle of the form*

$$\Pi_{\geq n} M \to M \to \Pi_{<n} M \to \Pi_{\geq n} M[1] \tag{5.1}$$

such that $\Pi_{\geq n} M$ lies in $DT_{\geq n}$ and $\Pi_{<n} M$ lies in $DT_{<n}$.

Proof: Set

$$\Pi_{\geq n}(M) = \underline{Hom}(\mathbf{Z}(n), M)(n)$$

and define $\Pi_{<n} M$ by the distinguished triangle

$$\Pi_{\geq n} M \to M \to \Pi_{<n} M \to \Pi_{\geq n} M[1]$$

where the first arrow is $e = ev_{\mathbf{Z}(n),M}$. Clearly, $\Pi_{\geq n} M$ lies in $DT_{\geq n}$. It remains to check that $\Pi_{<n} M$ lies in $DT_{<n}$. By Lemma 5.8 it is sufficient to check that $H_{*,i}(\Pi_{<n} M) = 0$ for all $i \geq n$, i.e. that e defines an isomorphism on $H_{*,i}(-)$ for $i \geq n$. In view of Proposition 8.5 and the Five Lemma it is sufficient to verify it for $M = \mathbf{Z}(q)[p]$, in which case it follows from Corollary 5.5. □

Remark 5.10 Note that the proof of Lemma 5.8 shows that in the distinguished triangle of Lemma 5.9, one may choose $M \to \Pi_{<n}M$ to be

$$\psi : M \to \underline{Hom}(\underline{Hom}(M, \mathbf{Z}(n-1)), \mathbf{Z}(n-1)).$$

Lemma 5.11 *Let $f : M_1 \to M_2$ be a morphism in DT and let*

$$\Pi_{\geq n}M_1 \to M_1 \to \Pi_{<n}M_1 \to \Pi_{\geq n}M_1[1]$$

$$\Pi_{\geq n}M_2 \to M_2 \to \Pi_{<n}M_2 \to \Pi_{\geq n}M_2[1]$$

be distinguished triangles satisfying the conditions of Lemma 5.9. Then there exists a unique morphism of triangles of the form

$$\begin{tikzcd} \Pi_{\geq n}M_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, "f"'] & \Pi_{<n}M_1 \arrow[r] \arrow[d, "h"'] & \Pi_{\geq n}M_1[1] \arrow[d] \\ \Pi_{\geq n}M_2 \arrow[r] & M_2 \arrow[r] & \Pi_{<n}M_2 \arrow[r] & \Pi_{\geq n}M_2[1]. \end{tikzcd} \tag{5.2}$$

Proof: Uniqueness follows from the fact that

$$Hom(\Pi_{\geq n}M_1[*], \Pi_{<n}M_2) = 0.$$

The same fact implies that

$$Hom(M_1, \Pi_{<n}M_2) = Hom(\Pi_{<n}M_1, \Pi_{<n}M_2)$$

and therefore there exists a morphism h which makes the middle square of (5.2) commutative. Extending this square to a morphism of distinguished triangles, we get the existence part of the lemma. $\square$

Lemma 5.12 *For any M and any triangle of the form (5.1) satisfying the conditions of Lemma 5.9, one has:*

1. *For any N in $DT_{<n}$ one has*

$$Hom(\Pi_{<n}M, N) = Hom(M, N)$$

$$Hom(\Pi_{\geq n}M, N) = 0.$$

2. *For any N in $DT_{\geq n}$ one has*

$$Hom(N, \Pi_{\geq n}M) = Hom(N, M)$$

$$Hom(N, \Pi_{<n}M) = 0.$$

Remark 5.13 The major difference between the slice filtrations in the triangulated category of motives and in the motivic stable homotopy category is that in the later case, Lemma 5.12 does not hold. For N in $SH_{\geq n}$ and M in $SH_{<n}$ one may have $Hom(N,M) \neq 0$. The Hopf map $S^1_t \to S^0$ is an example of a morphism of this kind.

Lemma 5.11 implies that the triangles of the form (5.1) are functorial in M. Choosing one such triangle for each M and each n, we get functors:

$$\Pi_{\geq n} : DT \to DT_{\geq n}$$

$$\Pi_{<n} : DT \to DT_{<n}.$$

Lemma 5.12 shows that $\Pi_{\geq n}$ is a right adjoint to the corresponding inclusion and $\Pi_{<n}$ is a left adjoint to the corresponding inclusion. We can also describe these functors in terms of the internal Hom-functors

$$\Pi_{\geq n}(M) = \underline{Hom}(\mathbf{Z}(n), M)(n)$$

$$\Pi_{<n}(M) = \underline{Hom}(\underline{Hom}(M, \mathbf{Z}(n-1)), \mathbf{Z}(n-1)).$$

By Proposition 8.5 we conclude that $\Pi_{\geq n}$ and $\Pi_{<n}$ are triangulated functors.

Applying Lemma 5.12 for $N = \Pi_{\geq (n+1)} M$ and $N = \Pi_{<(n-1)}$ we get canonical morphisms

$$\Pi_{\geq (n+1)} M \to \Pi_{\geq n} M$$

$$\Pi_{<n} \to \Pi_{<(n-1)} M.$$

We extend these morphisms to distinguished triangles

$$\Pi_{\geq (n+1)} M \to \Pi_{\geq n} M \to s_n(M) \to \Pi_{\geq (n+1)} M[1] \tag{5.3}$$

$$s'_{n-1}(M) \to \Pi_{<n} \to \Pi_{<(n-1)} M \to s'_{n-1}(M)[1]. \tag{5.4}$$

One observes easily that $s'_n(M) \cong s_n(M)$ and that this object lies in $DT_n = DT_{\geq n} \cap DT_{<n+1}$. Therefore, Lemma 5.11 is applicable to triangles (5.3) and (5.4) and we conclude that these triangles are functorial in M. Choosing one such triangle for each M and each n, we obtain functors

$$s_n : DT \to DT_n. \tag{5.5}$$

Since $s_n = \Pi_{<n+1} \Pi_{\geq n}$, these functors are triangulated. We set

$$s_* = \oplus_{n \geq 0} s_n : DT \to \oplus_{n \geq 0} DT_n. \tag{5.6}$$

Note that (5.6) makes sense, since for any M one has $s_n(M) = 0$ for all but finitely many n. The functors (5.5), (5.6) are called the *slice functors* over $\mathcal{X}$.

Lemma 5.14 *The functor s_* is conservative, i.e. if $s_*(M) = 0$ then $M = 0$.*

Proof: Follows easily by induction. □

Lemma 5.15 *Define a tensor product on $\oplus_n DT_n$ by the formula*

$$(M_i)_{i\geq 0}\otimes (M_j)_{j\geq 0} = (\oplus_{i+j=n} M_i\otimes M_j)_{n\geq 0}.$$

Then for any N, M there is a natural isomorphism

$$s_*(N\otimes M) = s_*(N)\otimes s_*(M).$$

Proof: For any M and N, the morphisms $\Pi_{\geq i}M\to M$ and $\Pi_{\geq j}N\to N$ define a morphism

$$s_{i+j}(\Pi_{\geq i}M\otimes \Pi_{\geq j}N)\to s_{i+j}(M\otimes N). \tag{5.7}$$

On the other hand the the projections $\Pi_{\geq i}M\to s_i(M)$ and $\Pi_{\geq j}N\to s_j(N)$ define a morphism

$$s_{i+j}(\Pi_{\geq i}M\otimes \Pi_{\geq j}N)\to s_{i+j}(s_i(M)\otimes s_j(N)) = s_i(M)\otimes s_j(N). \tag{5.8}$$

One can easily see that (5.8) is an isomorphism. The inverse to (5.8) together with (5.7) defines a natural morphism

$$\oplus_{i+j=n} s_i(M)\otimes s_j(N)\to s_n(M\otimes N).$$

One verifies easily that it is an isomorphism for $M = \mathbf{Z}(q)[p]$, $N = \mathbf{Z}(q')[p']$, which implies by the Five Lemma that it is an isomorphism for all M and N. □

Lemma 5.16 *The functors $\Pi_{\geq n}$, $\Pi_{<n}$ and s_* commute with the pull-back functors Lf^* for arbitrary morphisms of smooth simplicial schemes $f:\mathcal{X}\to\mathcal{Y}$.*

Proof: This follows immediately from the fact that the functor Lf^* takes $DT_{\geq n}$ to $DT_{\geq n}$ and $DT_{<n}$ to $DT_{<n}$. □

Lemma 5.17 *Let X, Y, P_1, P_2 be such that for some n and m, one has*

$$X\in DT_{\leq n}\;\; P_1\in DT_{\geq n}$$

$$Y\in DT_{\leq m}\;\; P_2\in DT_{\geq m}.$$

Then

$$(\underline{Hom}(X,P_1)\otimes \underline{Hom}(Y,P_2), ev_{X,P_1}\otimes ev_{Y,P_2})$$

is an internal Hom-object from $X\otimes Y$ to $P_1\otimes P_2$.

Proof: We need to verify that for any M the homomorphism

$$Hom(M, X' \otimes Y') \to Hom(M \otimes X \otimes Y, P_1 \otimes P_2)$$

defined by $ev_{X,P_1} \otimes ev_{Y,P_2}$ is a bijection. Using Proposition 8.5 and the Five Lemma we can reduce the problem to the case when M, X, Y, P_1 and P_2 are all motives of the form $\mathbf{Z}(q)[p]$ with the appropriate restrictions of q. In this case the statement follows from Corollary 5.5. □

Lemma 5.18 *Let $n \geq 0$ be an integer and*

$$M_0 \stackrel{a}{\to} M_1 \stackrel{b}{\to} M_2 \tag{5.9}$$

a sequence of morphisms in DT such that the following conditions hold:

1. *M_0 is in $DT_{\geq n}$ and $s_i(a)$ is an isomorphism for $i \geq n$,*
2. *M_2 is in $DT_{<n}$ and $s_i(b)$ is an isomorphism for $i < n$.*

Then there exists a unique morphism $M_2 \to M_0[1]$ such that the sequence

$$M_0 \stackrel{a}{\to} M_1 \stackrel{b}{\to} M_2 \to M_0[1]$$

is a distinguished triangle. This distinguished triangle is then isomorphic to the triangle

$$\Pi_{\geq n}(M_1) \to M_1 \to \Pi_{<n}(M_1) \to \Pi_{\geq n}(M_1)[1].$$

Proof: Note first that $Hom(M_0, M_2) = 0$ and therefore $b \circ a = 0$. Extending a to a distinguished triangle we get a factorization of b through a morphism $\phi : cone(a) \to M_2$. Our conditions imply that $s_*(\phi)$ is an isomorphism and we conclude by Lemma 5.14 that ϕ is an isomorphism and hence (5.9) extends to a distinguished triangle. The proof of the two other statements of the lemma is straightforward. □

Since the functor $X \mapsto X(n)$ from DT_0 to DT_n is an equivalence (by Corollary 5.5), we may consider s_* as a functor with values in $\oplus_n DT_0$. To describe the category DT_0 consider the projection

$$D(\mathcal{X}) \to DM_-^{eff}(\mathcal{X}) \tag{5.10}$$

from the derived category of presheaves with transfers over $\mathcal{X}$ to DM. Let us say that a presheaf with transfers (F_i) on $\mathcal{X}$ is *locally constant* if for every i the presheaf with transfers F_i on $Sm/\mathcal{X}_i$ is locally constant. Locally constant presheaves with transfers clearly form an abelian subcategory LC in the abelian category of presheaves with transfers.

Remark 5.19 Let $X \mapsto CC(X)$ be the functor which commutes with coproducts and takes a connected scheme to the point. Applying CC to a simplicial scheme $\mathcal{X}$ we get a simplicial set $CC(\mathcal{X})$. If $CC(\mathcal{X})$ is a connected simplicial set then $LC(\mathcal{X})$ is equivalent to the category of modules over $\pi_1(CC(\mathcal{X}))$.

Let DLC be full the subcategory in $D(\mathcal{X})$ which consists of complexes of presheaves with transfers with locally constant cohomology presheaves. Note that DLC is a thick subcategory. Let further DT_0' be the thick subcategory in DLC generated by the constant sheaf $\mathbf{Z}$. Note that the category DLC is Karoubian and therefore the same holds for DT_0'.

Proposition 5.20 *The projection (5.10) defines an equivalence between DT_0' and DT_0.*

Proof: Let us show first that the restriction of (5.10) to DT_0' is a full embedding. In order to do this, we have to show that objects of DT_0' are orthogonal to objects of $r_i(W^{el}(\mathcal{X}_i))$. In order to do this, it is enough to show that for a smooth scheme X the constant presheaf with transfers is orthogonal to complexes lying in $W^{el}(X)$, i.e. that for any such presheaf F and such a complex K one has

$$Hom_D(K,F) = 0.$$

This follows immediately from the fact that for a smooth X and constant F, one has

$$H^i_{Nis}(X,F) = 0 \,\mathrm{for\, i} > 0$$

and

$$F(X \times \mathbf{A}^1) = F(X).$$

To finish the proof of the proposition, it remains to check that the image of DT_0' lies in DT and that any object of DT_0 is isomorphic to the image of an object in DT_0'. The first statement is obvious from definitions. To see the second one, observe that since our functor is a full embedding and the source is Karoubian, its image is a thick subcategory. Since it contains $\mathbf{Z}$ it coincides with DT_0. □

Remark 5.21 The category DLC has a t-structure whose heart is the category $LC(\mathcal{X})$ of locally constant presheaves with transfers. It is equivalent to the derived category of $LC(\mathcal{X})$ if and only if $CC(\mathcal{X})$ is a $K(\pi,1)$.

Remark 5.22 The condition that the terms of $\mathcal{X}$ are disjoint unions of smooth schemes over a field is important for Proposition 5.20. More precisely what is required is that the terms of $\mathcal{X}$ are disjoint unions of geometrically unibranch (e.g.normal) schemes. If this condition does not hold the Nisnevich cohomology of the terms with the coefficients in constant sheaves may be non-zero.

6. Embedded simplicial schemes

In this section we consider a special case of the general theory developed above.

Definition 6.1 A smooth simplicial scheme $\mathcal{X}$ over k is called *embedded* (over k) if the morphisms

$$M(\mathcal{X} \times \mathcal{X}) \to M(\mathcal{X})$$

defined by the projections are isomorphisms.

In the following lemma we consider $\mathcal{X}$ as a simplicial presheaf on Sm/S and denote by $\pi_0(\mathcal{X})$ the Nisnevich sheaf associated with the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$. We also write pt for the final object in the category of sheaves, which is represented by S.

Lemma 6.2 *Let $\mathcal{X}$ be a smooth simplicial scheme over S such that $\mathcal{X} \to \pi_0(\mathcal{X})$ is a local equivalence in the Nisnevich topology, as a morphism of simplicial presheaves, and the morphism $\pi_0(\mathcal{X}) \to pt$ is a monomorphism. Then $\mathcal{X}$ is embedded.*

Proof: Our conditions imply that $pr : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a local equivalence. Therefore, $M(pr)$ is an isomorphism. □

Example 6.3 Let X be a smooth scheme over S and $\check{C}(X)$ the Cech simplicial scheme of X (see [6, Sec.9]). Then $\check{C}(X)$ is embedded. The sheaf $\pi_0(X)$ takes a smooth connected scheme U to pt, if for any point p of U there exists a morphism

$$Spec(\mathcal{O}^h_{U,p}) \to X,$$

and to $\emptyset$ otherwise.

Example 6.4 For any subpresheaf F of the constant sheaf pt, the standard simplicial resolution $G(F)$ of F is an embedded simplicial scheme. We will show below that for any embedded $\mathcal{X}$, there exists $F \subset pt$ such that $M(\mathcal{X}) \cong M(G(F))$. See Lemma 6.19.

Lemma 6.5 *Let $\mathcal{X}$ be an embedded simplicial scheme over S and*

$$a : c^* Lc_\# c^* \to c^*$$

the natural transformation defined by the adjunction. Then a is an isomorphism.

Proof: By the definition of adjoint functors, the obvious map $b : c^* \to c^* Lc_\# c^*$ is a section of a. Hence, it is sufficient to show that

$$b \circ a : c^* Lc_\# c^* \to c^* Lc_\# c^*$$

is the identity. This map is adjoint to the map

$$p_1 : Lc_\# c^* Lc_\# c^* \to Lc_\# c^*$$

which collapses the second copy of the composition $Lc_{\#}c^*$ to the identity. On the other hand the identity on $c^*Lc_{\#}c^*$ is adjoint in the same way to the map

$$p_2 : Lc_{\#}c^*Lc_{\#}c^* \to Lc_{\#}c^*$$

which collapses the first copy of the composition $Lc_{\#}c^*$ to the identity. It remains to show that $p_1 = p_2$. By Proposition 3.10 we have

$$Lc_{\#}c^*(N) = N \otimes M(\mathcal{X})$$

and one can easily see that p_1 and p_2 can be identified with the morphisms

$$N \otimes M(\mathcal{X}) \otimes M(\mathcal{X}) \to N \otimes M(\mathcal{X})$$

defined by the two projections

$$M(\mathcal{X}) \otimes M(\mathcal{X}) = M(\mathcal{X} \times \mathcal{X}) \to M(\mathcal{X}).$$

These two projections are isomorphisms by our assumption on $\mathcal{X}$ and since the diagonal is their common section we conclude that they are equal. $\square$

For any $\mathcal{X}$, let $DM_{\mathcal{X}}$ denote the localizing subcategory in $DM_-^{eff}(\mathcal{X})$, which is generated by objects of the form $c^*(M)$ for M in $DM_-^{eff}(S)$. Note that $DM_{\mathcal{X}}$ contains the category $DT(\mathcal{X})$ of Tate motives over $\mathcal{X}$.

Lemma 6.6 *If $\mathcal{X}$ is embedded then $Lc_{\#}$ defines a full embedding*

$$Lc_{\#} : DM_{\mathcal{X}} \to DM_-^{eff}(S).$$

Proof: It is sufficient to verify that for $M_1, M_2 \in DM_-^{eff}(S)$, one has

$$Hom_{DM(\mathcal{X})}(c^*M_1, c^*M_2) = Hom_{DM(S)}(Lc_{\#}c^*M_1, Lc_{\#}c^*M_2)$$

which immediately follows by adjunction from Lemma 6.5. $\square$

Lemma 6.7 *If $\mathcal{X}$ is embedded and M,N are objects of $DM_{\mathcal{X}}$ then the canonical morphism*

$$Lc_{\#}(M \otimes N) \to Lc_{\#}(M) \otimes Lc_{\#}(N) \tag{6.1}$$

is an isomorphism.

Proof: Note first that a natural morphism of the form (6.1) is defined by adjunction, since c^* commutes with the tensor products. Since both sides of (6.1) are triangulated functors in each of the arguments, the class of M and N such that (6.1) is an isomorphism is a localizing subcategory. It remains to check that it contains pairs of the form $c^*M(X)$, $c^*M(Y)$ where X, Y are smooth schemes over S. With respect to isomorphisms

$$Lc_\# c^*(M(X)) = M(X) \otimes M(\mathcal{X})$$

$$Lc_\# c^*(M(Y)) = M(Y) \otimes M(\mathcal{X})$$

$$Lc_\# c^*(M(X) \otimes M(Y)) = M(X) \otimes M(Y) \otimes M(\mathcal{X})$$

and the morphism (6.1) coincides with the morphism

$$M(X) \otimes M(Y) \otimes M(\mathcal{X}) \to M(X) \otimes M(\mathcal{X}) \otimes M(Y) \otimes M(\mathcal{X})$$

defined by the diagonal of $\mathcal{X}$. This morphism is an isomorphism, since $\mathcal{X}$ is embedded. $\square$

Lemma 6.7 shows that the restriction of $Lc_\#$ to $DM_{\mathcal{X}}$ is almost a tensor functor. Note that it is not really a tensor functor since

$$Lc_\#(\mathbf{Z}) = M(\mathcal{X}) \neq \mathbf{Z}.$$

We also have to distinguish the internal Hom-objects in $DM_{\mathcal{X}}$ and $DM_-^{eff}(S)$. See Example 6.23 below.

From now on we assume that $\mathcal{X}$ is embedded over S. We use Lemma 6.6 to identify $DM_{\mathcal{X}}$ with a full subcategory in $DM_-^{eff}(S)$. With respect to this identification the functor c^* takes M to $M \otimes M(\mathcal{X})$.

Lemma 6.8 *The subcategory $DM_{\mathcal{X}}$ coincides with the subcategory of objects M such that the morphism*

$$M \otimes M(\mathcal{X}) \to M \tag{6.2}$$

is an isomorphism.

Proof: Let D be the subcategory of M such that (6.2) is an isomorphism. As was mentioned above the functor c^* takes a motive M to $M \otimes M(\mathcal{X})$, so D is contained in $DM_{\mathcal{X}}$. Since D is a localizing subcategory and $DM_{\mathcal{X}}$ is generated by motives of the form $M(X) \otimes M(\mathcal{X})$, the opposite inclusion follows from the fact that for any X the morphism

$$M(X) \otimes M(\mathcal{X}) \otimes M(\mathcal{X}) \to M(X) \otimes M(\mathcal{X})$$

is an isomorphism. $\square$

Remark 6.9 Lemma 6.8 show that $DM_{\mathcal{X}}$ is an ideal in $DM_{-}^{eff}(S)$, i.e. for any K and any M in $DM_{\mathcal{X}}$ the tensor product $K \otimes M$ is in $DM_{\mathcal{X}}$.

Lemma 6.10 *For M in $DM_{\mathcal{X}}$ and $N \in DM_{-}^{eff}(S)$, the natural map*

$$Hom(M, N \otimes M(\mathcal{X})) \to Hom(M, N) \tag{6.3}$$

is an isomorphism.

Proof: Consider the map

$$Hom(M, N) \to Hom(M, N \otimes M(\mathcal{X}))$$

which takes f to $(f \otimes Id_{M(\mathcal{X})}) \circ \phi^{-1}$ where ϕ is the morphism of the form (6.2). One can easily see that this map is both the right and the left inverse to (6.3). □

Lemma 6.10 has the following straightforward corollary.

Lemma 6.11 *Let M, N be objects of $DM_{\mathcal{X}}$, P an object of $DM_{-}^{eff}(S)$ and*

$$e : M \otimes N \to P$$

a morphism such that (N, e) is an internal Hom-object from M to P in $DM_{-}^{eff}(S)$. Define

$$e_{\mathcal{X}} : M \otimes N \to P \otimes M(\mathcal{X})$$

as the morphism corresponding to e, by Lemma 6.10. Then $(N, e_{\mathcal{X}})$ is an internal Hom-object from M to $P \otimes M(\mathcal{X})$ in $DM_{\mathcal{X}}$.

Definition 6.12 An object M in $DM_{\mathcal{X}}$ is called *restricted*, if for any N in $DM_{-}^{eff}(S)$ the natural map

$$Hom(N, M) \to Hom(N \otimes M(\mathcal{X}), M) \tag{6.4}$$

is an isomorphism.

For our next result, we need to recall the motivic duality theorem. For a smooth variety X over a field k and a smooth subvariety Z of X of pure codimension d, the Gysin distinguished triangle defines the motivic cohomology class of Z in X of the form $M(X) \to \mathbf{Z}(d)[2d]$. In particular for X of pure dimension d, the diagonal gives a morphism $M(X) \otimes M(X) \to \mathbf{Z}(d)[2d]$ which we denote by Δ^*. The following motivic duality theorem is proved in [5, Th. 4.3.2, p.234].

Theorem 6.13 *For a smooth projective variety X of pure dimension d over a perfect field k, the pair $(M(X), \Delta^*)$ is the internal Hom-object from $M(X)$ to $\mathbf{Z}(d)[2d]$ (see Appendix).*

Lemma 6.14 *Let $S = Spec(k)$ where k is a perfect field and let X be a smooth projective variety such that $M(X)$ lies in $DM_{\mathcal{X}}$. Then $M(X)$ is restricted.*

Proof: We may clearly assume that X has pure dimension d for some $d \geq 0$. Theorem 6.13 implies that for $M = M(X)$, the morphism (6.4) is isomorphic to the morphism

$$Hom(N \otimes M(X), \mathbf{Z}(d)[2d]) \to Hom(N \otimes M(\mathcal{X}) \otimes M(X), \mathbf{Z}(d)[2d])$$

which is an isomorphism by Lemma 6.8 and our assumption that $M(X)$ lies in $DM_{\mathcal{X}}$. □

Example 6.15 The unit object $\mathbf{Z}_{\mathcal{X}} = M(\mathcal{X})$ of $DM_{\mathcal{X}}$ is usually not restricted. Consider for example the case when $\mathcal{X} = \check{C}(Spec(E))$ where E is a Galois extension of k with Galois group G. Then

$$Hom(\mathbf{Z}[i], M(\mathcal{X})) = H_i(G, \mathbf{Z})$$

and this group may be non-zero for $i > 0$. If $M(\mathcal{X})$ were restricted, this group would be equal to

$$Hom(M(\mathcal{X})[i], M(\mathcal{X})) = Hom(M(\mathcal{X})[i], \mathbf{Z}) = H^{-i,0}(\mathcal{X}, \mathbf{Z})$$

which is zero for $i > 0$.

Lemma 6.16 *Let M, N be objects of $DM_{\mathcal{X}}$, P an object of $DM_{-}^{eff}(S)$ and*

$$e_{\mathcal{X}} : M \otimes N \to P \otimes M(\mathcal{X})$$

a morphism such that $(N, e_{\mathcal{X}})$ is an internal Hom-object from M to $P \otimes M(\mathcal{X})$ in $DM_{\mathcal{X}}$. Assume further that N is restricted. Then one has:

1. *$(N, e_{\mathcal{X}})$ is an internal Hom-object from M to $P \otimes M(\mathcal{X})$ in $DM_{-}^{eff}(S)$,*
2. *if e is the composition*

$$M \otimes N \to P \otimes M(\mathcal{X}) \to P$$

then (N, e) is an internal Hom-object from M to P in $DM_{-}^{eff}(S)$.

Proof: To prove the first statement, we have to show that the map

$$Hom(K, N) \to Hom(K \otimes M, P \otimes M(\mathcal{X})) \tag{6.5}$$

is a bijection for any K in $DM_{-}^{eff}(S)$. Since N is restricted and M is in $DM_{\mathcal{X}}$, this map is isomorphic to the map

$$Hom(K \otimes M(\mathcal{X}), N) \to Hom(K \otimes M \otimes M(\mathcal{X}), P \otimes M(\mathcal{X}))$$

which is a bijection since N is an internal Hom-object in $DM_{\mathcal{X}}$.

To prove the second part, we have to show that the composition of (6.5) with the map

$$Hom(K \otimes M, P \otimes M(\mathcal{X})) \to Hom(K \otimes M, P) \tag{6.6}$$

is a bijection. This follows from the first part and the fact that (6.6) is a bijection by Lemma 6.10. □

Lemma 6.17 *Let X be a smooth scheme over S. Then $M(X)$ lies in $DM_{\mathcal{X}}$ if and only if the canonical morphism $u : M(X) \to \mathbf{Z}$ factors through the canonical morphism $v : M(\mathcal{X}) \to \mathbf{Z}$.*

Proof: If $M(X)$ is in $DM_{\mathcal{X}}$ then (6.2) is an isomorphism, which immediately implies that u factors through v. On the other hand if $u = v \circ w$ where w is a morphism $M(X) \to M(\mathcal{X})$ then

$$(Id \otimes w) \circ \Delta : M(X) \to M(X) \otimes M(X) \to M(X) \otimes M(\mathcal{X})$$

is a section of the projection (6.2). Therefore, $M(X)$ is a direct summand of an object of $DM_{\mathcal{X}}$ and since $DM_{\mathcal{X}}$ is closed under direct summands, we conclude that $M(X)$ is in $DM_{\mathcal{X}}$. □

Example 6.18 If $\mathcal{X} = \check{C}(X)$ and Y is any smooth scheme such that

$$Hom(Y, X) \neq \emptyset$$

then Lemma 6.17 shows that $M(Y)$ lies in $DM_{\mathcal{X}}$. In particular $M(X)$ lies in $DM_{\mathcal{X}}$. More generally, for any $\mathcal{X}$ over a perfect field one can deduce from Lemma 6.17 that $M(Y)$ lies in $DM_{\mathcal{X}}$ if and only if for any point y of Y there exists a morphism $Spec(\mathcal{O}^h_{Y,y}) \to \mathbf{Z}_{tr}(\mathcal{X}_0)$ such that the composition

$$Spec(\mathcal{O}^h_{Y,y}) \to \mathbf{Z}_{tr}(\mathcal{X}_0) \to \mathbf{Z}$$

equals 1.

For an embedded $\mathcal{X}$, let

$$\check{\mathcal{X}} = \check{C}(\mathcal{X}_0)$$

where $\mathcal{X}_0$ is the zero term of $\mathcal{X}$.

Lemma 6.19 *There is an isomorphism $M(\mathcal{X}) \to M(\check{\mathcal{X}})$.*

Proof: Let us show that both projections

$$M(\mathcal{X}) \otimes M(\check{\mathcal{X}}) \to M(\mathcal{X})$$

and

$$M(\mathcal{X}) \otimes M(\check{\mathcal{X}}) \to M(\check{\mathcal{X}})$$

are isomorphisms. Since both $\mathcal{X}$ and $\check{\mathcal{X}}$ are embedded, it is sufficient by Lemma 6.8 to show that one has

$$M(\mathcal{X}) \in DM_{\check{\mathcal{X}}}$$

and

$$M(\check{\mathcal{X}}) \in DM_{\mathcal{X}}.$$

The terms of $\mathcal{X}$ are smooth schemes $\mathcal{X}_i$ and for each i we have

$$Hom(\mathcal{X}_i, \mathcal{X}_0) \neq \emptyset.$$

We conclude by Example 6.18 that $M(\mathcal{X}_i)$ are in $DM_{\check{\mathcal{X}}}$ and therefore $M(\mathcal{X})$ is in $DM_{\check{\mathcal{X}}}$. To see the second inclusion, it is sufficient to show that $M(\mathcal{X}_0)$ is in $DM(\mathcal{X})$. This follows from Lemma 6.17, since the morphism $\mathcal{X}_0 \to S$ factors through the morphism $\mathcal{X} \to S$ in the obvious way. □

Remark 6.20 Let $S = Spec(k)$ where k is a field. Recall from [6] that for X such that $X(k) \neq \emptyset$ the projection $\check{C}(X) \to Spec(k)$ is a local equivalence. Since a non-empty smooth scheme over a field always has a point over a finite separable extension of this field, we conclude from Lemma 6.19 that for any embedded $\mathcal{X}$ such that $M(\mathcal{X}) \neq 0$ there exists a finite separable field extension E/k such that the pull-back of $M(\mathcal{X}) \to \mathbf{Z}$ to E is an isomorphism.

Lemma 6.21 *Let $\mathcal{X}$ be an embedded simplicial scheme and X a smooth scheme over S. Assume that the following conditions hold:*

1. $M(X) \in DM_{\mathcal{X}}$,
2. *for any Y such that $M(Y) \in DM_{\mathcal{X}}$, there exists a Nisnevich covering $U \to X$ of X and a morphism $M(U) \to M(Y)$ such that the square*

$$\begin{array}{ccc} M(U) & \longrightarrow & M(Y) \\ \downarrow & & \downarrow \\ \mathbf{Z} & \xrightarrow{Id} & \mathbf{Z} \end{array}$$

   commutes.

Then $M(\check{C}(X)) \cong M(\mathcal{X})$.

Proof: We need to verify that the projections

$$M(\mathcal{X} \times \check{C}(X)) \to M(\mathcal{X})$$

and

$$M(\mathcal{X} \times \check{C}(X)) \to M(\check{C}(X))$$

are isomorphisms. The second one is an isomorphism by Lemma 6.8, since $M(X) \in DM_{\mathcal{X}}$ and therefore $M(\check{C}(X)) \in DM_{\mathcal{X}}$. To check that the first projection defines an isomorphism, it is sufficient by the same lemma to verify that $M(\mathcal{X})$ is in $DM_{\check{C}(X)}$. In view of Lemma 6.19 it is sufficient to check that $M(\mathcal{X}_0)$ is in $DM_{\check{C}(X)}$. Since $\mathcal{X}_0$ is a disjoint union of smooth varieties of finite type Y such that $M(Y)$ is in $DM_{\mathcal{X}}$, it remains to check that for such Y one has

$$M(Y) \in DM_{\check{C}(X)}. \tag{6.7}$$

One can easily see (cf. [6]) that for a Nisnevich covering $U \to Y$ the corresponding map $\check{C}(U) \to \check{C}(Y)$ is a local equivalence. Hence we may assume that $U = Y$, i.e. that there is a morphism $M(Y) \to M(X)$ over $\mathbf{Z}$. Then the morphism $M(Y) \to \mathbf{Z}$ factors throught $M(\check{C}(X))$ and we conclude by Lemma 6.17 that (6.7) holds. $\square$

Remark 6.22 One can show that (at least over a perfect field) the conditions of Lemma 6.21 are in fact equivalent to the condition that $M(\check{C}(X)) \cong M(\mathcal{X})$.

Example 6.23 In the notation of Example 6.15, consider the pair $(M(\mathcal{X}), e)$ where e is the canonical morphism

$$M(\mathcal{X}) \otimes M(\mathcal{X}) \to M(\mathcal{X}).$$

Since $M(\mathcal{X})$ is the unit of $DM_{\mathcal{X}}$, this pair is an internal Hom-object from $M(\mathcal{X})$ to itself in $DM_{\mathcal{X}}$. However it is not an internal Hom-object from $M(\mathcal{X})$ to itself in $DM_{-}^{eff}(k)$, since if it were we would have

$$Hom(M, M(\mathcal{X})) = Hom(M \otimes M(\mathcal{X}), M(\mathcal{X}))$$

for all M in $DM_{-}^{eff}(k)$ and we know that this equality does not hold for $M = \mathbf{Z}$.

Example 6.24 Since for the terms $\mathcal{X}_i$ of $\mathcal{X}$ we have

$$M(\mathcal{X}_i) \in DM_{\mathcal{X}},$$

the motive $M(\mathcal{X})$ lies in the localizing subcategory generated by motives of $\mathcal{X}_i$. If all $\mathcal{X}_i$ are smooth projective varieties, this implies by Lemma 6.14 that $M(\mathcal{X})$ lies in the localizing subcategory generated by restricted motives. Together with the previous example this shows that the category of restricted motives is not localizing. Indeed, one can easily see that it is closed under triangles and direct summands but not necessarily under infinite direct sums.

7. Coefficients

All the results of Sections 2-6 can be immediately reformulated in the R-linear context where R is any commutative ring with unit. Note that the notion of embedded simplicial scheme depends on the choice of coefficients.

If we consider motives with coefficients in R where R is of characteristic zero then the pull-back with respect to a finite separable field is a conservative functor (i.e. it reflects isomorphisms). Therefore Remark 6.20 implies that for $S = Spec(k)$ and motives with coefficients in a ring R of characteristic zero, one has

$$M(\mathcal{X}) \cong R$$

for any non-empty embedded simplicial scheme $\mathcal{X}$. This means that in the case of motives over a field, the theory of Section 6 is interesting only if we consider torsion phenomena.

8. Appendix: Internal Hom-objects

Recall that for two objects X, S in a tensor category, an *internal Hom-object* from X to S is a pair (X', e) where X' is an object and $e : X' \otimes X \to S$ a morphism such that for any Q the map

$$Hom(Q, X') \to Hom(Q \otimes X, S) \tag{8.1}$$

given by $f \mapsto e \circ (f \otimes Id_X)$ is a bijection.

If (X', e_X) is an internal Hom-object from X to S and (Y', e_Y) an internal Hom-object from Y to S and if we have a morphism $f : X \to Y$ then the composition

$$Y' \otimes X \to Y' \otimes Y \xrightarrow{e_Y} S$$

is the image under (8.1) of a unque morphism $Y' \to X'$ which we denote by $D_{S,e_X,e_Y} f$ or simply Df, if S, e_X and e_Y are clear from the context. One verifies easily that $D(gf) = DfDg$, if all the required morphisms are defined. The same is true with respect to the functoriality of internal Hom-objects in S.

The internal Hom-objects are unique up to a canonical isomorphism in the following sense.

Lemma 8.1 *Let (X', e'), (X'', e'') be internal Hom-objects from X to S. Then there is a unique isomorphism $\phi : X' \to X''$ such that $e' = e'' \circ (\phi \otimes Id_X)$.*

If (Y', e_Y) is an internal Hom-object from Y to S and $f : Y \to X$ a morphism then

$$(D'f : X' \to Y') = (X' \xrightarrow{\phi} X'' \xrightarrow{D''f} Y')$$

where D' is the dual with respect to e' and e_Y and D'' the dual with respect to e'' and e_Y. A similar property holds for morphisms $X \to Y$ and for morphisms in S.

A *specification* of internal Hom-objects for a tensor category $\mathcal{C}$ is a choice of one internal Hom-object for each pair (X,S) such that there exists an internal Hom-object from X to S. We will always assume below that a specification of internal Hom-objects is fixed. The distinguished internal Hom-object from X to S with respect to this specification will be denoted by $(\underline{Hom}(X,S), ev_{X,S})$.

The construction of $D_S f$ described above shows that for each S,

$$X \mapsto \underline{Hom}(X,S) \tag{8.2}$$

is a contravariant functor from the full subcategory of $\mathcal{C}$ consisting of X such that $\underline{Hom}(X,S)$ exists to $\mathcal{C}$. Lemma 8.1 shows that different choices of specifications of internal Hom-objects lead to isomorphic functors of the form (8.2). The same holds for functoriality in S.

Consider now the case of a tensor triangulated category $\mathcal{C}$ which satisfies the obvious axioms (TC1), (TC2a) connecting the tensor and the triangulated structure. We want to investigate how internal Hom-objects behave with respect to the shift functor and distinguished triangles.

Let X, S be a pair of objects of $\mathcal{C}$ and $(X', e : X' \otimes X \to S)$ an internal Hom-object from X to S. Consider the pair $(X'[-1], X'[-1] \otimes X[1] \to S)$ where the morphism is the composition

$$X'[-1] \otimes X[1] \to X'[-1][1] \otimes X \to X' \otimes X \to S. \tag{8.3}$$

One verifies easily that this pair is an internal Hom-object from $X[1]$ to S. Similar behavior exists with respect to shifts of S. Together with Lemma 8.1, this shows that for a given specification of internal Hom-objects there are canonical isomorphisms

$$\underline{Hom}(X[1],S) \to \underline{Hom}(X,S)[-1] \tag{8.4}$$

$$\underline{Hom}(X,S[1]) \to \underline{Hom}(X,S)[1]. \tag{8.5}$$

Remark 8.2 There is another possibility for the pairing (8.3), which one gets by moving $[-1]$ to X instead of $[1]$ to X'. It differs from (8.3) by sign and also makes $X'[-1]$ into an internal Hom-object from $X[1]$ to S. The isomorphisms (8.4), (8.5) constructed using different pairings (8.3) will differ by sign.

If $h : Z \to X[1]$ is a morphism and $\underline{Hom}(Z,S)$ and $\underline{Hom}(X,S)$ exist then the composition of Dh with (8.4) gives a morphism $\underline{Hom}(X,S)[-1] \to \underline{Hom}(Z,S)$ which we will also denote by Dh. This does not lead to any problems since it is always possible to choose a specification of internal Hom-object such that the morphisms (8.4) and (8.5) are identities.

Theorem 8.3 *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{8.6}$$

be a distinguished triangle in a tensor triangulated category satisfying axioms (TC1), (TC2a) and (TC3) of [2, p.47-49] and such that $\underline{Hom}(X,S)$ *and* $\underline{Hom}(Z,S)$ *exist. Then for any distinguished triangle of the form*

$$\underline{Hom}(Z,S) \xrightarrow{g'} Y' \xrightarrow{f'} \underline{Hom}(X,S) \xrightarrow{Dh[1]} \underline{Hom}(Z,S)[1]$$

there exists a morphism $e_Y : Y' \otimes Y \to S$ *such that* (Y', e_Y) *is an internal Hom-object from* Y *to* S *and one has* $g' = Dg$, $f' = Df$.

Proof: To simplify notation, set

$$X' = \underline{Hom}(X,S) \; e_X = ev_{X,S}$$

$$Z' = \underline{Hom}(Z,S) \; e_Z = ev_{Z,S}.$$

We want to find e_Y such that for any Q, the map

$$Hom(Q,Y') \to Hom(Q \otimes Y, S) \tag{8.7}$$

given by $f \mapsto e_Y \circ (f \otimes Id_Y)$ is a bijection and the induced maps Dg and Df coincide with g' and f', respectively. Consider the diagram

$$\begin{tikzcd} Hom(Q,Z') \arrow[r] \arrow[d] & Hom(Q,Y') \arrow[r] & Hom(Q,X') \arrow[d] \\ Hom(Q \otimes Z, S) \arrow[r] & Hom(Q \otimes Y, S) \arrow[r] & Hom(Q \otimes X, S). \end{tikzcd} \tag{8.8}$$

If we can find e_Y such that the corresponding map (8.7) subdivides this diagram into two commutative squares then this map will be a bijection by the Five Lemma. In addition setting $Q = Z'$ and using the commutativity of the left square on $Id_{Z'}$, we will get $g' = Dg$ and setting $Q = Y'$ and using the commutativity of the right square on $Id_{Y'}$, we will get $f' = Df$. It suffices therefore to find e_Y which for any Q splits (8.8) into two commutative squares.

A simple diagram chase shows that the commutativity of the left square is equivalent to the commutativity of the square

$$\begin{tikzcd} Z' \otimes Y \arrow[r] \arrow[d] & Y' \otimes Y \arrow[d, "e_Y"] \\ Z' \otimes Z \arrow[r, "e_Z"] & S \end{tikzcd}$$

and the commutativity of the right square to the commutativity of the square

$$\begin{tikzcd}
Y' \otimes X \arrow[r] \arrow[d] & Y' \otimes Y \arrow[d, "e_Y"] \\
X' \otimes X \arrow[r, "e_X"] & S.
\end{tikzcd}$$

Together, we may express our condition as the commutativity of the square

$$\begin{tikzcd}
(Y' \otimes X) \oplus (Z' \otimes Y) \arrow[r] \arrow[d] & Y' \otimes Y \arrow[d, "e_Y"] \\
(X' \otimes X) \oplus (Z' \otimes Z) \arrow[r, "e_X + e_Z"] & S.
\end{tikzcd}$$

Applying Axiom TC3' ([2]) to our triangles, we see that there is an object W which fits into a commutative diagram

$$\begin{tikzcd}
(Y' \otimes X) \oplus (Z' \otimes Y) \arrow[r] \arrow[d] & Y' \otimes Y \arrow[d, "k_2"] \\
(X' \otimes X) \oplus (Z' \otimes Z) \arrow[r, "k_3 + k_1"] & W.
\end{tikzcd}$$

It remains to show that $e_X + e_Z$ factors through $k_3 + k_1$. By [2, Lemma 4.9] the lower side of this square extends to an exact triangle of the form

$$(X' \otimes Z)[-1] \to (X' \otimes X) \oplus (Z' \otimes Z) \stackrel{k_3+k_1}{\to} W \to X' \otimes Z.$$

Therefore it is sufficient to show that the diagram

$$\begin{tikzcd}
(X' \otimes Z)[-1] \arrow[r] \arrow[d] & X' \otimes X` \arrow[d, "e_X"] \\
Z' \otimes Z \arrow[r, "e_Z"] & S
\end{tikzcd}$$

anticommutes. A diagram of this form can be defined for any morphism of the form $X \to Z[1]$ and its anticommutativity follows easily from the elementary axioms. $\square$

Remark 8.4 Applying Theorem 8.3 to the category opposite to $\mathcal{C}$, one concludes that a similar result holds for distinguished triangles with respect to the second argument of $\underline{Hom}$.

Theorem 8.3 together with the preceeding discussion of internal Hom-objects and the shift functor, implies in particular that for a given S (resp. given X) the subcategory $\mathcal{C}_{(-,S)}$ (resp. $\mathcal{C}_{(X,-)}$), consisting of all X (resp. all S) such that $\underline{Hom}(X,S)$ exists, is a triangulated subcategory.

Proposition 8.5 *The functors*

$$\underline{Hom}(-,S) : \mathcal{C}_{(-,S)} \to \mathcal{C}$$

$$\underline{Hom}(X,-) : \mathcal{C}_{(X,-)} \to \mathcal{C}$$

considered together with the canonical isomorphisms (8.4), (8.5) are triangulated functors.

Proof: It clearly suffices to prove the part of the proposition related to $\underline{Hom}(-,S)$, i.e. to show that this functor takes distinguished triangles to distinguished triangles. Consider a distinguished triangle of the form (8.6) and the resulting triangle

$$\underline{Hom}(Z,S) \overset{Dg}{\to} \underline{Hom}(Y,S) \overset{Df}{\to} \underline{Hom}(X,S) \overset{Dh[1]}{\to} \underline{Hom}(Z,S)[1]. \qquad (8.9)$$

In view of Theorem 8.3, there exists an internal Hom-object $(\tilde{Y}', \tilde{e}_Y)$ from Y to S such that the triangle formed by $\tilde{D}g$, $\tilde{D}f$ and $Dh[1]$ is distinguished. By Lemma 8.1, there is an isomorphism $\tilde{Y}' \to \underline{Hom}(Y,S)$ which extends to an isomorphism of triangles. We conclude that (8.9) is isomorphic to a distinguished triangle and therefore is distinguished. □

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