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∞-Groupoids as a model for a homotopy category

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It is known [4] that CW-complexes X such that $\pi_i(X) = 0$ for $i \geqslant 2$ can be described by groupoids from the homotopy point of view. In the unpublished paper "Pursuing stacks" Grothendieck proposed the idea of a multi-dimensional generalization of this connection that used polycategories. The present note is devoted to a realization of this idea.

1. A spherical ∞-category C consists (see [1]–[3]) of a collection of sets C_i, $i \in \mathbb{Z}_+$, maps $s_i, t_i: C_k \to C_i$, $\mathbb{1} = \mathbb{1}_k: C_i \to C_k$ defined for $i \leqslant k$, and partial composition operations $(a, b) \mapsto a \underset{i}{\circ} b$ on C_k defined for $i \leqslant k-1$ in the case when $s_i(a) = t_i(b)$. A list of axioms for these data is given in [1] (see also [2]–[3]), where D_i^0, D_i^1, and E_k are used instead of our notation s_i, t_i, and $\mathbb{1}_k$. It follows from these axioms, in particular, that for $i \leqslant k-1$ the operation $\underset{i}{\circ}$ endows C_k with the structure of a category with the set C_i of objects. If $C_{n+i} = \mathbb{1}(C_n)$ for $i \geqslant 0$, then C is called an *n-category*. In particular, a 1-category is the same as an ordinary category. All ∞-categories form the (1-) category Cat_∞. For an ∞-category C the elements of C_i are called *i-morphisms* of C. The 0-morphisms are called *objects*.

2. An ∞-category C is called an *∞-groupoid* if the following conditions (GR'_{ik}), (GR''_{ik}) hold for all $i < k$:

$(GR'_{ik}$, $i < k-1)$. For every $a \in C_{i+1}$, $b \in C_k$, and $v, u \in C_{k-1}$ with $s_i(a) = t_i(u) = t_i(v)$, $a \underset{i}{\circ} u = s_{k-1}(b)$, and $a \underset{i}{\circ} v = t_{k-1}(b)$ there exist an $x \in C_k$ and a $\varphi \in C_{k+1}$ such that $s_k(\varphi) = a \underset{i}{\circ} x$, $t_k(\varphi) = b$, $s_{k-1}(x) = u$, and $t_{k-1}(x) = v$.

$(GR'_{k-1,k})$. For every $a, b \in C_k$ with $t_{k-1}(a) = t_{k-1}(b)$ there exist an $x \in C_k$ and a $\varphi \in C_{k+1}$ such that $s_k(\varphi) = a \underset{k-1}{\circ} x$ and $t_k(\varphi) = b$.

$(GR''_{ik}$, $i < k-1)$. For every $a \in C_{i+1}$, $b \in C_k$, and $v, u \in C_{k-1}$ with $t_i(a) = s_i(u) = s_i(v)$, $u \underset{i}{\circ} a = s_{k-1}(b)$, and $v \underset{i}{\circ} a = t_{k-1}(b)$ there exixst an $x \in C_k$ and a $\varphi \in C_{k+1}$ such that $s_k(\varphi) = x \underset{i}{\circ} a$, $t_k(\varphi) = b$, $s_{k-1}(z) = u$, and $t_{k-1}(x) = v$.

$(GR''_{k-1,k})$. For every $a, b \in C_k$ with $s_{k-1}(a) = s_{k-1}(b)$ there exist an $x \in C_k$ and a $\varphi \in C_{k+1}$ such that $s_k(\varphi) = x \underset{k-1}{\circ} a$ and $t_k(\varphi) = b$.

In an informal sense, the conditions amount to weak (to within a "homotopy" φ) solubility of all equations of the form $a \underset{i}{\circ} x = b$ and $x \underset{i}{\circ} a = b$ in the cases when such equations make sense. We define an *n-groupoid* to be an n-category that is an ∞-groupoid. Let $\mathrm{Gr}_n \subset \mathrm{Gr}_\infty \subset \mathrm{Cat}_\infty$ be the full subcategories of n-groupoids and ∞-groupoids.

3. Let $G \in \mathrm{Gr}_\infty$, and let $x \in G_0$ be an object. For $i > 0$ we denote by $\pi_i(G, x)$ the quotient set of $\{z \in G_i: s_{i-1}(z) = t_{i-1}(z) = \mathbb{1}_{i-1}(x)\}$ with respect to the following equivalence relation: $z \sim w$ if there is a $y \in G_{i+1}$ such that $s_i(y) = z$ and $t_i(y) = w$. Also, let $\pi_0(G)$ be the quotient of G_0 with respect to the following equivalence relation: $x \sim x'$ if there is a $y \in G_1$ such that $s_0(y) = x$ and $t_0(y) = x'$.

***Proposition* 1.** *For $i \geqslant 1$ the operation $\underset{i-1}{\circ}$ endows $\pi_i(G, x)$ with the structure of a group that is commutative for $i \geqslant 2$.*

We denote by W (respectively, W_n) the class of morphisms $f: G \to G'$ of the category Gr_∞ (respectively, Gr_n) that induce bijections $\pi_0(G) \to \pi_0(G')$ and $\pi_i(G, x) \to \pi_i(G', f(x))$ for all $x \in G_0$ and $i > 0$. Let $\mathrm{Gr}_\infty[W^{-1}]$ be the category of fractions [4]. Also, let Hot denote the homotopy category of CW-complexes, and $\mathrm{Hot}_{\leqslant n} \subset \mathrm{Hot}$ the full subcategory of complexes X such that $\pi_i(X, x) = 0$ for all $i > n$ and $x \in X$.

***Theorem* 2.** *The following equivalences of categories are valid*:

$$\mathrm{Gr}_\infty\,[W^{-1}] \simeq \mathrm{Hot},\ \ \mathrm{Gr}_n\,[W_n^{-1}] \simeq \mathrm{Hot}_{\leqslant n}.$$

4. In one direction the equivalence in Theorem 2 is supplied by the nerve functor for ∞-categories in [2]. This functor associates with an ∞-category C the simplicial set $\mathrm{Nerv}(C)$, whose p-simplexes are the "weakly commutative p-dimensional simplexes" in C. The following facts are proved in the proof of Theorem 2.

***Theorem* 3.** *Every CW-complex is homotopically equivalent to the nerve of some ∞-groupoid that is unique to within an isomorphism in the category $\mathrm{Gr}_\infty[W^{-1}]$. Every CW-complex X such that $\pi_i(X, x) = 0$ for all $i > n$ and $x \in X$ is homotopically equivalent to the nerve of some n-groupoid that is unique to within an isomorphism in the category $\mathrm{Gr}_n[W_n^{-1}]$.*

***Theorem* 4.** a) *For every ∞-groupoid G, its nerve is a complete simplicial set in the Kan sense (see* [4]). *In particular, for all $x \in G_0$ there is a natural isomorphism $\pi_i(G, x) \simeq \pi_i(|\mathrm{Nerv}(G)|, x)$, where the usual homotopy groups are on the right-hand side, and $|\cdot|$ denotes the geometric realization of a simplicial set.*

b) *Conversely, if the nerve of an ∞-category C is a complete simplicial set, then C is an ∞-groupoid.*

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