From motives to motivic homotopy types.

1. 60-ies and 70-ies : study of individual cohomology theories and particular connections between them. Formulation of "standard conjectures".

   First (mostly unsuccesful) attempt of systematization - Grothendieck "motives"

2. ~~late~~ mid 80 ies : the idea of motivic cohomology; reformulation of the standard conjectures in new terms;

   Second attempt of systematization - Beilinson's and Deligne's "mixed motives" and "motivic sheaves". Conjectural.

3. late 80ies - mid 90ies: construction of motivic cohomology; establishment of their "easy" properties.
Third attempt of systematization - - triangulated categories of motives. Sucsesful but of limited use.

4. mid 90 ies - <u>now</u> : construction of stable and partly unstable motivic homotopy theories.
Stable homotopy groups, algebraic cobordism, Steenrod operations, duality. Very good progress on all ~~conjectures~~ standard conjectures related to <u>finite</u> coefficients.

Cohomology theories $Alg/k \to Ab$ defined and studied in 60-70ies

1. l-adic (etale) cohomology
2. Quillen - Thomason K-theory
3. Etale K-theory (later)
4. Chow groups (only for smooth varieties)
5. Milnor's K-theory (only for regular local rings)
6. The Witt ring of quadratic forms (same restriction as for the Milnor K-theory)

These theories are connected to each other by a collection of natural transformations which looks as follows:

$$\begin{tikzcd}
\text{Milnor's K-theory} \arrow[r, "\ell=2"] \arrow[d] \arrow[drr, "\ell>2"] & \text{Witt ring (graded)} \arrow[dr, "\ell=2"] & \\
\text{Quillen-Thomason K-theory} \arrow[r] \arrow[dr, dashed, "\text{only on } K_0"'] & \text{Etale K-theory} \arrow[r] & \ell\text{-adic cohomology} \\
& \text{Chow groups} \arrow[ur] &
\end{tikzcd}$$

Study of concrete examples led to a number of conjectrues about the properties of these transformations

They are called the <u>standard conjectures</u> and can be divided into 3 blocks:

1. The Grothendieck standard conjectures

2. Vanishing and rigidity of Beilinson and Soule

3. Finite coefficients conjectures of Milnor, Quillen Lichtenbaum, Bloch, Kato.

Grothendieck's standard conjectures (late 60ies)

These conjectures are about the natural transformation from

$$\boxed{\begin{array}{c}\text{Chow groups}\\ \downarrow \\ l\text{-adic cohomology}\end{array}}$$

to for smooth projective varieties

G1 The "hard" L. conjecture

G2 Homological equivalence =
= numerical equivalence.

No progress made.

Vanishing and rigidity.

(mid 80ies)

These conjectures are about the natural transformation from to

Quillen-Thomason K-theory
↓
l-adic cohomology.

for ~~all~~ ~~smooth~~ smooth varieties. The theories are considered with rational coefficients.

No progress made

Finite coefficients conjectures

(70ies)

1. The Milnor conjecture - about the natural transformations

Milnor's K-theory	$\longrightarrow$	The Witt ring (graded)	$\longrightarrow$	2-adic cohomology

2. The Bloch - Kato conjecture - about the natural transformation

Milnor's K-theory	$\longrightarrow$	ℓ-adic cohomology

all ℓ $\neq$ char (k)

3. The Quillen - Lichtenbaum conjecture - - about the natural transformation

Quillen-Thomason K-theory	$\longrightarrow$	Etale K - theory

Motivic cohomology of Beilinson and Lichtenbaum

In late 80-ies B. and L. suggested that there should exist a new theory $H^{p,q}_{\mathcal{M}}$ which is bigraded and gave explicite sets of conditions which it should satisfy.

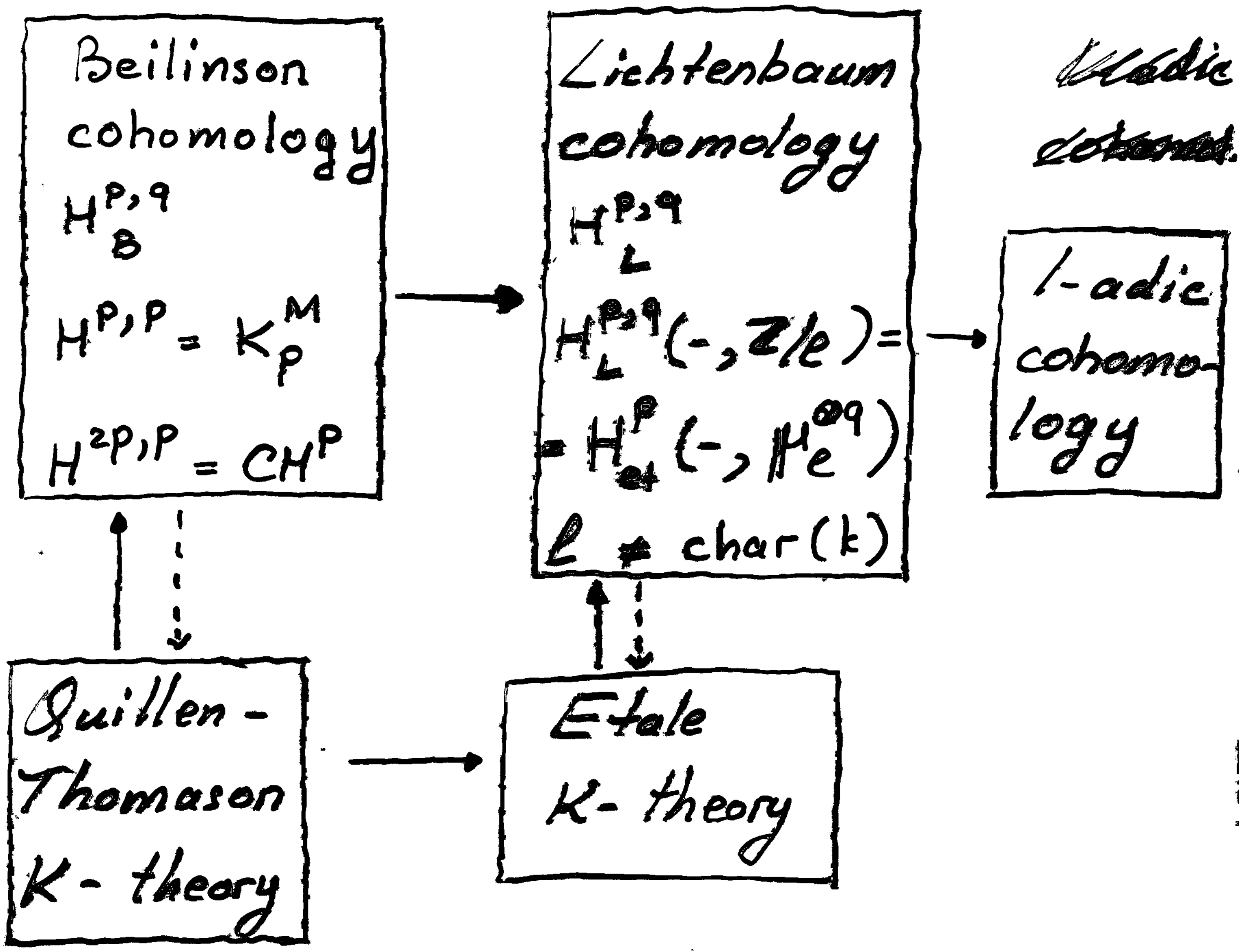
As we know now it gives two theories called Beilinson's and Lichtenbaum's motivic cohomology which are connected by a natural transformation

$$\begin{matrix} \text{Beilinson} & & \text{Lichtenbaum} \\ \text{cohomology} & \longrightarrow & \text{cohomology} \\ H^{p,q}_{B} & & H^{p,q}_{L} \end{matrix}$$

The first succesful definition of motivic cohomology was given by S. Bloch. Later other definitions were given by A. Suslin and V. Voevodsky and E. M. Friedlander.

Today we know that for varieties over any field all definitions of motivic cohomology agree.

The relations between motivic cohomology and the standard theories.



Reformulations of the standard conjectures.

Instead of 3 blocks we have 3 conjectures:

1. Nilpotence (V. Voevodsky)

Let X be smooth projective of dimension d and $x \in H^{2i,i}(X,\mathbb{Q})$ an element such that $(x,y)=0$ for all y where $(-,-)$ is the intersection pairing

$$H^{2i,i} \otimes H^{2d-2i,\,d-i} \to H^{0,0}$$

Then $x^{\boxtimes d} = 0$ in $H^{2id,\,id}(X^d,\mathbb{Q})$.

2. Vanishing (A. Beilinson)

Let C be a geometrically connected proj. sm. curve over a field K. Then

$$H^{i,j}(\operatorname{Spec} K,\mathbb{Q}) \to H^{i,j}(C,\mathbb{Q})$$

is an iso. for $i \leq 1$.

Standard conjectures in terms of motivic cohomology

1. The Grothendieck standard conjectures follow from the "nilpotence conjecture"

2. Vanishing and rigidity can be naturally rewritten for motivic cohomology

3. All the finite coefficients conjectures follow from the Beilinson - Lichtenbaum conjecture:

~~[illegible]~~

Beilinson - Lichtenbaum conjecture:

$H_B^{i,j} \to H_L^{i,j}$ is an isomorphism for $i \le j$.

3.1. Trivial for rational coefficients.

3.2. Proved for $\mathbb{Z}_{(2)}$ - coefficients

3.3. Proved (?) for $\mathbb{Z}_{(p)}$ - coef. $p = \mathrm{char}(k)$

3.4. In the works for $\mathbb{Z}_{(\ell)}$ $\ell > 2$, $\ell \neq \mathrm{char}(k)$.

Sch/S the category of schemes of finite type over a Noetherian scheme S.

Ex: $S = \operatorname{Spec} \mathbb{Z}$

For each $X \in Sch/S$ we have a category $SHot(X)$ - the stable homotopy category of schemes over X.

This category has two autoequivalences (commuting) denoted by: Σ_s and Σ_t and a distinguished object $\mathbb{1}_X$.

For each morphism $f: X \to Y$ there are four functors

$$f^*: SHot(Y) \to SHot(X)$$
$$f_*: SHot(X) \to SHot(Y)$$
$$f^!: SHot(Y) \to SHot(X)$$
$$f_!: SHot(X) \to SHot(Y)$$

Every object E of $SHot(S)$ definies 4 " theories "

$$E^{p,q}(X) = Hom(\mathbb{1}_S, \Sigma_t^q \Sigma_s^{p-q} f_* f^* E)$$
$$E_c^{p,q}(X) = Hom(\mathbb{1}_S, \Sigma_t^q \Sigma_s^{p-q} f_! f^* E)$$
$$E_{p,q}(X) = Hom(\mathbb{1}_S, \Sigma_t^q \Sigma_s^{p-q} f_! f^! E)$$
$$E_{p,q}^{BM}(X) = Hom(\mathbb{1}_S, \Sigma_t^q \Sigma_s^{p-q} f_* f^! E)$$

where $f: X \to S$ the canonical m.

The functors $f^*, f_*, f^!, f_!$ are connected by a number of natural transformation and form a "2-theory"

For each of the theories called "standard" there is $E \in \mathrm{SHot}(\mathrm{Spec}\,\mathbb{Z})$ such that this theory is a part of $E^{p,q}(-)$.