FREE n-CATEGORY GENERATED BY A CUBE, ORIENTED MATROIDS, AND HIGHER BRUHAT ORDERS

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In investigating multidimensional generalizations of the Yang–Baxter equation, in [1-3] Manin and Shekhtman introduced partially ordered sets B(n, k) called higher Bruhat orders. The set B(n, 1) is the symmetric group S_n with its weak Bruhat order, and B(n, k + 1) is obtained by a factorization from the set of maximal chains in B(n, k). In [1-3] the connection was pointed out for B(n, k) with arrangements of n hyperplanes in $\mathbf{R}^k$ in the general situation and also with the structure of the convex hull of a general orbit of S_n in $\mathbf{R}^n$. In the present note we give an interpretation of B(n, k) as the sets of some k-dimensional strips in an n-dimensional cube. This permits us to clarify the above-mentioned connections, in particular to prove a hypothesis from [1, 3] and disprove another hypothesis from [1, 2]. Our approach is based on a consideration of n-category Q_n "freely generated" by the faces of an n-dimensional cube. It provides a natural "unlooping" of the (n − 1)-category S_n, introduced in [3] and generating higher Bruhat orders.

1. We introduce the definition of sets B(n, k) from [1-3]. Let C(n, k) be the set of all k-element subsets of {1,...,n}. For each $\sigma = \{\sigma_1 < \ldots < \sigma_{k+1}\} \in C(n, k+1)$ we have $\partial_i(\sigma) = \sigma - \{\sigma_i\} \in C(n, k)$. We shall call a complete order ρ on C(n, k) "admissible" if for each $\sigma \in C(n, k+1)$ either $\partial_1(\sigma)\rho\partial_2(\sigma)\rho\ldots\rho\partial_{k+1}(\sigma)$, or $\partial_{k+1}(\sigma)\rho\ldots\rho\partial_1(\sigma)$. Let A(n, k) be the set of all admissible orders of C(n, k). Two orders $\rho, \rho' \in A(n, k)$ are called adjacent if ρ' is obtained from ρ by a permutation of two neighboring elements $a, b \in C(n, k)$ such that $|a \cap b| < k - 1$. Let ~ be the equivalence relation generated by adjacency. A higher Bruhat order is the set $B(n, k) := A(n, k)/\sim$. For the construction of a partial order on B(n, k) see [1-3].

2. Let $I^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n\colon\ 0 \leqslant x_i \leqslant 1\}$ be an n-dimensional cube, and α = (0,...,0), ω = (1,...,1) its antipodal vertices. Let numbers $t_1 \ll t_2 \ll \ldots \ll t_n \in \mathbf{R}$ be given. We define a linear map $p_k\colon \mathbf{R}^n \to \mathbf{R}^k$, taking the i-th basis vector to $(t_i, t_i^2, \ldots, t_i^k)$. We define a polyhedron $Z(n, k) = p_k(I^n)$.

THEOREM 1. B(n, k) is identified with the set of subcomplexes (closed subsets that are unions of faces) $\Sigma \subset I^n$ such that $p_k\colon \Sigma \to Z(n, k)$ is a bijection.

For example, elements of B(n, 1) = S_n correspond to the n! paths in I^n from α to ω, going along the edges.

For $\Sigma \in B(n, k)$, the images of faces Σ are some cubilage of polyhedron Z(n, k). Considering the block-decomposition of Z(n, k) dual to this cubilage, we obtain an arrangement of multifaceted hypersurfaces in Z(n, k). Elements of this arrangement intersect such that if they were hyperplanes in the general situation, then one could construct an oriented matroid by it [8]. We give the exact result for the case k = 2, using the equivalent language of arrangements of pseudospheres instead of oriented matroids [8].

We shall define arrangements of pseudoneighborhoods in sphere S^2 as the collection of connected closed piecewise-smooth centrally symmetric curves $\xi = \{C_i, i \in I\}$, $C_i \subset S^2$, such that each $C_i \cap C_j$ consists of two centrally symmetric points. Arrangements will be called general if for distinct i, j, k, we have $C_i \cap C_j \cap C_k = \varnothing$. A cell of an arrangement shall be defined as the cell originating from a block decomposition of S^2. An isomorphism of arrangements $\xi = \{C_i, i \in I\}$ and $\xi' = \{C'_j, j \in J\}$ is the homeomorphism f: $S^2 \to S^2$ commuting with the central symmetry and taking each C_i to some C'_j and defining the bijection I → J.

THEOREM 2. B(n, 2) is in a bijection with the set of isomorphism classes of pairs (ξ, F), where ξ is the general arrangement of n + 1 pseudoneighborhoods in S^2 and $F = (F_0 \subset F_1 \subset F_2)$ is the total flag of cells of ξ.

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Ringel [9] gave an example of a general arrangement of nine pseudoneighborhoods in S^2 not realized by large circles. Thus we obtain a contradiction to the hypotheses of [1, 2]: the corresponding element of B(8, 2) cannot be realized by a general arrangement of lines in $\mathbf{R}^2$.

3. The proof of Theorem 1 and a description in geometric terms of the order on B(n, k) are based on the technique of polycategories. A (spherical) n-category C consists (see [4, Sec. 2; 3, Sec. 3]) of a collection of sets $C_i = \mathrm{Mor}_i C$ $(i = 0, 1, \ldots, n)$, mappings

$$D_i^p\colon C_k \to C_i,\ E_k\colon C_i \to C_k,\ 0 \leqslant i \leqslant k \leqslant n,\ p \in \{0, 1\},$$

and partial operations of composition $(x, y) \to x \underset{i}{\circ} y$ on C_k, defined for $0 \leqslant i \leqslant k-1$ in the case when $D_i^0(x) = D_i^1(y)$. These data must satisfy the axioms given in [4, Secs. 2.1-2.4], from which in particular it follows that $\underset{i}{\circ}$ provides C_k with the structure of a category with set of objects C_i. Elements of C_k are called k-morphisms (polymorphisms for unspecified k), 0-morphisms are called objects; the 1-category is the usual category (see also [5]). Intuitively, a k-morphism of x can be represented as an oriented k-dmensional strip (an arrow for k = 1), "joining" the (k − 1)-dimensional strips $D_{k-1}^0(x)$ and $D_{k-1}^1(x)$ having a common "boundary."

For any objects x and y of n-category C, an (n − 1)-category $\mathrm{Hom}_C(x, y)$ is defined, the objects of which are 1-morphisms $f \in C_1$ such that $D_0^0(f) = x$, $D_0^1(f) = y$ (see [5]).

4. Let $\mathcal{F}_k$ be the set of k-dimensional faces of n-dimensional cube I^n. Faces of I^n have the form $F(Y, Z) = \{x \in I^n\colon x_i = 0,\ \forall i \in Y,\ x_j = 1,\ \forall j \in Z\}$ for $Y, Z \subset \{1, \ldots, n\}$, $Y \cap Z = \varnothing$. Let $k = \dim F(Y, Z) = n - |Y| - |Z|$ and $a_1 < \ldots < a_k$ be all the elements from $\{1, \ldots, n\} - Y - Z$. We define cubic operators of faces [6] by formulas $\partial_i^0 F(Y, Z) = F(Y \cup \{a_i\}, Z)$, $\partial_i^1 F(Y, Z) = F(Y, Z \cup \{a_i\})$. For each $j \in \mathbf{Z}$ let $\bar{j} \in \{0, 1\}$ be such that $\bar{j} \equiv j \pmod 2$.

For each $\gamma \in \mathcal{F}_k$ and $i \leqslant k$, $p \in \{0, 1\}$, we define a subset $\mathcal{D}_i^p(\gamma) \subset \mathcal{F}_i$, consisting of faces of the type

$$(\overline{\partial_{i+1}^{p+i}})^{k-C_i} (\overline{\partial_i^{p+i-1}})^{C_i - C_{i-1} - 1} (\overline{\partial_{i-1}^{p+i-2}})^{C_{i-1} - C_{i-2} - 1} \cdot \ldots \cdot (\partial_1^p)^{C_1 - 1}, \qquad (\gamma)$$

for all sequences $1 \leqslant C_1 < \ldots < C_i \leqslant k$. In particular, $\mathcal{D}_{k-1}^p(\gamma)$ consists of $\partial^q(\gamma)$ such that $j + q + p \equiv 1 \pmod 2$. In other words, $\mathcal{D}_{k-1}^0(\gamma)$ ($\mathcal{S}_{k-1}^1(\gamma)$) consists of faces $\delta \subset \gamma$ of codimension 1, whose standard orientation agrees (does not agree) with the standard orientation of γ.

5. We define an n-category Q_n whose k-morphisms are the "correct" subcomplexes in I^n, and compositions are defined by union. The analogous category for a simplex (instead of a cube) was introduced by Street in [5]. We shall call a set $A \subset \mathcal{F}_m$ admissible if for γ, $\delta \in A$, $\gamma \neq \delta$ we have $\mathcal{S}_{m-1}^p(\gamma) \cap \mathcal{S}_{m-1}^p(\delta) = \varnothing$, $p \in \{0, 1\}$. Our n-category Q_n has for k-morphisms the sets $\{A_i^p\}$ $(p \in \{0, 1\},\ i \in \{0, 1, \ldots, k\})$, where $A_i^p \subset \mathcal{F}_i$ are admissible subsets satisfying the conditions

$$A_i^p = \Big(\bigcup_{\gamma \in A_{i+1}^q} \mathcal{S}_i^p(\gamma) - \bigcup_{\gamma \in A_{i+1}^q} \mathcal{D}_i^{1-p}(\gamma)\Big) \cup (A_i^0 \cap A_i^1) \quad \forall i < k,\ p \in \{0, 1\},$$

and $A_k^0 = A_k^1$.

Let $D_j^q(\{A_i^p\}) = \{B_i^p\}$, where $B_i^p = A_i^p$ for $i < j$, $B^0 = B^1 = A^q$. For $\ell < k$ and $\{A_i^p\} \in \mathrm{Mor}_l Q_n$, we set $E_k(\{A_i^p\}) = \{B_i^p\} \in \mathrm{Mor}_k Q_n$, where $B_i^p = \varnothing$ for $i > \ell$, and $B_i^p = A_i^p$ for $i \leq \ell$. If $D^0(\{A_i^p\}) = D_j^1(\{B_i^p\})$, then we set $\{A_i^p\} \underset{j}{\circ} \{B_i^p\} = \{C_i^p\}$, where $C_i^p = A_i^p \cup B_i^p$ for $i \neq j$, $C_j^0 = A^0$, $C^1 = B^1$.

Proposition 1. The given D_j^q, E_k, and $\underset{j}{\circ}$ define on Q_n the structure of an n-category.

To each face $\gamma \in \mathcal{F}_k$ there corresponds a set $[\gamma] = \{\mathcal{S}_i^p(\gamma)\} \in \mathrm{Mor}_k Q_n$. Each polymorphism of Q_n is the product of polymorphisms of type $[\gamma]$. Moreover, Q_n is freely generated by such polymorphisms in the sense of [5].

THEOREM 3. The set B(n, k) is in a bijection with the set of k-morphisms of x of n-category Q_n such that $D_{k-1}^p(x) = D_{k-1}^p([I^n])$ $(p \in \{0, 1\})$. A partial order relation on B(n, k) is induced by relation $\mathrm{Hom}(x, y) := \{z \in \mathrm{Mor}_{k+1} Q_n\colon D_k^0(z) = x,\ D_k^1(z) = y\} \neq \varnothing$.

With the help of reconstructions of elements of the higher Bruhat orders an (n − 1)-category S_n with set of objects S_n was introduced in [3].

THEOREM 4. An isomorphism of (n − 1)-categories $S_n \simeq \mathrm{Hom}_{Q_n}(\alpha, \omega)$ exists, where $\alpha = (0,\ldots,0)$, $\omega = (1,\ldots,1)$ are considered objects of Q_n.

6. Let $P_n \subset \mathbf{R}^n$ be an (n − 1)-dimensional permutohedron, i.e., the convex hull of the orbit of a point $(x_1 < \ldots < x_n)$ with respect to S_n (see [7, 10]). From the description of the faces of P_n (see [7]) it is easy to derive

Proposition 2. The r-dimensional faces of P_n are in a bijection with polymorphisms on Q_n of type $[\gamma_1] \underset{0}{\circ} \ldots \underset{0}{\circ} [\gamma_s]$, where γ_j are faces of the cube, $D_0^0(\gamma_1) = \{\alpha\}$, $D_0^1(\gamma_s) = \{\omega\}$, $\Sigma(\dim \gamma_j - 1) = r$. A face corresponding to such a polymorphism is isomorphic to $P_{\dim \gamma_1} \times \ldots \times P_{\dim \gamma_s}$.

COROLLARY. There exists a natural identification of the following three sets:

a) uncompressed chains of length 1 in B(n, k − 1);

b) irreducible k-morphisms of (n − 1)-category $S_n = \mathrm{Hom}_{Q_n}(\alpha, \omega)$;

c) k-dimensional faces of P_n isomorphic to P_{k+1}.

This assertion was stated in [1, 3] as a hypothesis. For k = 2 the bijection a) → c) is a known interpretation of the weak Bruhat order on S_n (see [10]).

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