Notes on framed correspondences

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1 Motivation

The simplest way to motivated the necessity of the motivic stable homotopy category is to refer to the fact that algebraic K-theory does not extend to the the triangulated category of motives. In particular the triangulated category of motives does not provide a sufficient framework to study such a fundamental object as the motivic spectral sequence. The other motivation comes from the cohomological operations but for historic reasons it seems to be less useful.

In the theory of ordinary motivic cohomology there are two "conflicting" groups of results. In the first group are the "computation" results comparing motivic cohomology to other theories. As an example one may state that $H^{n,n} = K_n^M$ for a field or that $H^{p,q} = 0$ for $q < 0$. In the second group there are "good behavior" results such as the Mayer-Vietoris property or the projective bundle theorem. There are constructions of the motivic category (or of the motivic cohomology) which make the results of either of the two groups easy to prove but in each case the results of the opposite group require a lot of technical work.

At the moment we have only one approach to the stable homotopy category which is an analog of the "good behavior" approach to the ordinary cohomology. As a result very little is known about the simplest cases of motivic stable homotopy groups and other related groups. As an outstanding exmple consider the fact that while there is (at least over fields of characteristics zero) the motivic Adams spectral sequence we can not prove that it converges even for S^0.

I hope that the constructions described in this notes will lead to a new model of the stable homotopy theory which will be more friendly for computations questions. Of course one expects that it will be non-trivial to show that the new and the old models agree.

2 Framed correspondences

Fix a noetherian scheme S and let Sch/S be the category of separated schemes of finite type over S. We define a rational function on a scheme X as an equivalence class of invertible functions on dense open subsets of X identifying a function with its restriction to a smaller subset. Rational functions are contravariantly functorial for dominant morphisms and form a sheaf $\mathcal{M}^*$ on the small etale site X_{et} of X. For any rational function f there exists a maximal open subset U_f where f is an invertible function. We let $Supp(f)$ denote the closed complement to U_f.

Let Z be a closed subset in X. A *framing* of Z of level n is a collection $\phi_1, \ldots, \phi_n$ of rational functions on X such that $\cap_{i=1}^{n} Supp(\phi_i) = Z$. For a scheme X over S a framing of Z over S is a framing of Z such that the closed subsets $Supp(\phi_i)$ do not contain the generic points of the fibers of $X \to S$.

For schemes X, Y over S and $n \geq 0$ an *explicit* framed correspondence of level n is the following collection of data:

1. a closed subset Z in $\mathbf{A}^n_X$ which is finite over X
2. an etale neighborhood $p : U \to \mathbf{A}^n_X$ of Z
3. a framing $\phi_1, \ldots, \phi_n$ of level n of Z in U over X
4. a morphism $g : U \to Y$

The subset Z is called the support of the correspondence. Note that the existence of framing of level n implies that Z is equidimensional over X of relative dimension zero. Two explict framed correspondences Φ and Φ' of level n are called equivalent if they have the same support and there exists an open neighborhood V of Z in $U \times_{\mathbf{A}^n_X} U'$ such that on V, $g \circ pr$ agrees with $g' \circ pr'$ and $\phi \circ pr$ agree with $\phi' \circ pr'$. A *framed correspondence of level* n is an equivalence class of explicit framed correspondences of level n.

We let $F_n^{gl}(X,Y)$ denote the set of globally framed correspondences from X to Y. We consider it as a pointed set with the distinguished point being the class 0_n of the explicit correspondence with $U = \emptyset$.

Example 2.1 The set $F_0^{gl}(X,Y)$ coincides with the set of pointed morphisms $X_+ \to Y_+$. In particular, for a connected scheme X one has

$$F_0^{gl}(X,Y) = Hom(X,Y) \amalg \{0_0\}$$

Example 2.2 Let $p_1, \ldots, p_n$ be a collection of pair-wise disjoint X-points in $\mathbf{A}^m_X$. We denote by $\Phi(p_1, \ldots, p_n)$ the framed correspondence from X to $\coprod_{i=1}^n X$ with the support in $p_1(X) \cup \cdots \cup p_n(X)$ and the framing given in a neighborhood of $p_i(X)$ by $\phi_j = pr_j - pr_j \circ p_i$.

If $f : X' \to X$ is a morphism of schemes and $\Phi = (U, \phi, g)$ an explicit correspondence from X to Y then $(U' = U \times_X X', \phi \circ pr, g \circ pr)$ is an explicit correspondence from X' to Y which we denote by $f^*(\Phi)$. Let (U, ϕ, g) be an explicit correspondence of level n from X to Y and (V, ψ, g) an explicit correspondence of level m from Y to Z. We define their composition as follows. Consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{pr_V} & V \\ {\scriptstyle pr_U}\downarrow & & \downarrow \\ U & \xrightarrow{g} & Y \\ \downarrow & & \\ X & & \end{array} \qquad (2.2.1)$$

where W is the fiber product of V and U. Let us verify that $(W, (\phi \circ pr_U, \psi \circ pr_V), g \circ pr_V)$ is an explicit correspondence of level $n+m$ from X to Z. Note first that there is a diagram

$$\begin{array}{ccc} W & \xrightarrow{pr_V} & V \\ \downarrow & & \downarrow \\ U \times \mathbf{A}^m & \xrightarrow{g\times Id} & Y \times \mathbf{A}^m \\ \downarrow & & \\ X \times \mathbf{A}^n \times \mathbf{A}^m & & \end{array} \qquad (2.2.2)$$

where the square is cartesian which shows that W is etale over $\mathbf{A}_X^{n+m}$. The rational functions $\psi_j \circ pr_V$ are defined because of the property 2.i of the definition of an explicit correspondence and the functions $\phi_i \circ pr_U$ are defined because pr_U is smooth. It remains to verify that the closed subset

$$(\cap_i Supp(\phi_i \circ pr_U)) \cap (\cap_j Supp(\psi_j \circ pr_V))$$

is finite over X and maps monomorphically to $X \times \mathbf{A}^n \times \mathbf{A}^m$. The first statement follows immediately from (2.2.1) since $Supp(\phi_i \circ pr_U) = pr_U^{-1}(Supp(\phi_i))$ and $Supp(\psi_j \circ pr_V) \subset pr_V^{-1}(Supp(\psi_j))$.

The second statement follows in a similar manner from the diagram (2.2.2).

The composition of explicit correspondences clearly respects the equivalences and defines a map $F_n(X,Y) \times F_m(Y,Z) \to F_{n+m}(X,Z)$. The following lemma is straightforward.

Lemma 2.3 *The composition of the equivalence classes of explicit correspondences is associative.*

For a pair of schemes X, Y denote by $F_*^{gl}(X,Y)$ the set $\amalg_n F_n^{gl}(X,Y)$. Composition of framed correspondences defines a category $F_*^{gl}(S)$ with the same objects as Sch/S and morphisms given by $F_*^{gl}(X,Y)$. Since morphisms of schemes can be identified with special framed correspondences of level zero we get a functror $Sch/S \to F_*^{gl}(S)$ and one can easily see that for a framed correspondence $\Phi : X \to Y$ and a morphism $f : X' \to X$ one has $f^*(\Phi) = \Phi \circ f$.

Remark 2.4 *The category $F_*^{gl}(S)$ has neither an initial nor a final object. Endomorphisms of the empty scheme in $F_*^{gl}(S)$ are of the form $F_*^{gl}(\emptyset,\emptyset) = \{0_0, \ldots, 0_n, \ldots\}$. Observe also that the disjoint union of schemes is not their coproduct in $F_*^{gl}(S)$.*

3 Framed functors

Definition 3.1 *A globally framed functor F on Sch/S is a contravariant functor from $F_*^{gl}(S)$ to the category of pointed sets such that $F(\emptyset) = pt$ and $F(X \coprod Y) = F(X) \times F(Y)$.*

Note that the representable functors on $F_*^{gl}(S)$ are not framed functors. To associate a framed functor to a scheme one needs a slightly more involved construction. Denote by σ_X the framed correspondence of level 1 from X to X given by $(\mathbf{A}^1_X \xrightarrow{Id} \mathbf{A}^1_X, id, pr_X)$. For any morphism of pointed schemes $f : X_+ \to Y_+$ one has $f\sigma_X = \sigma_Y f$ by for a general globally framed correspondence Φ one has $\Phi\sigma_X \neq \sigma_Y \Phi$.

Let h_X be the functor represented by X on $F_*^{gl}(S)$. Then σ_X defines an endomorphism of h_X and we set:

$$Q^{gl}X_+ := colim_{\sigma_X : h_X \to h_X} h_X$$

Note that the functors $Q^{gl}X_+$ are functorial with respect to morphisms of pointed schemes but not with respect to general framed correspondences.

Lemma 3.2 *For any X the functor $Q^{gl}X_+$ is a globally framed functor.*

Proof: We have $h_X(\emptyset) = F_*^{gl}(\emptyset, X) = \coprod\{0_n\}$. Composition with σ_X takes $\{0\}_n$ to $\{0\}_{n+1}$ which implies that $Q^{gl}X_+(\emptyset) = pt$. Given two framed correspondences $\Phi : X \to Y$, $\Phi' : X' \to Y$ of the same level we may consider the framed correspondence $\Phi \coprod \Phi' : X \amalg X' \to Y$. This construction defines in the obvious manner a map $Q^{gl}Y_+(X) \times Q^{gl}Y_+(X') \to Q^{gl}Y_+(X \amalg X')$ which is clearly bijective.

Definition 3.3 *A globally framed functor is called stable if for any X one has $F(\sigma_X) = Id_F(X)$.*

Note that functors of the form $Q^{gl}X_+$ are not stable. As always a framed functor is called homotopy invariant if $F(X) = F(X \times \mathbf{A}^1)$ for all X.

Lemma 3.4 *Let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are two sets of pair-wise disjoint X-points in $\mathbf{A}^m_X$. Then the framed correspondences $\Phi(p_1, \ldots, p_n)$ and $\Phi(q_1, \ldots, q_n)$ defined in Example 2.2 are $\mathbf{A}^1$-homotopic.*

Proof: ???

Lemma 3.5 *Let $x_1, \ldots, x_n$ be a collection of pair-wise disjoint X-points on $\mathbf{A}^1_X$ and let $(\phi_1, \ldots, \phi_n)$, $(\psi_1, \ldots, \psi_m)$ be two collections of rational functions in neighborhoods of the points $x_1, \ldots, x_n$ respectively which define framed correspondences*

$$\Phi, \Psi : X \to \amalg_{i=1}^n X$$

with the support in $x_1(X) \cup \cdots \cup x_n(X)$. Assume that for each i the function ϕ_i/ψ_i is regular in a neighborhood of $x_i(X)$ and equals 1 on $x_i(X)$. Then Φ and Ψ are $\mathbf{A}^1$-homotopic.

Proof: The homotopy is given by the framed correspondence $X \times \mathbf{A}^1 \to \amalg_{i=1}^n X$ with the support in $(x_1(X) \cup \cdots \cup x_n(X)) \times \mathbf{A}^1$ and the framing given in a neighborhood of $x_i(X) \times \mathbf{A}^1$ by the rational function $\phi_i + t(\phi_i - \psi_i)$ where t is the projection to the $\mathbf{A}^1$.

For any X denote by δ the explicit correspondence from X to $X \amalg X$ with $U = (\mathbf{A}^1 - \{0\} \amalg \mathbf{A}^1 - \{1\})_X$, $\phi = t \amalg (t-1)$ where $t : \mathbf{A}^1_X \to X$ is the projection and $g : (\mathbf{A}^1 - \{0\} \amalg \mathbf{A}^1 - \{1\})_X \to X \amalg X$. For any framed presheaf F it defines maps $F(X) \times F(X) \to F(X)$.

Theorem 3.6 *Let F be a stable homotopy invariant framed presheaf. Then the maps $F(X) \to F(X) \times F(X)$ make F into a framed presheaf of abelain groups.*

Proof: For a framed functor F and a pair of correspondences Φ_1, Φ_2 of the same level we have $F(\Phi_1 \coprod \Phi_2) = F(\Phi_1) \times F(\Phi_2)$. Therefore, to check that our operation is associative we need to check that the following diagram in F_*^{gl} commute up to an $\mathbf{A}^1$-homotopy:

$$\begin{array}{ccc} X & \xrightarrow{\delta} & X \amalg X \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle \sigma_X \amalg \delta} \\ X \amalg X & \xrightarrow{\delta \amalg \sigma_X} & X \amalg X \amalg X \end{array}$$

The two paths in this diagram are represented by correspondences of the form $\Phi(p_1, p_2, p_3)$ where in one case $p_1 = (0,0); p_2 = (0,1); p_3 = (1,0)$ and in another $p_1 = (0,0); p_2 = (1,0); p_3 = (1,1)$. They are homotopic by Lemma 3.4. Similarly one establishes the commutativity of our operation. The fact that the distinguished point represents the right unit is obvious and the fact that it is also the left unit follows from the commutativity.

It remains to show that there is an inverse. Consider the framed correspondence m of level 1 from X to X with the support $X \times \{0\}$ and framing $-1/t$. We claim that for $a \in F(X)$, $m^*(a)$ is the inverse to a. It is enough to show that the framed correspondence $X \stackrel{\delta}{\to} X \coprod X \stackrel{\sigma \coprod m}{\to} X$ is $\mathbf{A}^1$-homotopic to 0_2. This composition is a correspondence of level 2 with the support in $(0,0) \amalg (1,0)$ and the framing given by (x_1, x_2) in the neighborhood of the first point and $(x_1 - 1, -1/x_2)$ in the neighborhood of the second. Observe first that we can move the second point first to the point $(1,1)$ with framing $(x_1 - 1, 1/(1-x_2))$ and then to $(0,1)$ with framing $(x_1, 1/(1-x_2))$. Applying Lemma 3.5 with respect to x_2 we further conclude that our correspondence is homotopic to the correspondence with the support $(0,0) \amalg (0,1)$ and framing given by $(x_1, x_2/(1-x_2))$ in the neighborhood of both points. The homotopy of the last correspondence to zero is given by the correspondence over $\mathbf{A}^1_X$ with the support $\{0\} \times Supp(f)$ where $f(x_2, t) = (x_2 - t)/(1 - x_2)$ and the framing (x_1, f). Theorem 3.6 is proved.

4 Framed correspondences and the Nisnevich topology

For a scheme X we let Et/X denote the category of schemes *separated* and etale over X.

Theorem 4.1 *Let X be a normal scheme. Then for any scheme Y the functor $U \mapsto F_n(U, Y)$ from Et/X to $Sets_\bullet$ is a sheaf in the etale topology.*

Proof:

Lemma 4.2 *Let X be a normal scheme. Then the category Et/X has finite colimits and the functor from Et/X to Sch/X preserves these colimits.*

Proof: The statement of the lemma clearly holds for finite coproducts. It remians to check it for coequalizers. Let $f, g : U \to V$ be a pair of morphisms in Et/X. Let $\bar{U}$ and $\bar{V}$ be normal schemes finite over X which contain U and V respectively as dense open subschemes. The f and g extend to a pait of morphisms $\bar{f}, \bar{g} : \bar{U} \to \bar{V}$. Let $\bar{W}$ be the spectrum of the equalizer of the corresponding pair of homomorphisms of the sheaves of rings $\mathcal{O}(\bar{V}) \to \mathcal{O}(\bar{U})$. Clearly, $\bar{W}$ is the coequalizer of $\bar{f}$ and $\bar{g}$ in the category of affine schemes over X. The universal properties of $\bar{W}$ and the universal properties of normalization imply immediately that $\bar{W}$ is normal and equidimensional over X. The morphism $\bar{V} \to \bar{W}$ being a finite morphism of normal schemes equidimensional over X is equidimnsional and therefore open. Let W be the image of V in $\bar{W}$. Since W is normal and quasi-finite over X and there is a surjective morphism $V \to W$ with V etale over X we conclude that W is etale over X.

It remains to verify that W is the coequalizer of f and g in the category of separated schemes of finite type over X. We have an etale covering $p : V \to W$ and it is sufficient to check that for Y in Sch/X and a map $u : V \to Y$ such that $uf = ug$ we have $u\,pr_1 = u\,pr_2$ where pr_i are the projections $V \times_W V \to V$. Since Y is separated and all connected components of $V \times_W V$ dominate X it is sufficient to verify this property over the generic points of X. Over these points we have $\bar{f} = f$, $\bar{g} = g$ and $\bar{W} = W$ and since $\bar{W}$ is the coequalizer of $\bar{f}$ and $\bar{g}$ the morphism u descends to W and in particular one has $u\,pr_1 = u\,pr_2$.

Example 4.3 Lemma 4.2 is false without the assumption that X is normal. Let X be the union of two copies of $\mathbf{A}^1 - \{0\}$ glued together in points $\{1\}$. Let U be the etale covering of X of degree 2 obtained by taking two copies of the map $z \mapsto z^2$ and gluing these copies together over $\{1\}$. Let V be the open subscheme in U which is the preimage of one of the open subschemes $\mathbf{A}^1 - \{0, 1\}$ in X. Let us show that the colimit $U/V = U \cap_V X$ in the category of separated schemes is not etale over X. Observe that the colimit in the category of separated schemes does not change if we replace V by its closure in U. In this case we get a diagram of schemes finite over X and its colimit is the union of X and the scheme obtained from the double cover of $\mathbf{A}^1 - 0$ by contracting the preimage of 1 to a point. Since it has only one geometric point over the singular point of X and two everywhere else it is not etale over X.

We now proceed to the proof of Theorem 4.1. First of all one observes easily that framed correspondences with values in a scheme Y form a separated presheaf for the etale topology. It remains to check that for a normal scheme X, an etale covering $N \to X$ of X and a framed correspondence Φ_N from N to Y such that the pull-backs of Φ_N with respect to the two projections $N \times_X N \to N$ agree there exists a framed correspondence Φ from X to Y which restricts to Φ_N on N.

A framed correspondence from N to Y is the equivalence class of an explicit correspondence $(Z_N, U_N \to \mathbf{A}^n_N, \phi_N, g : U_N \to Y)$. The condition that the pull-backs of Φ_N with respect to the two projections $N \times_X N \to N$

agree mean that $pr_1^{-1}(Z_N) = pr_2^{-1}(Z_N)$ and that there is an etale neghborhood V of this closed subset in $\mathbf{A}^n_{N\times N}$ together with two maps $V \to U_N$ such that the pull-backs of ϕ and g with respect to these maps agree. The first condition implies that Z_N is the pull-back of a closed subset Z in $\mathbf{A}^n_X$. Since Z_N is finite over X so is Z. Consider the coequalizer W of the maps $V \to U_N$ in the category of schemes separated and etale over $\mathbf{A}^n_X$ which exist by Lemma 4.2. The fiber Z' of W over Z is covered by Z_N and the map $Z_N \times_Z Z_N \to Z'$ equalizes the two projections. This implies immediately that $Z' = Z$ i.e. that W is an etale neghborhood of Z. The functions ϕ_N and the morphism g_N descend to functions ϕ and a morphism g on W. One verifies immediately that ϕ is a framing for Z. We obtained a framed correspondence $\Phi = (Z, W \to \mathbf{A}^n_X, \phi, g)$. The pull-back Φ' of Φ to N is, as an explicit framed correspondence, $(Z_N, W \times_X N, \phi' \circ pr_W, g' \circ pr_W)$. The morphism $U_N \to W \times_X N$ takes $\phi' \circ pr_W$ to ϕ_N and $g \circ pr_W$ to g_N. Therefore, Φ' is equivalent to Φ_N. Theorem is proved.

Recall that for a morphism $f : X \to Y$ we denote by $\check{C}(f)$ or $\check{C}(Y/X)$ the Cech simplicial object defined by f (see []).

Theorem 4.4 *Let $U \to X$ be an etale (resp. Nisnevich) covering of a scheme X. Then for any n the morphism of simplicial presheaves*

$$F_n(-, \check{C}(U/X)) \to F_n(-, X)$$

is a local equivalence in the etale (resp. Nisnevich) topology.

Proof: Let us consider the Nisnevich case. The etale case is similar. We have to show that for a local henselian scheme Y the map of simplicial sets $F_n(Y, \check{C}(U/X)) \to F_n(Y, X)$ is a weak equivalence.

The functor $F_n(Y, -)$ is a coproduct of functors of the form $F_n^{Z,\phi}(Y, -)$ where Z is a closed subset of $\mathbf{A}^n_Y$ finite over Y and ϕ is an equivalence class of framings of Z in its etale neighborhoods in $\mathbf{A}^n_Y$. Therefore it is sufficient to show that for any Z and ϕ the map

$$F_n^{Z,\phi}(Y, \check{C}(U/X)) \to F_n^{Z,\phi}(Y, X) \tag{4.4.1}$$

is a weak equivalence. The functors $F^{Z,\phi}(Y, -)$ commutes with fiber products and therefore we have

$$F_n^{Z,\phi}(Y, \check{C}(U/X)) = \check{C}(F_n^{Z,\phi}(Y, U) \xrightarrow{p} F_n^{Z,\phi}(Y, X)).$$

It remains to note that if Y is henselian then so is Z and which implies that the map p is surjective and therefore (4.4.1) is a simplicial homotopy equivalence.

Corollary 4.5 *Let F be a framed presheaf. Then the associated sheaf in the etale (resp. Nisnevich) topology has a unique structure of a framed presheaf such that the map $F \to a_{et}F$ is a map of framed presheaves.*

Proof: Let us consider the Nisnevich case. The etale case is similar. Let us show first that the separated presheaf associated with F has a structure of a framed presheaf. Let X be a scheme, $U \to X$ a Nisnevich covering and $u : Y \to X$ be a framed correspondence of level n. Let f, g two elements of $F(X)$ which become equal on U. We have to verify that $u^*(f)$ and $u^*(g)$ become equal on a Nisnevich covering of Y. Theorem 4.4 asserts in particular that the map of presheaves $F_n(-, U) \to F_n(-, X)$ is an epimorphism in the Nisnevich topology. Therefore, there is a Nisnevich covering $V \to Y$ and a framed correspondence $V \to U$ such that the square

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & V \end{array} \tag{4.5.1}$$

commutes. This implies that the pull-backs of f and g become equal on V. The same diagram implies the uniqueness part of the corollary.

Consider a section of F on U whose restrictions with respect to the two projections $U \times_X U \to U$ agree. This section defines a morphism of presheaves $F_n(-, U) \to F$ which factors through the coequalizer of the maps $F_n(-, U \times_X U) \to F_n(-, U)$ defined by the two projections. The sheaf associated with this coequalizer is, by Theorem 4.4, $F_n(-, X)$ and we get a map $F_n(-, X) \to a_{Nis}F$, i.e. for any framed correspondence $Y \to X$ of level n we got a section of $a_{Nis}F$ on Y. Using again (4.5.1) one verifies that this construction commutes with compositions of correspondences. Corollary is proved.

5 A construction of framed correspondences

6 A construction of framed functors

For any X the endomorphism σ_X defines an endomorphism of the corresponding representable functor $Fr_*(-, X)$ on $Fr_*(S)$. We define $Fr(-, X)$ as the colimit of the sequence $Fr_*(-,X) \xrightarrow{\sigma_X} Fr_*(-,X) \to \dots$. It is a functor on $Fr_*(S)$ and one can easily see that $Fr(U \amalg V, X) = Fr(U,X) \amalg Fr(V,X)$ which implies that $Fr(-,X)$ is a *framed functor* on Sch/S. Note that a framed correspondence $X \to Y$ does not in general define a morphism of framed functors $Fr(-,X) \to Fr(-,Y)$. Note also that $Fr(-,X)$ is not stable in the sense of Definition **??**.

For a scheme X and an explicit correspondence $(U,\phi,g) : Y \to Z$ we define an explicit correspondence $Id_X \times (U,\phi,g) : X \times Y \to X \times Z$ as $(X \times U, \phi \circ pr_U, Id_X \times g)$. Let $f : X_1 \to X_2$ be a morphism of schemes. Then the diagram of framed correspondences:

$$\begin{array}{ccc} X_1 \times Y & \xrightarrow{Id_{X_1} \times (U,\phi,g)} & X_1 \times Z \\ {\scriptstyle f \times Id_Y}\downarrow & & \downarrow{\scriptstyle f \times Id_Z} \\ X_2 \times Y & \xrightarrow{Id_{X_2} \times (U,\phi,g)} & X_2 \times Z \end{array} \tag{6.0.2}$$

commutes. This shows that the category $Fr_*(S)$ has a "module" structure over the category Sch/S: for a morphism f of schemes and a framed correspondence G one defines $f \times G$ to be the framed correspondence given by the diagonal in (6.0.2).

We will use two actions of $GL_n(X)$ on $Fr_n(X,Y)$. If $\Phi = (p : U \to \mathbf{A}^n_X, \phi, g)$ is a framed correspondence and a is an element of $GL_n(X)$ (or more generally an element of $Aut(\mathbf{A}^n)(X)$), then $a\Phi = (a \circ p : U \to \mathbf{A}^n_X, \phi, g)$ is again a framed correspondence. This defines one action of $GL_n(X)$ on $Fr_n(X,Y)$. To define the second action consider the collection of rational functions $\phi = (\phi_1, \dots, \phi_n)$ as a collection of regular functions on a dense open subset of U. Applying a to ϕ we get a new sequence of regular functions $(\phi'_1, \dots, \phi'_n)$. Let us show that there is an open neighborhood V this functions are not identically zero ...

An elementary $\mathbf{A}^1$-homotopy from a framed correspondence f to a framed correspondence g is a framed correspondence $h : X \times \mathbf{A}^1 \to Y$ such that $h \circ (Id_X \times \{0\}) = f$ and $h \circ (Id_X \times \{1\}) = g$ Two framed correspondences $f, g : X \to Y$ are called $\mathbf{A}^1$-homotopic (or simply homotopic if no confusion is possible) if they can be connected by a chain of elementary homotopies. One verifies easily that the composition of framed correspondences preserves the homotopy relation. The category whose objects are schemes and morphisms are the homotopy classes of framed correspondences will be denoted by $\pi_0 Fr_*(S)$.

Lemma 6.1 *For any framed correspondence $f : X \to Y$ the framed correspondences $f \circ \sigma_X$ and $\sigma_Y \circ f$ are canonically homotopic.*

Proof: Assume that f is of level n. For any X and Y we have an action of the group $GL_n(S) \times GL_n(S)$ (where S is the base scheme) on $Fr_n(X, Y)$ of the form:

$$(A, B)(p : U \to \mathbf{A}^n_X, \phi, g) = (A \circ p, B \circ \phi, g)$$

Let δ_n be the permutation $(t_1, \dots, t_n, s) \mapsto (s, t_1, \dots, t_n)$. One can easily see that $f \circ \sigma_X = (\delta_n, \delta_n)(\sigma_Y \circ f)$. The permutation δ_n is homotopic in GL_n to transformation

$$a : (x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n, -y)$$

this shows that $f \circ \sigma_X$ is canonically homotopic to $(s, s)(f \circ \sigma_X)$. It remains to note that $(s, s)(f \circ \sigma_X) = (f \circ \sigma_X)$: if (U, ϕ, g) is an explicit correspondence representing f then the automorphism

$$U \times \mathbf{A}^1 \xrightarrow{(u,t)\mapsto(u,-t)} U \times \mathbf{A}^1$$

defines an equivalence between $(U, \phi, g) \circ \sigma_X$ and $(s, s)((U, \phi, g) \circ \sigma_X)$.

For a framed functor F denote by $C_*(F)$, as always, the simplicial framed functor of the form $U \mapsto F(U \times \Delta^\bullet)$ where $\Delta^\bullet$ is the standard cosimplicial object in Sch/S. Conisder the simplicial set

$$C_*(Fr(-, Y))(X) = Fr(X \times \Delta^\bullet, Y)$$

Since the morphisms σ_X commute with the morphisms of level zero the morphisms $\sigma : X \times \Delta^n \to X \times \Delta^n$ define an endomorphism of $C_*(Fr(-, Y))(X)$ which we denote σ.

Lemma 6.2 *There is a pointed simplicial homotopy from σ to the idenity.*

Proof: The construction of Lemma 6.1 gives a map $h : Fr(X, Y) \to Fr(X \times \mathbf{A}^1, Y)$ natural with respect to morphsms of level zero in X and such that for any $f : X \to Y$ one has

$$h(f)_{|X\times\{0\}} = \sigma_Y \circ f$$

$$h(f)_{|X\times\{1\}} = f \circ \sigma_X$$

The maps h for $X \times \Delta^n$ give a map of simplicial sets $C_*(Fr(Y))(X) \to C_*(Fr(Y))(X \times \mathbf{A}^1)$ whose compositions with the restrictions to $X \times \{0\}$ and $X \times \{1\}$ are σ_l and σ_r respectively. Our result follows from the fact that these restrictions are simplicially homotopic (see [,]).

7 Framed sheaves

A framed presheaf on a category of schemes is just a framed functor. For a topology on the category of schemes a framed sheaf with respect to this topology is a framed presheaf which is a sheaf with respect to this topology.

Proposition 7.1 *Let F be a framed presheaf. Then the associated sheaf in the Nisnevich (resp. etale) topology has a unique staructure of a framed functor such that the map $F \to aF$ is a map of framed functors.*

For a scheme X we denote by Fr_X the sheaf in the Nisnevich topology associated with the presheaf $Y \mapsto Fr_*(Y, X)$.

8 Traces of framed correspondences

Let $\Phi = (U \to \mathbf{A}^n \times X, \phi, g : U \to Y)$ be a framed correspodence from X to Y with the support $Z \subset U$ which we consider as a closed reduced subscheme. The trace of Φ is the triple $(U \to \mathbf{A}^n \times X, \phi, g_{|Z}Z \to Y)$. A framed correspondence is called smooth if in any point z of Z the functions ϕ_i are regular and the divisors $\phi_i^{-1}(0)$ are smooth and intersect transversally. For a smooth correspondence Z is a smooth closed subvariety of U.

Proposition 8.1 *Let Φ and Φ' be framed correspondences from X to Y. Assume that they are smooth and that their traces coincide. Then the sections of $a_{Nis}Fr_*(-, Y)$ on X defined by Φ and Φ' are $\mathbf{A}^1$-homotopic.*

Proof: ???