Докл. Акад. Наук СССР
Том 304 (1989), № 2

Soviet Math. Dokl.
Vol. 39 (1989), No. 1

EQUILATERAL TRIANGULATIONS OF RIEMANN SURFACES, AND CURVES OVER ALGEBRAIC NUMBER FIELDS

UDC 513.6

V. A. VOEVODSKIĬ AND G. B. SHABAT

The introduction of a Euclidean structure on the faces of a triangulated compact orientable surface (i.e., the specification of lengths for all the edges in such a way that all the triangle inequalities are satisfied), like the specification of a Riemannian metric on a smooth surface, permits the definition of a complex structure on the surface. In this note we consider equilateral triangulations—the lengths of all the edges are equal to 1; our goal is to prove that the collection of Riemann surfaces thus obtained, regarded as algebraic curves over **C**, coincides precisely with the collection of algebraic curves defined over the algebraic closure of $\overline{\mathbf{Q}}$ of the field of rational numbers.

In the case of genus 0, equilateral triangulations have been considered by physicists (see, for example, [2]) in connection with the investigation of discrete analogues of string theory; our result is thus fundamental for the use of arithmetic methods in string theory.

The proof makes essential use of ideas of Grothendieck in his unpublished paper "Esquisse d'un programme."

1. Recall that a two-dimensional simplicial scheme S is defined to be a triple consisting of a finite set $J_0(S)$ (the "vertices") and collections $J_1(S)$ and $J_2(S)$ of two-element and three-element subsets, respectively, (the "edges" and "faces") of it; all the two-element subsets of an arbitrary element in $J_2(S)$ are required to be in $J_1(S)$.

The realization $|S|$ of a simplicial scheme S is defined to be the polyhedron in $\mathbf{R}^{J_0(S)}$ constructed as follows: the elements are imbedded in $\mathbf{R}^{J_0(S)}$ as the coordinate unit vectors, and then the convex hulls of the images of the subsets in $J_1(S)$ and $J_2(S)$ are taken.

An important role in what follows will be played by the flag set $\mathscr{F}(S)$, which consists of the tuples $\alpha \times \beta \times \gamma \in J_0(S) \times J_1(S) \times J_2(S)$, such that $\alpha \subset \beta \subset \gamma$. Let $\pi_q : \mathscr{F}(S) \to J_q(S)$ be the corresponding projections.

2. The *two-dimensional cartographical group* $\mathscr{C}_2$ is defined to be the abstract group given by the generators σ_0, σ_1, and σ_2 and the relations $\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = (\sigma_0\sigma_2)^2 = 1$.

3. It is convenient to formulate conditions singling out those two-dimensional simplicial schemes S with realizations $|S|$ homeomorphic to Riemann surfaces in terms of the action of $\mathscr{C}_2$ on $\mathscr{F}(S)$. We postulate the existence of an action $\mathscr{C}_2 : \mathscr{F}(S)$ satisfying the following basic property.

(3.0) For all $p, q \in \{0, 1, 2\}$ and for all $F \in \mathscr{F}(S)$

$$\pi_q(\sigma_p \cdot F) = \pi_q(F) \Longleftrightarrow p \neq q.$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30F99, 14H15, 57Q15; Secondary 83E99.

Two more properties are required:

(3.1) (Connectedness). The action $\mathscr{C}_2: \mathscr{F}(S)$ is transitive.

(3.2) (Orientability). There exists a function $o: \mathscr{F}(S) \to \{\pm 1\}$, satisfying the equalities $o(\sigma_q \cdot F) = -o(F)$ for all $q \in \{0, 1, 2\}$ and $F \in \mathscr{F}(S)$.

It can be shown that if $|S|$ is homeomorphic to a compact Riemann surface, then there exists an action $\mathscr{C}_2: \mathscr{F}(S)$ satisfying (3.0)–(3.2). Such simplicial schemes S will be called *Riemannian* in what follows.

4. Let S be a Riemannian simplical scheme with a fixed orientation $o: \mathscr{F}(S) \to \{\pm 1\}$. Along with $|S|$ we construct another topological realization of it on which it is convenient to define a complex structure.

Let $T^\pm \subset \mathbf{C}$ denote closed equilateral triangles with the real segment [0,1] as one of the sides; T^+ lies in the upper half-plane, and T^- in the lower half-plane. The basic new realization is the disconnected set of triangles

$$\mathscr{P}(S) = \{z \times F \in \mathbf{C} \times \mathscr{F}(S) | z \in T^\pm \text{ for } o(F) = \pm 1\}.$$

Defined on it are two equivalence relations R_1 and R_2 specifying pastings together along edges and faces, respectively. By definition, a pair $z \times F$ is R_1-equivalent to a pair $z' \times F'$ if and only if $F' = \sigma_2 \cdot F$ and $z = z' \in \mathbf{R}$. To define the relation R_2 we denote by $\mathscr{C}_1$ the subgroup of $\mathscr{C}_2$ generated by the elements σ_0 and σ_1. The group $\mathscr{C}_1$ acts on $\mathscr{P}(S)$ as follows:

$$\sigma_0 \cdot (z \times F) = ((1 - z) \times (\sigma_0 \cdot F)),$$
$$\sigma_1 \cdot (z \times F) = ((e^{-io(F)\pi/3} z) \times (\sigma_1 \cdot F)).$$

The equivalence relation R_2 is determined by this action $\mathscr{C}_1: \mathscr{P}(S)$.

Denote by R the weakest equivalence relation on $\mathscr{P}(S)$ that is stronger than R_1 and R_2 ($p \sim_R p'$ if and only if there exists a chain $p_0 = p, p_1, \ldots, p_n = p'$, where $p_j \sim_{R_1} p_{j-1}$ or $p_j \sim_{R_2} p_{j-1}$ for all $j = 1, \ldots, n$).

Denote by $X(S)$ the quotient $\mathscr{P}(S)/R$ and by κ_S the natural projection $\mathscr{P}(S) \to X(S)$.

LEMMA. *The space $X(S)$ is homeomorphic to the realization $|S|$.*

PROOF. Each flag $F \in \mathscr{F}(S)$ can be uniquely represented in the form

$$\alpha \times \{\alpha, \alpha'\} \times \{\alpha, \alpha', \alpha''\} \in J_0(S) \times J_1(S) \times J_2(S),$$

where $\alpha, \alpha', \alpha'' \in J_0(S)$. Therefore, an affine mapping of the triangle $T^\pm \times F$ into $|S|$ carrying 0 into α, 1 into α', and the remaining vertex into α'' is uniquely determined. This determines a "piecewise linear" mapping $\mathscr{P}(S) \to |S|$; one can easily verify that it factors through $X(S)$ by means of κ_S and gives the required homeomorphism.

5. Denote by $\mathring{T}^\pm$ the interiors of the triangles $T^\pm$, and let

$$\mathring{\mathscr{P}}(S) = \{(z \times F) \in \mathscr{P}(S) | z \in \mathring{T}^\pm\}.$$

There is a natural complex structure on this set. For an open set $U \subset X(S)$ let $\mathscr{O}_U$ denote the ring of continuous functions $\varphi: U \to \mathbf{C}$ for which the restriction $\kappa_S^* \varphi | (\kappa^{-1} U) \cap \mathring{\mathscr{P}}(S)$ is holomorphic.

ASSERTION. *The correspondence $U \to \mathscr{O}_U$ determines on $X(S)$ a sheaf $\mathscr{O}$ of rings that turns $X(S)$ into a compact Riemann surface.*

PROOF. Denote by $\dot{X}(S)$ the surface $X(S)$ with the vertices removed, i.e., without the images under the mapping $\kappa_S: \mathscr{P}(S) \to X(S)$ of the vertices of the triangles

$T^{\pm} \times F$. The fact that the restriction of the sheaf $\mathscr{O}$ to $\dot{X}(S)$ turns $\dot{X}(S)$ into a complex curve follows from the fact that $\dot{X}(S)$ is covered by the "rhombuses"

$$\mathscr{R}_F = \operatorname{Int}(\kappa_S(T^{\pm} \times F) \cup \kappa_S(T^{\mp} \times (\sigma_2 \cdot F))),$$

and in each such rhombus the function $\operatorname{pr}_{\mathbf{C}} \circ \kappa_S^{-1}$ is single-valued and lies in $\mathscr{O}_{\mathscr{R}_F}$.

To prove the assertion it remains to establish that each vertex $P \in X(S)$ has a neighborhood $U \subset X(S)$ such that $\dot{U} = U \setminus \{P\}$ is biholomorphically equivalent to a punctured disk. For this we choose a flag $F \in \mathscr{F}(S)$ such that $P = \kappa_S(0 \times F)$, and denote by n the smallest positive integer such that $(\sigma_1\sigma_2)^n \cdot F = F$. For $\nu = 1, \dots, n$ let V_ν be the "angle" $\kappa_S(T^{\pm} \times (\sigma_1\sigma_2)^\nu F)$ (the sign on $T^{\pm}$ is determined by the orientation $O(F)$), and let $z_\nu \colon V_\nu \to \mathbf{C}$ be the composition $\operatorname{pr}_{\mathbf{C}} \circ \kappa_S^{-1}|_{V_\nu}$. The desired biholomorphic equivalence is given by the function Z determined by the condition $Z|_{V_\nu} = e^{2\pi i\nu/n} Z_\nu^{6/n}$.

The complex structure determined on $X(S)$ by the sheaf $\mathscr{O}$ will be called the *equilateral* complex structure.

6. We are now ready to state our main result.

THEOREM. *Let X be a complete nonsingular complex algebraic curve. It is defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers if and only if there exists a Riemannian simplicial scheme S for which X is biholomorphically equivalent to the Riemann surface $X(S)$ with the equilateral complex structure.*

The proof of both parts uses a theorem of Belyĭ [1] asserting that a compact Riemann surface is defined over $\overline{\mathbf{Q}}$ if and only if there exists a meromorphic function on it with three critical values. Such functions will be called *Belyĭ functions*.

PROOF. "*If*". It suffices to construct a Belyĭ function on the surface $X(S)$. This can be done as follows. In each triangle $T^{\pm} \times F$, $F \in \mathscr{F}(S)$, we draw the medians dividing $T^{\pm}$ into six small triangles. This procedure corresponds to the barycentric subdivision of the triangulated surface $X(S)$.

The group $\mathscr{C}_1$, acting on $\mathscr{P}(S)$, acts also on the set of small triangles; the orbits of this action are in a one-to-one correspondence with the flag set $\mathscr{F}(S)$, since in such an orbit there exists a unique small triangle having the real segment $[0,1/2]$ as a side. This implies that the images of the small triangles under the mapping $\kappa_S \colon \mathscr{P}(S) \to X(S)$ are also in a one-to-one correspondence with the elements of $\mathscr{F}(S)$; we call a small triangle *white* (*black*) if the corresponding flag $F \in \mathscr{F}(S)$ has orientation $o(F) = 1$ (respectively, $o(F) = -1$). We "paint" the small triangles on $\mathscr{P}(S)$ according to the "colors" of their images in $X(S)$. Let us now construct a continuous mapping of $\mathscr{P}(S)$ onto the complex projective line $\mathbf{P}^1(\mathbf{C})$, transferring each white (black) small triangle into the upper (lower) half-plane by a conformal mapping under which the vertex at the 30° angle goes into 0, the vertex at the 60° angle goes into ∞, and the vertex at the 90° angle goes into 1. This mapping factors through $X(S)$ by means of $\kappa_S \colon \mathscr{P}(S) \to X(S)$ and determines the required Belyĭ function $X(S) \to \mathbf{P}^1(\mathbf{C})$.

"*Only if*". Suppose that the curve X is defined over $\overline{\mathbf{Q}}$; by Belyĭ's theorem, there exists a Belyĭ function $b \colon X \to \mathbf{P}^1(\mathbf{C})$ on it. We assume that the critical values of b are $0, 1$, and ∞.

We need to define a Riemannian simplicial scheme S with the help of b. It is determined by the fact that the one-dimensional skeleton of the corresponding triangulation of X is $b^{-1}(\mathbf{P}^1(\mathbf{R}))$. More precisely, $J_0(S)$ is by definition in one-to-one correspondence with $b^{-1}\{0, 1, \infty\}$, and $J_2(S)$ in one-to-one correspondence with the connected components of $b^{-1}(\mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(R))$. Let T be such a connected component;

the mapping $b\colon T \to \mathbf{P}^1(\mathbf{C})$ can be extended to the closure of T, defining a homeomorphism of $\overline{T}$ with the closed upper half-plane or the closed lower half-plane; therefore, there are exactly three elements in $b^{-1}\{0, 1, \infty\}$ on the boundary ∂T. This correspondence between $J_2(S)$ and the three-element subsets of $J_0(S)$ determines the desired Riemannian simplicial scheme S. The biholomorphic equivalence of X and $X(S)$ can be established in essentially the same way as in the proof of the preceding part of the theorem, except that it is pasted together from conformal equivalences of the half-planes to equilateral triangles instead of 30°–60°–90° triangles.

The theorem is proved.

The authors thank A. A. Migdal, Al. B. Zamolodchikov, V. A. Kazakov, and D. V. Bulatov for their useful advice and demonstrated interest.

Steklov Mathematical Institute
Academy of Sciences of the USSR
Moscow

Received 1/JULY/87

BIBLIOGRAPHY

1. G. V. Bekyĭ, Izv. Akad. Nauk SSSR Ser. Math. **43** (1979), 267–276; English transl. in Math. USSR Izv. **14** (1980).

2. V. D. Boulatov [Bulatov] et al., Nuclear Phys. B **275** (1986), 641–686.

Translated by H. H. McFADEN