# Exercising Nuprl's Open-Endedness

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# Nuprl in a Nutshell

Similar to Coq and Agda

Extensional Intuitionistic Type Theory with partial types

Consistency proof in Coq:
`https://github.com/vrahli/NuprlInCoq`
We've proved the validity of about half the rules

Cloud based & virtual machines: `http://www.nuprl.org`

# Motivation

Correctness

Exploring type theory

Consistency of Nuprl relative to the one of Coq

Nuprl serves as a metatheory for Cubical Type Theory (see Mark's talk)

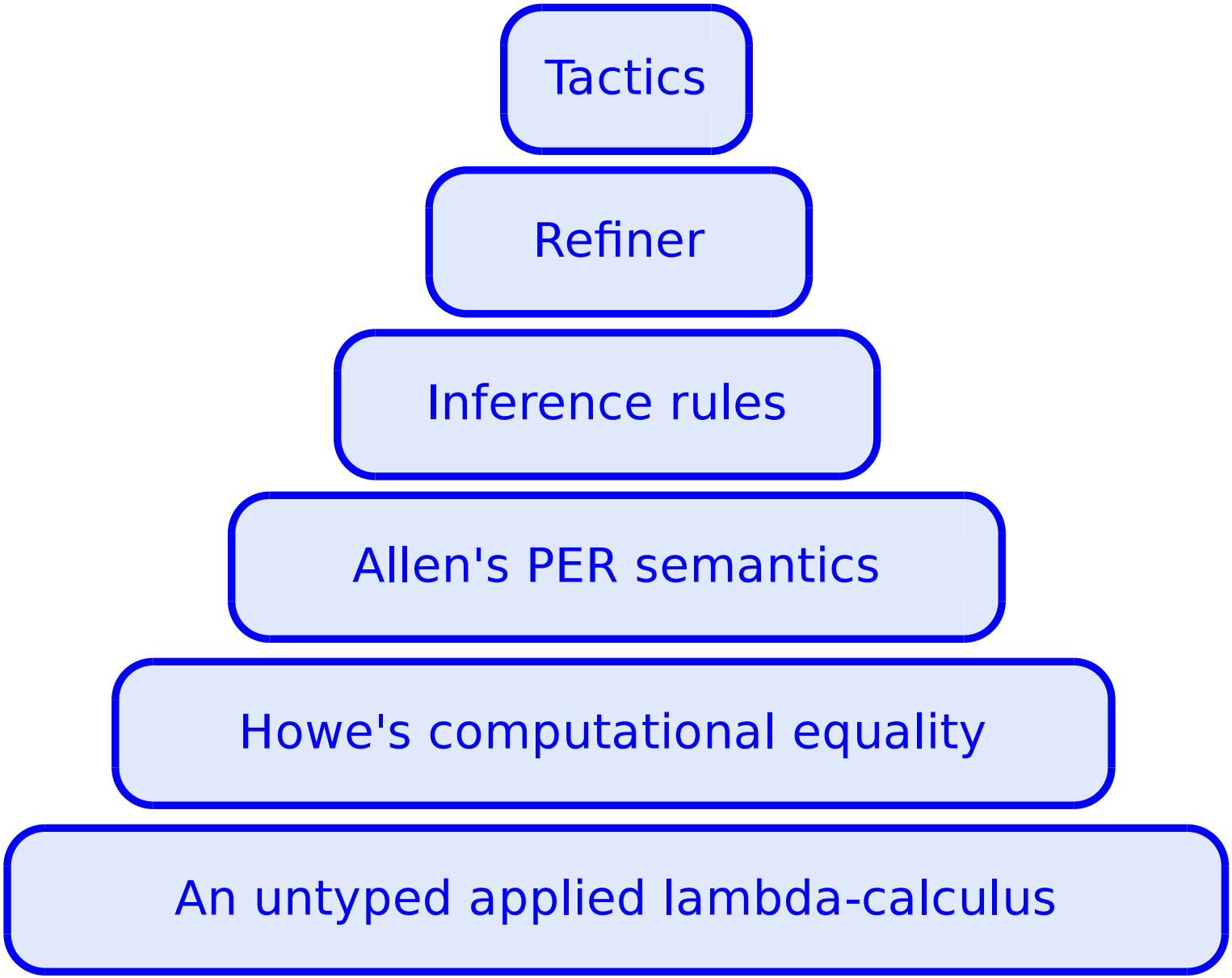
# Other Verified Systems

Verified version of HOL light (Myreen et al.)

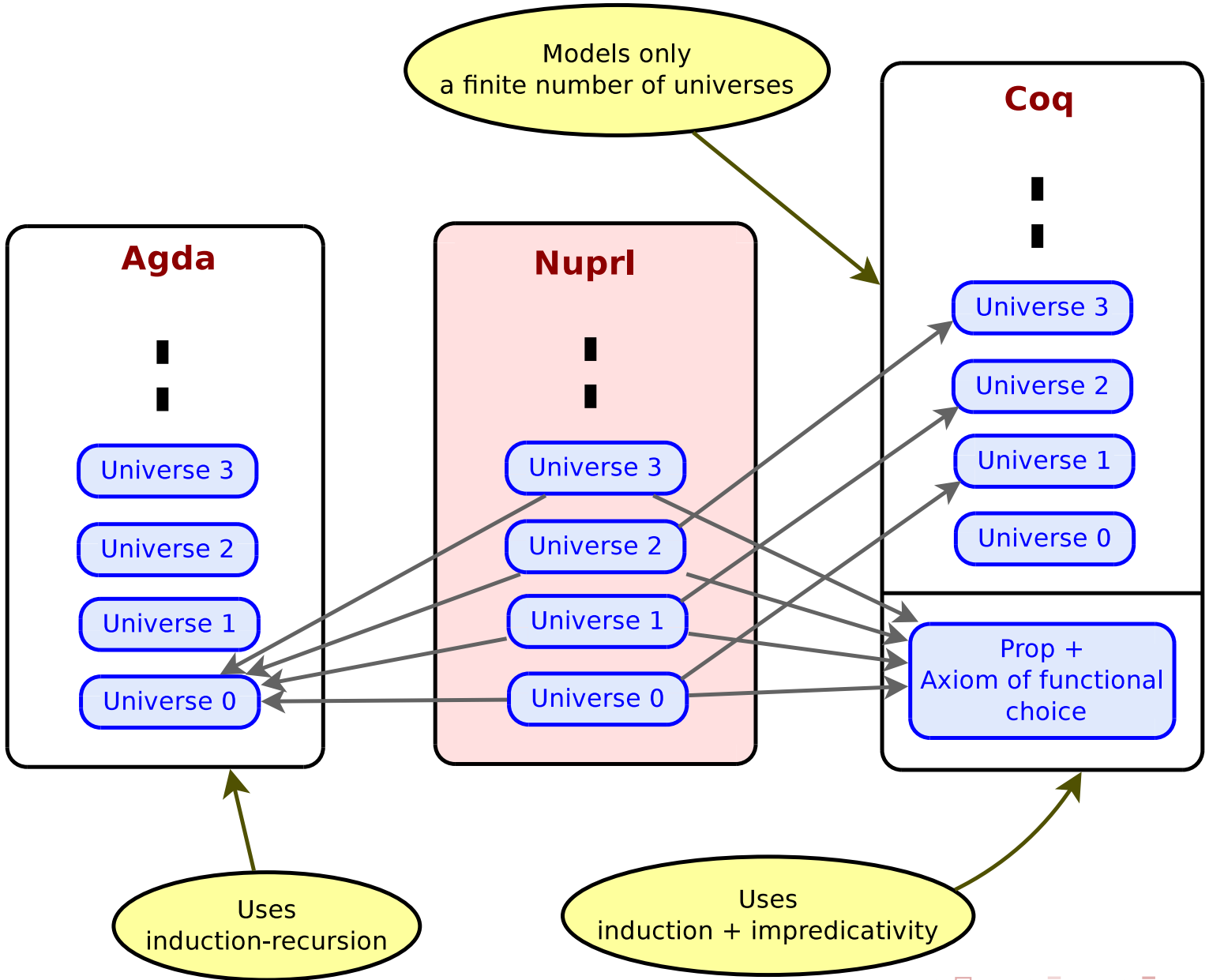
Milawa: Verified ACL2-like prover (Myreen & Davis)

Coq in Coq (Barras)

# Nuprl Stack


Tactics
Refiner
Inference rules
Allen's PER semantics
Howe's computational equality
An untyped applied lambda-calculus

# Nuprl PER Semantics Implemented in Coq



# Nuprl Types

Based on Martin-Löf's extensional type theory

**Equality**: $a = b \in T$

**Dependent product**: $\mathbf{\Pi} a{:}A.B[a]$

**Dependent sum**: $\mathbf{\Sigma} a{:}A.B[a]$

**Universe**: $\mathbb{U}_i$

# Nuprl Types

Less "conventional types"

**Partial**: $\overline{A}$

**Disjoint union**: $A+B$

**Intersection**: $\bigcap a{:}A.B[a]$

**Union**: $\bigcup a{:}A.B[a]$

**Subset**: $\{a : A \mid B[a]\}$

**Quotient**: $T//E$

**Domain**: $\mathtt{Base}$

**Simulation**: $t_1 \preceq t_2$

**Bisimulation**: $t_1 \simeq t_2$

**Image**: $\mathtt{Img}(A, f)$

**PER**: $\mathtt{per}(R)$

# Two Equalities

## Propositional/Definitional equality

$$(\lambda x.x + 1) = (\lambda x.1 + x) \in \mathbb{N}^{\mathbb{N}}$$

## Howe's computational equivalence relation

$$\mathtt{fix}(\lambda x.x) \simeq \mathtt{fix}(\lambda x.x(x))$$

$$\mathtt{fix}(\lambda x.\langle 0, x\rangle) \simeq \mathtt{fix}(\lambda x.\langle 0, \langle 0, x\rangle\rangle)$$

# Allen's PER semantics

Inductive (type equality/$\_\equiv\_$)
- Recursive (equality in a type/$\_\equiv\_\in\_$)

$$x_1{:}A_1 \to B_1 \equiv x_2{:}A_2 \to B_2$$

$$A_1 \equiv A_2$$
$$\wedge\ \forall a_1, a_2.\ a_1 \equiv a_2 \in A_1 \Rightarrow B_1[x_1 \backslash a_1] \equiv B_2[x_2 \backslash a_2]$$

$$f_1 \equiv f_2 \in x{:}A \to B$$

$$\texttt{type}((x{:}A \to B))$$
$$\wedge\ \forall a_1, a_2.\ a_1 \equiv a_2 \in A \Rightarrow f_1(a_1) \equiv f_2(a_2) \in B[x \backslash a_1]$$

# Allen's PER semantics—Fun Facts

These are Nuprl types:

$$n{:}\mathbb{N} \to \mathbb{U}_n$$

$$\bigcup n{:}\mathbb{N}.\mathbb{U}_n$$

# The More Types & Inference Rules the Better!

All verified (our goal)

Expose more of the metatheory

Encode Mathematical knowledge

# Open-Endedness

Nuprl was **designed to be open-ended**:
Theorems about computations and types hold for a broad class of extensions to the system

For example:

**Howe characterized the computations that can be added to Nuprl** in order to preserve the congruence of its computational equivalence relation

**Allen characterized the types that can be added to Nuprl** in order to preserve the fact that it is a type system

# Open-Endedness

New computations:

- named exceptions

$$1 + \mathtt{exc}(\boldsymbol{a}, 0) \mapsto^* \mathtt{exc}(\boldsymbol{a}, 0)$$

- a fresh operator

$$\boldsymbol{\nu} x.\mathtt{if}\ x{=}\boldsymbol{a}\ \mathtt{then}\ 0\ \mathtt{else}\ 1 \mapsto^* 1$$

- choice sequences

$$\mathtt{seq}(\mathtt{f})(1) \mapsto^* \mathtt{f}(1)$$

Where `f` is a Coq function on numbers.

Only used on the metatheory, never get exposed in the theory.

# Open-Endedness

New types:

- generalized $\mathbf{\Pi}$ types

$$\bot \in \mathbb{N}^{\mathtt{False}}$$

- generalized equality types

$$(\lambda x.x \in B^A) \in \mathtt{Type}$$

- added PER types

$$T//E = \mathtt{per}(\lambda x, y.x \in T \wedge y \in T \wedge (E\ x\ y))$$

# Open-Endedness

New inference rules:

- continuity
- bar induction
- **axiom of choice**

# Truncation

$\downarrow T$ $\quad \{\texttt{Unit} \mid T\}$

$x = y \in \downarrow T$ iff $x$ and $y$ compute to $\star$
and $T$ "is true"
...but we don't remember the reason why

$\downarrow T$ $\quad T//\texttt{True}$

$x = y \in \downarrow T$ iff $x, y \in T$

# Axiom of Choice

## Trivial

$$\mathbf{\Pi} a{:}A.\mathbf{\Sigma} b{:}B.P\ a\ b\ \Rightarrow\ \mathbf{\Sigma} f{:}B^{A}.\mathbf{\Pi} a{:}A.P\ a\ f(a)$$

## Proved in Nuprl (non-trivial): $\mathsf{AC}_{0,n}$ and $\mathsf{AC}_{1,n}$

$$\mathbf{\Pi} a{:}\mathbb{N}.\lfloor\mathbf{\Sigma} b{:}B.P\ a\ b\ \Rightarrow\ \lfloor\mathbf{\Sigma} f{:}B^{\mathbb{N}}.\mathbf{\Pi} a{:}\mathbb{N}.P\ a\ f(a)$$

$$\mathbf{\Pi} a{:}\mathcal{B}.\lfloor\mathbf{\Sigma} b{:}B.P\ a\ b\ \Rightarrow\ \lfloor\mathbf{\Sigma} f{:}B^{\mathcal{B}}.\mathbf{\Pi} a{:}\mathcal{B}.P\ a\ f(a)$$

where $\mathcal{B} = \mathbb{N}^{\mathbb{N}}$

# Axiom of Choice

Proved in Coq model: $\mathsf{AC}_{0,n}$

$$\mathbf{\Pi} a{:}\mathbb{N}.{\downarrow}\mathbf{\Sigma} b{:}\mathbb{N}.P\ a\ b \Rightarrow {\downarrow}\mathbf{\Sigma} f{:}\mathbb{N}^{\mathbb{N}}.\mathbf{\Pi} a{:}\mathbb{N}.P\ a\ f(a)$$

$$\mathbf{\Pi} a{:}\mathbb{N}.{\downarrow}\mathbf{\Sigma} b{:}\mathtt{NBase}.P\ a\ b \Rightarrow {\downarrow}\mathbf{\Sigma} f{:}\mathtt{NBase}^{\mathbb{N}}.\mathbf{\Pi} a{:}\mathbb{N}.P\ a\ f(a)$$

where $\mathtt{NBase} = \{t : \mathtt{Base} \mid (t : \mathtt{Base})\#\}$
(name-free members of Base)

Uses the axiom of choice and choice sequences

# Axiom of Choice

## Negation of $AC_{2,n}$ in Nuprl:

$$\mathbf{\Pi} a{:}\mathbb{N}^{\mathcal{B}}.\downarrow\mathbf{\Sigma} b{:}B.P\ a\ b \Rightarrow \downarrow\mathbf{\Sigma} f{:}B^{(\mathbb{N}^{\mathcal{B}})}.\mathbf{\Pi} a{:}\mathbb{N}^{\mathcal{B}}.P\ a\ f(a)$$

$$\mathbf{\Pi} a{:}\mathbb{N}^{\mathcal{B}}.\downarrow\mathbf{\Sigma} b{:}B.P\ a\ b \Rightarrow \downarrow\mathbf{\Sigma} f{:}B^{(\mathbb{N}^{\mathcal{B}})}.\mathbf{\Pi} a{:}\mathbb{N}^{\mathcal{B}}.P\ a\ f(a)$$

## Using continuity

# Open-ended?

How can we know to what extent Nuprl is open-ended?

Parameterize! Enough?

How can we know we haven't made decisions that unexpectedly limit Nuprl's open-endedness?

By requiring some properties of the computation system.

By adding operators that preclude us from adding others.