IMRN International Mathematics Research Notices
1995, No. 4

A Nilpotence Theorem for Cycles Algebraically Equivalent to Zero

V. Voevodsky

1 Introduction

In this paper we prove that a correspondence from a smooth projective variety over a field to itself, which is algebraically equivalent to zero, is nilpotent in the ring of correspondences modulo rational equivalence (with rational coefficients). We also show that a little more general result holds, namely, that for any algebraic cycle Z on a smooth projective X which is algebraically equivalent to zero, there exists an $N > 0$ such that the cycle $Z^{\otimes N}$ on X^N is zero in the corresponding Chow group of X^N with rational coefficients.

In the first section, we recall the definition and some elementary properties of the additive category of Chow motives over a field. In the second section, we prove our nilpotence theorems for cycles algebraically equivalent to zero. Finally, in the third section, we formulate a very strong nilpotence conjecture for algebraic cycles and explain how it is related to some other known conjectures. We did not try to give very accurate proofs in this last section, since almost all the statements there are conditional anyway, and the only reason to include this section at all was to illustrate the importance of nilpotence results for the theory of algebraic cycles.

Everywhere in this paper, except for the first section, we consider the $\mathbb{Q}$-linear situation, i.e., we completely ignore all torsion and cotorsion effects.

2 Chow motives

Let k be a field. Denote by $SmProj/k$ the category of smooth projective varieties over k. (We do not require objects of this category to be connected.) For any X in $SmProj/k$ and

Received 13 March 1995. Revision received 17 March 1995.
Communicated by Yasutaka Ihara.

any $n \geq 0$, denote by $A_n(X)$ the Chow group of cycles of dimension n on X modulo rational equivalence. For a pair of objects X, Y, we set

$$A(X, Y) = \oplus_{X_i} A_{\dim(X_i)}(X_i \times Y)$$

where X_i are the connected components of X. The classical construction of composition of correspondences gives us pairings

$$A(X, Y) \otimes A(Y, Z) \to A(X, Z)$$

such that the usual associativity property holds. We can define now the category of Chow correspondences $Cor_{Chow}(k)$ over k as the additive category such that

(1) $Ob(Cor_{Chow}(k)) = Ob(SmProj/k)$,

(2) $Hom_{Cor_{Chow}(k)}(X, Y) = A(X, Y)$,

and the composition is given by composition of correspondences. Note that there is an obvious functor $SmProj/k \to Cor_{Chow}(k)$, which is the identity on objects, and which takes a morphism to its graph. For any commutative ring R, we can further define an R-linear category $Cor_{Chow}(k, R)$ by setting

$$Hom_{Cor_{Chow}(k,R)}(X, Y) = A(X, Y) \otimes R.$$

Definition 2.1. The category of effective Chow motives $Chow^{eff}(k, R)$ over k with coefficients in R is the pseudoabelian envelope[1] of the category $Cor_{Chow}(k, R)$. For an object X in $SmProj/k$, we denote the corresponding object of $Chow^{eff}(k, R)$ by $[X]$.

One can easily see that there is a natural tensor structure on $Chow^{eff}(k, R)$ such that $[X \otimes Y] = [X] \times [Y]$. We denote the unit object of this tensor structure (i.e., $[Spec(k)]$) by R or $R\{0\}$. Consider the morphism $[\mathbf{P}^1] \to R$, which corresponds to the canonical morphism $\mathbf{P}^1 \to Spec(k)$. It splits by any point on $\mathbf{P}^1$, and therefore its kernel is a well-defined object of $Cor^{eff}(k, R)$. This object is called the Tate object, and we denote it by $R\{1\}$. (Note that since the canonical functor from varieties to $Chow^{eff}(k, R)$ is covariant, we indeed get in this way the Tate object as opposed to the Lefschetz object $R\{-1\}$, which one gets by considering contravariant theories.) We set further

$$R\{n\} = (R\{1\})^{\otimes n},$$

and for any object X of our category,

$$X\{n\} = X \otimes R\{n\}.$$

[1] A pseudoabelian envelope of an additive category $\mathcal{A}$ is an additive category obtained from $\mathcal{A}$ by formal enlargement of its class of objects so as to include kernels of all projectors.

Remark. Suppose that there exists an abelian category MM_k of mixed motives over k (with coefficients in R). Then the category $Chow^{eff}(k, R)$ of effective Chow motives is equivalent to the full subcategory in the *derived* category $D(MM_k)$ of MM_k generated by objects which correspond to smooth projective varieties over k (cf. [7]). The embedding $Chow^{eff}(k, R) \to D(MM_k)$ takes our Tate object $R\{n\}$ to the object $R(n)[2n]$, i.e., to the Tate motive of weight n placed in (homological) degree $2n$, which explains why we use the notation $R\{n\}$ instead of the standard $R(n)$.

Lemma 2.2. For any objects X, Y in $Cor^{eff}(k, R)$, the homomorphism

$$* \otimes Id_{R\{1\}} : Hom(X, Y) \to Hom(X\{1\}, Y\{1\})$$

is an isomorphism. □

Proof. It follows easily from the fact that $A_n(X \times \mathbf{P}^1) = A_n(X) \oplus A_{n-1}(X)$. ■

Finally we define the category $Chow(k, R)$ of Chow motives over k with coefficients in R as the category obtained from $Chow^{eff}(k, R)$ by formal inverting of $R\{1\}$. Lemma 2.2 implies that the obvious functor

$$Chow^{eff}(k, R) \to Chow(k, R)$$

is a full embedding.

The following proposition summarizes some elementary properties of the category $Chow(k, R)$. All of them follow easily from our definitions and standard facts about algebraic cycles modulo rational equivalence (see [6]).

Proposition 2.3. For any field k and any commutative ring R one has:

(1) $Chow(k, R)$ is a rigid tensor additive category. More precisely, the following statements hold:

(a) For any two objects X, Y, there is an internal Hom-object $\underline{Hom}(X, Y)$ such that, for any Z, there is a canonical isomorphism
$Hom(Z, \underline{Hom}(X, Y)) = Hom(Z \otimes X, Y)$.

(b) Denote the object $\underline{Hom}(X, R\{0\})$ by X^*. Then one has
$\underline{Hom}(X, Y) = X^* \otimes Y$
$(X \otimes Y)^* = X^* \otimes Y^*$
$(X^*)^* = X$.

(2) For any smooth projective equidimensional variety X of dimension n over k, one has a canonical isomorphism in $Chow(k, R)$ of the form $[X]^* = [X]\{-n\}$.

(3) For any smooth projective variety X over k and any $n \in \mathbf{Z}$, one has a canonical isomorphism

$$A_n(X) \otimes R = Hom_{Chow}(R\{n\}, [X]).$$

(For $n < 0$, it means that the right-hand-side group is zero.)

(4) For any smooth projective variety X over k and a vector bundle $\mathcal{E}$ over X, denote by $\mathbf{P}(\mathcal{E})$ the projectivization of $\mathcal{E}$. If $\mathcal{E}$ is of pure dimension n, there is a canonical isomorphism

$$[\mathbf{P}(\mathcal{E})] = \oplus_{i=0}^{n-1}[X]\{i\}.$$

(5) Let X be a smooth projective variety over k and let $Z \subset X$ be a smooth subvariety in X of pure codimension c. Denote by X_Z the blow-up of Z in X. Then there is a canonical isomorphism

$$[X_Z] = [X] \oplus (\oplus_{i=0}^{c-2}[Z]\{i\}).$$

(6) The objects $R\{i\}$ are mutually orthogonal; i.e., one has

$$\mathrm{Hom}(R\{i\}, R\{j\}) = \begin{cases} 0 & \text{for } i \neq j \\ R & \text{for } i = j. \end{cases}$$ □

The following fact, which will be used in the next section, is also a simple reformulation of well-known properties of Chow groups.

Proposition 2.4. Let X be a smooth projective variety, and let G be a finite group which acts on X. Further, let R be a commutative ring such that the order of G is invertible in R. Then there is an object $[X]/G$ in $\mathrm{Chow}(k, R)$ such that, for any object Y in $\mathrm{Chow}(k, R)$, one has

$$\mathrm{Hom}([X]/G, Y) = \mathrm{Hom}([X], Y)^G.$$

If the categorical quotient X/G exists in the category of schemes over k, one has canonical isomorphisms

$$\mathrm{Hom}(R\{n\}, [X]/G) = A_n(X/G) \otimes R.$$

If, in addition, X/G is smooth and projective, one has a canonical isomorphism

$$[X]/G = [X/G].$$ □

Definition 2.5. A morphism $f : X \to Y$ in a tensor additive category is called smash nilpotent if there exists an $N \geq 0$ such that the morphism

$$f^{\otimes N} : X^{\otimes N} \to Y^{\otimes N}$$

equals zero.

Remark. Note that a tensor functor from an additive tensor category to the category of K-vector spaces for a field K takes any smash nilpotent morphism to zero. In particular, any smash nilpotent morphism in a category with a fiber functor is zero.

We will show, in the next section, that if $f : \mathbf{Q}\{n\} \to [X]$ is a morphism in the category of Chow motives over k with rational coefficients such that the corresponding cycle of dimension n on X is algebraically equivalent to zero, then f is smash nilpotent.

We will use the following two trivial lemmas.

Lemma 2.6. For any tensor additive category, the subset of smash nilpotent morphisms $X \to Y$ is a subgroup in $\mathrm{Hom}(X, Y)$. □

Lemma 2.7. Let $\mathcal{A}$ be a rigid tensor additive category with the unit object $\mathbf{1}$, and let $f : X \to X$ be an endomorphism such that the adjoint morphism $\mathbf{1} \to X^* \otimes X$ is smash nilpotent. Then f is nilpotent in the ring $\mathrm{End}(X)$. □

3 Nilpotence theorems for cycles algebraically equivalent to zero

Proposition 3.1. Let $Z = \sum n_i z_i$ be a cycle of degree zero on a smooth projective curve X over k. Then the cycle $Z^{\otimes N}$ on X^N is rationally equivalent to zero for $N > 2g - 1$. □

Proof. Consider the morphism

$$\phi_Z : \mathbf{Q} \to [X]$$

in the category of Chow motives, which corresponds to the cycle Z. We have to show that the morphism

$$\phi_Z^{\otimes N} : \mathbf{Q} \to [X^N]$$

is zero for $N > 2g - 1$. By Proposition 2.4, we have a decomposition

$$[X^N] = [S^N X] \oplus ?$$

where $S^N X$ is the symmetric product of X. Since $\phi_Z^{\otimes N}$ is invariant under the action of the symmetric group, we have

$$\mathrm{pr}_1(\phi_Z^{\otimes N}) = \phi_Z^{\otimes N}.$$

(Here pr_1 is the projector which corresponds to the direct summand $[S^N X]$.) It is sufficient therefore to show that

$$S^N(\phi_{\mathcal{Z}}) : \mathbf{Q} \to [S^N X]$$

is zero.

Since we are working with rational coefficients, we may assume that the base field k is algebraically closed. Let $x : \mathrm{Spec}(k) \to X$ be a point of X. It defines a decomposition in $\mathrm{Chow}(k, \mathbf{Q})$ of the form $[X] = [\tilde{X}] \oplus \mathbf{Q}$. Since $\mathcal{Z}$ is a cycle of degree zero, the morphism $\phi_{\mathcal{Z}}$ can be factored through $[\tilde{X}]$. We have, further,

$$[S^N X] = S^N[X] = \sum_{i=0}^{N} S^i[\tilde{X}].$$

With respect to this decomposition, the morphism $S^N(\phi_{\mathcal{Z}})$ factors through $S^N[\tilde{X}]$. It is sufficient to show now that

$$\mathrm{Hom}(\mathbf{Q}, S^N[\tilde{X}]) = 0.$$

The left-hand-side group is the same as $A_0(S^N X)/A_0(S^{N-1}X)$, where the morphism $A_0(S^{N-1}X) \to A_0(S^N X)$ corresponds to the embedding

$$S^{N-1}X \to S^N X$$

given by x. Consider the commutative diagram

$$\begin{array}{ccc} S^{N-1}X & \longrightarrow & S^N X \\ \downarrow & \swarrow & \\ \mathrm{Jac}(X). & & \end{array}$$

Then, for $N > 2g - 1$, both $S^{N-1}X$ and S^X are projective bundles over the Jacobian $\mathrm{Jac}(X)$, and therefore the vertical arrows give isomorphisms on the Chow groups of zero cycles. This proves the proposition. ■

Corollary 3.2. Let X be a smooth projective variety over a field k, and let $\mathcal{Z}$ be a cycle of dimension d on X which is algebraically equivalent to zero. Then there exists an $N > 0$ such that the cycle $\mathcal{Z}^{\otimes N}$ equals zero in $A_d(X^N) \otimes \mathbf{Q}$. □

Proof. We may assume that k is algebraically closed. By the definition of algebraic equivalence, there exists a sequence of cycles $\mathcal{Z}_1, \ldots, \mathcal{Z}_k$ on X such that the following conditions hold:

(1) $\mathcal{Z}_1 = \mathcal{Z}$.

(2) $\mathcal{Z}_k = 0$.

(3) For any $i = 1, \dots, k-1$, there is a smooth projective curve C_i, a pair of points x_i, y_i on C, and a cycle $\mathcal{Y}_i$ on $X \times C$ such that

$$\mathcal{Y}_i \cap (X \times \{x_i\}) = \mathcal{Z}_i$$

$$\mathcal{Y}_i \cap (X \times \{y_i\}) = \mathcal{Z}_{i+1}.$$

Rewriting $\mathcal{Z}$ in the form

$$\mathcal{Z} = \sum_{i=1}^{k-1} (\mathcal{Z}_i - \mathcal{Z}_{i+1}),$$

we may assume by Lemma 2.6 that $k = 1$.

Consider the morphism $\phi_{\mathcal{Z}} : \mathbf{Q}\{d\} \to [X]$ in the category of Chow motives which corresponds to $\mathcal{Z}$. Then we have a factorization of the form

$$\begin{array}{ccccc} & \mathbf{Q}\{d\} & \xrightarrow{\phi} & [X] & \\ f & \searrow & & \nearrow & g \\ & & [C]\{d\} & & \end{array}$$

where f is the tensor product of the morphism $\mathbf{Q} \to [C]$ defined by the cycle $x_1 - y_1$ with the identity morphism of $\mathbf{Q}\{d\}$, and g is defined by the cycle $\mathcal{Y}_1$. Our statement follows now from Proposition 3.1, since the morphism $\phi_{\mathcal{Z}}^{\otimes N}$ which corresponds to the cycle $\mathcal{Z}^{\otimes N}$ on X^N can be factored through $f^{\otimes N}$. ■

Corollary 3.3. Let X be a smooth projective variety of dimension n, and let $\mathcal{Z}$ be a cycle of dimension n on $X \times X$ which is algebraically equivalent to zero. Then $\mathcal{Z}$ is nilpotent in the ring of End([X]) of correspondences from X to X modulo rational equivalence. □

Proof. This follows from Corollary 3.2 and Lemma 2.7. ■

Corollary 3.4. Let X be a smooth projective variety over a field k, and let $\mathcal{Z}$ be a correspondence from X to itself which is algebraically equivalent to the identity correspondence. Then $\mathcal{Z}$ induces automorphisms on the Chow groups $A_n(X)$ of cycles on X modulo rational equivalence. □

4 The nilpotence conjecture and theories of motivic type

Consider again the category Chow$(K, \mathbf{Q})$ of Chow motives over a field k with rational coefficients. This category is far from being abelian. It was Grothendieck's idea to obtain an abelian (and in fact abelian and semisimple) category out of it in the following way.

By duality, we have

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(\mathbf{Q}, X^* \otimes Y)$$

$$\mathrm{Hom}(Y, X) = \mathrm{Hom}(\mathbf{Q}, Y^* \otimes X) = \mathrm{Hom}(X^* \otimes Y, \mathbf{Q}).$$

Since $\mathrm{Hom}(\mathbf{Q}, \mathbf{Q}) = \mathbf{Q}$, it gives us a pairing

$$(-,-) : \mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, X) \to \mathbf{Q}.$$

We say that a morphism $f : X \to Y$ is numerically equivalent to zero if, for any $g : Y \to X$, one has $(f, g) = 0$. Denote by $\mathrm{Hom}_{\mathrm{num}}(X, Y)$ the quotient group of $\mathrm{Hom}(X, Y)$ with respect to the subgroup of morphisms numerically equivalent to zero. One can verify easily that there is a rigid tensor additive category with the same class of objects as $\mathrm{Chow}(k, \mathbf{Q})$ and morphisms given by the groups $\mathrm{Hom}_{\mathrm{num}}(-,-)$. Its pseudoabelian envelope is called the *category of Grothendieck motives* over k with coefficients in $\mathbf{Q}$. We denote this category by $\mathrm{GM}(k, \mathbf{Q})$. The following result is due to U. Jannsen [5].

Theorem 4.1. For any field k, the category $\mathrm{GM}(k, \mathbf{Q})$ is a semisimple abelian category. □

The following *nilpotence conjecture* relates the category of Chow motives and the category of Grothendieck motives in a different way.

Conjecture 4.2. Let k be a field. A morphism $f : X \to Y$ in $\mathrm{Chow}(k, \mathbf{Q})$ is numerically equivalent to zero if and only if it is smash nilpotent. □

Note that this conjecture is formulated in purely algebrogeometrical terms. In particular, it does not refer to any specific (co-)homology theories on algebraic varieties. On the other hand, it would clearly imply the Grothendieck standard conjecture, which says that numerical equivalence on cycles coincides with homological equivalence [4].

Another conjecture which would follow from Conjecture 4.2 is the Bloch conjecture on zero cycles on surfaces with $p_g = 0$ (see [1]), which says that, for a smooth projective surface X over an algebraically closed field k such that $H^2(X)$ is generated by classes of divisors, the Albanese kernel $A_0(X) \to \mathrm{Alb}(X)(k)$ is zero. In fact, it is not hard to show that under our assumptions on X, there exists a correspondence $\mathcal{Z}$ from X to itself such that the following two conditions hold:

(1) $\mathcal{Z}$ is numerically equivalent to the identity correspondence.

(2) $\mathcal{Z}$ acts trivially on the Albanese kernel $\ker(A_0(X) \to \mathrm{Alb}(X)(k))$.

If Conjecture 4.2 were true, it would imply that $\mathcal{Z}$ acts on Chow groups by automorphisms and in particular that the Albanese kernel is zero.

The results of Section 3 show that, to prove the nilpotence conjecture, it is sufficient to show that, for a cycle $\mathcal{Z}$ on a smooth projective variety X which is numerically equivalent to zero, there exists an N such that $\mathcal{Z}^{\otimes N}$ is algebraically equivalent to zero on X^N.

It makes sense to believe that the nilpotence conjecture is true, because it would follow from the existence of a theory of mixed motives. To make this statement a little more precise, we need the following definition.

Let k be a field. We fix an algebraic closure $\bar{k}$ of k and denote, for any scheme X over k by $\bar{X}$, the scheme $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$.

A theory of motivic type (with rational coefficients) over k is the following collection of data:

(1) a Tannakian category $\underline{C}$ over $\mathbf{Q}$ (see [3]);

(2) a contravariant functor $M\colon \mathrm{Sch}/k \to D^b(\underline{C})$ from the category of schemes of finite type over k to the derived category of bounded complexes over $\underline{C}$;

(3) for any prime l not equal to the characteristic of k, an exact tensor functor
$F_l : \underline{C} \to \mathbf{Q}_l - \mathrm{Vect}$
from $\underline{C}$ to the category of $\mathbf{Q}_l$-vector spaces together with natural isomorphisms of $\mathbf{Q}_l$-vector spaces
$\psi_X : H^i_{\acute{e}t}(\bar{X}, \mathbf{Q}_l) \to H^i(M(X)) \otimes \mathbf{Q}_l$
given for all schemes X of finite type over k;

(4) for any scheme of finite type X and a closed covering $X = X_1 \cup X_2$, a morphism
$M(X_1 \cap X_2) \to M(X)[1]$,
which is compatible by means of isomorphisms ψ_X with the corresponding standard homomorphisms on étale cohomology;

(5) for any two schemes of finite type over k, an isomorphism
$M(X \times Y) \to M(X) \otimes M(Y)$
compatible in the obvious sense with Kunneth isomorphisms in étale cohomology.

It can be shown that any theory of motivic type can be extended to a tensor triangulated functor from the triangulated category of mixed motives $DM_{gm}(k)$ to $D^b(\underline{C})$. (See [7] for a construction of $DM_{gm}(k)$.) For any theory of motivic type over k, denote by $\mathbf{Q}(-1)_{(M,\underline{C})}$ the object $H^2(M(\mathbf{P}^1_k))$ of $\underline{C}$. Let, further, $\mathbf{Q}(n)_{(M,\underline{C})} = (\mathbf{Q}(1)_{(M,\underline{C})})^{\otimes n}$. Then, for any theory of motivic type, there are characteristic classes

$$c_{ij} : K_i(X) \to \mathrm{Hom}_{D^b(\underline{C})}(\mathbf{Q}(-j)[i-2j], M(X)).$$

In particular, for an equidimensional smooth projective variety X of dimension d, we get a cycle map

$$A^i(X) = A_{d-i}(X) \to \mathrm{Hom}_{D^b(\underline{C})}(\mathbf{Q}(-i)[-2i], M(X)).$$

Definition 4.3. A theory of motivic type $(M, \underline{C})$ over k is called a theory of mixed motives over k if, for any smooth variety X over k, the Chern characters

$$ch_i : K_i(X) \to \mathrm{Hom}_{D^b(\underline{C})}(\oplus_{j=0}^{i+\dim(X)} \mathbf{Q}(-j)[i-2j], M(X))$$

associated with the corresponding characteristic classes are isomorphisms.

Remark. Note that, in particular, for a theory of mixed motives $(M, \underline{C})$, we have

$$A^i(X) \otimes \mathbf{Q} = \mathrm{Hom}_{D^b(\underline{C})}(\mathbf{Q}(-i)[-2i], M(X)).$$

At the present moment, it is quite unclear whether a theory of mixed motives over any field exists or not. What can be proven is that, if such a theory exists, then it is unique up to a canonical equivalence. Moreover, the corresponding category $D^b(\underline{C})$ is equivalent in this case to the triangulated category of mixed motives $DM_{gm}(k) \otimes \mathbf{Q}$ constructed in [7].

On the other hand, even if the theory of mixed motives does not exist, existence of theories of motivic type with certain properties would imply many of the "standard conjectures" on algebraic cycles. Let us consider the following two examples—the second of them shows, in particular, that the existence of mixed motives would imply the nilpotence conjecture.

Proposition 4.4. Suppose that for a field k there is a theory of motivic type $(M, \underline{C})$ such that the cycle maps

$$A^i(X) \to \mathrm{Hom}_{D^b(\underline{C})}(\mathbf{Q}(-i)[-2i], M(X))$$

are surjective. Then Grothendieck's Standard Conjecture B holds for varieties over k (see [4]). □

Proof. Note that by general properties of rigid tensor categories [3], the functors F_l are faithful. In particular, a morphism f in $D^b(\underline{C})$ is an isomorphism if and only if $F_l(f)$ is an isomorphism for some l.

We will need the following lemma.

Lemma 4.5. Let $(M, \underline{C})$ be a theory of motivic type. Then, for any smooth projective variety X over k, a hyperplane section H of X defines an isomorphism in $D^b(\underline{C})$ of the form

$$M(X) = \oplus_{i=0}^{2n} H^i(M(X))[-i].$$ □

Proof. By induction on $d = \dim(X)$, we may assume that this decomposition is already constructed for $M(H)$. Then, by the hyperplane section theorem, we get a decomposition

$$M(X) = \oplus_{i=0}^{d-1} H^i(M(X))[-i] \oplus N \oplus_{i=d+1}^{2d} H^i(M(X))[-i].$$

It is sufficient now to decompose N. By duality arguments, it is sufficient to construct a morphism $N(-1)[-2] \to N$ which induces an isomorphism

$$H^{d-1}(N)(1) \to H^{d+1}(N).$$

This morphism can be obtained from the morphism $M(X)(-1)[-2] \to M(X)$, which corresponds to the multiplication by the class of H. The fact that it induces the required isomorphism follows from the Deligne theorem [2]. ■

To finish the proof of the proposition, note that Lemma 4.5 implies, in particular, that the cycle map

$$A^i(X) \to \mathrm{Hom}_{\underline{C}}(\mathbf{Q}(-i), H^{2i}(M(X)))$$

is surjective. Applying again Deligne's theorem, we see that the morphisms $H^{n-j}(M(X)) \to H^{n+j}(M(X))$, given by multiplication by the jth power of the class of a hyperplane section H, are isomorphisms. Together with the fact that the cycle map in the above form is surjective, it implies that the inverse isomorphisms on cohomology groups take classes of algebraic cycles to classes of algebraic cycles, i.e., that the Standard Conjecture B holds for X. ■

Remark. Using the theorem of Jannsen, one can show that the opposite to Proposition 4.4 is true, i.e., that if we know that homological equivalence coincides with numerical equivalence, then one can construct a theory of motivic type with surjective cycle maps.

Proposition 4.6. Suppose that for a field k there is a theory of motivic type $(M, \underline{C})$ such that the cycle maps

$$A^i(X) \to \mathrm{Hom}_{D^b(\underline{C})}(\mathbf{Q}(-i)[-2i], M(X))$$

are surjective, and their kernels consist of smash nilpotent elements. Then the nilpotence conjecture holds for varieties over k. □

Proof. This follows immediately from Proposition 4.4 and the following simple lemma. ■

Lemma 4.7. Let $\underline{C}$ be a Tannakian category over a field E of characteristic zero, let X and Y be objects of the derived category of bounded complexes over $\underline{C}$, and let $f : X \to Y$ be a morphism in the derived category. Then it is smash nilpotent (i.e., $f^{\otimes N} = 0$ for some N) if and only if it is zero on cohomology objects of X and Y. □

Proof. Since $\underline{C}$ is a Tannakian category, there exists a field extension E' of E and an exact tensor functor

$$F : \underline{C} \to E' - \text{Vect},$$

which is faithful. Thus any smash-nilpotent morphism is zero on cohomology by obvious reasons. Suppose that $f : X \to Y$ is zero on cohomology objects. Using duality, we may assume that $X = \mathbf{1}$ is the unit object of $\underline{C}$. Our condition on f implies further that we may assume that $H_i(Y) = 0$ for $i < 1$ (here $H_i(-) = H^{-i}(-)$). Then there is a surjection $p : V \to \mathbf{1}$ such that $f \circ p = 0$, and therefore f can be factored through the first extension $f_0 : \mathbf{1} \to \ker(p)[1]$ given by V. It is sufficient to show that f_0 is smash nilpotent. One can easily see that, for any $n \geq 0$, the morphism

$$f_0^{\otimes n} : \mathbf{1} \to \ker(p)^{\otimes n}[n]$$

can be factored through the external product $\Lambda^n(\ker(p))$. Thus, for $n > \dim(F(\ker(p)))$, we have $f_0^{\otimes n} = 0$. The lemma is proven. ■

Remark. Proposition 4.6 shows that, to prove the nilpotence conjecture, it would be sufficient to construct a theory of motivic type $(M, \underline{C})$ such that, for any smooth projective variety over k, one of the following conditions holds:

(1) $\mathrm{Hom}_{D^b(\underline{C})}(M(X), \mathbf{Q}(-n)[-2n]) = A_n(X) \otimes \mathbf{Q}$;

(2) $\mathrm{Hom}_{D^b(\underline{C})}(M(X), \mathbf{Q}(-n)[-2n]) = B_n(X) \otimes \mathbf{Q}$ where $B_n(X)$ is the group of cycles of dimension n on X modulo algebraic equivalence.

In particular, it shows that, at least for some purposes, it would be enough to construct a "theory of mixed motives modulo algebraic equivalence." We will discuss how such a theory should look in another paper.

References

[1] S. Bloch, *Lectures on Algebraic Cycles*, Duke Univ. Mathematics Series **4**, Duke Univ. Dept. of Mathematics, Durham, NC, 1980.

[2] P. Deligne, *La conjecture de Weil II*, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 5–136.

[3] P. Deligne and J. S. Milne, "Tannakian categories" in *Hodge cycles*, *Motives and Shimura Varieties*, Lecture Notes in Math. **900**, Springer-Verlag, Berlin, 1982, 101–228.

[4] A. Grothendieck, "Standard conjectures on algebraic cycles" in *Algebraic Geometry*, *International Colloq.*, *Tata Inst.*, *Bombay 1968*, Oxford Univ. Press, London, 1969, 193–199.

[5] U. Jannsen, *Motives*, *numerical equivalence and semisimplicity*, Invent. Math. **107** (1992), 447–452.

[6] Yu. I. Manin, *Motifs and monoidal transformations*, Math. USSR-Sb. **6** (1968), 439–470.

[7] V. Voevodsky, *Triangulated categories of motives over a field*, preprint, 1994.

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138, USA