Chicago. Fall 1998.

Motivic homotopy type?

Motivic cohomology:

$$\begin{array}{ccc} Var/k & \longrightarrow & R\text{-mod} \\ \Psi & & \Psi \\ X & \longmapsto & H^{i,j}(X,R) \end{array}$$

Properties:

1. Mayer - Vietoris for open and closed coverings
2. $\mathbb{A}^1$ - homotopy invariance
3. Blow-up exact sequence

Th. 1 $H^{i,i}(\mathrm{Spec}\, k, R)$ is a quadratic algebra for any field k and coefficients ring R

Th. 2 $H^{i,i}(\mathrm{Spec}\, k, \mathbb{Z}/2) = H^i(k, \mathbb{Z}/2)$ for char $k \neq 2$.

Cor: $H^*(k, \mathbb{Z}/2)$ is a quadratic algebra.

Conjecture:

$$H^{i,j}(X, R) = 0$$

for $i < 0$ and any X, R.

Another approach to motivic cohomology

– categories $DM(X, R)$

<u>In topology</u>:

$H^i(X, R) = Hom(R, R[i])$

<u>In motives</u>:

$H^{i,j}(X, R) =$

$= Hom_{DM(X,R)}(R, R(j)[i])$

<u>Objects</u> $DM^{eff}(X, R)$

are complexes

$$0 \to \dots \to Y_n \to Y_{n-1} \to \dots \to 0$$

where Y_n are smooth algebraic varieties over X and differencials are finite correspondences over X with coef. R.

The complex

$$0 \to \mathbb{P}^1 \times X \to X \to 0$$

is called the Tate object and denoted $R(1)$.

The complex

$$0 \to X \to 0$$

denoted R or $R(0)$ is the unit of the obvious tensor structure

One sets:

$$R(n) \overset{df}{=} R(1)^{\otimes n}$$

and defines the triangulated category of Tate motives over X with coef. in R as the triangulated sub. cat. generated by $R(n)$'s.

Conjectrue: For any field k and a $\mathbb{Q}$-algebra R there exists a pro-algebraic group $Gal_{Tell}(k)$ ov.. R together with a distinguished representation $R(1): Gal \to \mathbb{G}_m$ such that the triangulated subcategory of $DM(\mathrm{Spec}\, k, R)$ generated by $R(n)$'s is equivalent to the derived category of representations of $Gal_{Tm}(k)$.

Conj:

$$D^b(\mathrm{Rep}\,\mathrm{Gal}_{\mathrm{Tu}}(k)) \xrightarrow{\ s\ } DT(\mathrm{Spec}\,k,\,R)$$

s $\otimes$, triang.

Cor. 1 $H^{i,i}(\mathrm{Spec}\,k, R)$ is quadratic

Cor. 2 $H^{i,j}(\mathrm{Spec}\,k, R) = 0$ for $i < 0$.

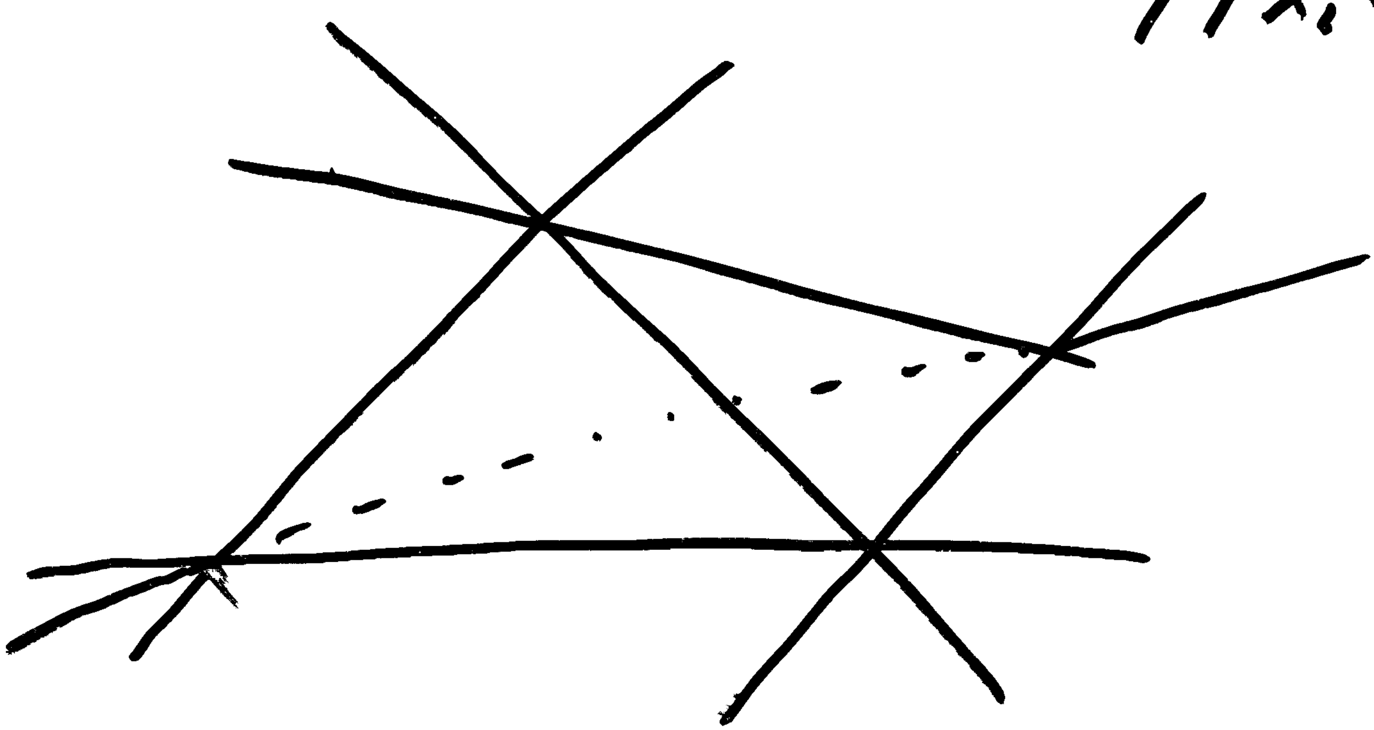
Conjecture <u>fails</u>

for general X or R which are not $\mathbb{Q}$-algebras.

Example:

$$X = \partial \Delta^3 =$$

$$= \operatorname{Spec} \mathbb{C}[x_0, \ldots x_3] \Big/ \begin{matrix} \sum x_i = 1 \\ \prod x_i = 0 \end{matrix}$$



$H^{i,j}(X, \mathbb{Q}) = 0$ for $j < 0$

(true for all X)

$$H^{i,0}(X, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ 0 & i = 1 \\ \mathbb{Q} & i = 2 \\ 0 & i \neq 0, 2 \end{cases}$$

(from closed M.-V. and $\mathbb{A}^1$-homotopy invariance)

Discrete groups

$\cap$

Pointed homotopy types

$$\boxed{G \longmapsto BG}$$

Algebraic groups

$\cap$

? ?

Models for algebraic homotopy types –

– commutative, associative cosimplicial algebras.

Discrete homotopy type represented by a simplicial set $X.$

$$\cdots\; X_2 \equiv X_1 = X_0$$

maps to the algebra of functions on $X.$ with values in R with pointwise multiplication. $\boxed{X_\cdot \longrightarrow R(X_\cdot)}$

Algebraic group G maps to the algebra of functions on the simplicial variety BG

$$\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; G \times G \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; G \rightrightarrows \bullet$$

$$\boxed{G \longmapsto \mathcal{O}(B_\bullet G)}$$

One can define trian-gulated category of modules over cosimpli-cial algebra such that :

$$D(\mathcal{O}(B.G)\text{-mod}) \cong D(\mathrm{Rep}\,G)$$

$$D(R(X.)) = D_{lc}(X.)$$

Conjecture: Let X be a variety over k and R a $\mathbb{Q}$-algebra. There exists $\mathcal{A}^{\bullet}_{\mathrm{Tw}}(k, R)$ such that

$$D(\mathcal{A}^{\bullet}_{\mathrm{Tw}}(k, R)) \cong DT(k, R)$$

In particular:

$$\bigoplus_{j \in \mathbb{Z}} H^{i,j}(X, R) =$$

$$= H^i(\mathcal{N}(\mathcal{A}^\bullet(X, R)))$$

All standard constructions give

$$\bigoplus_{i,j} H^{i,j} = \bigoplus H^i(\mathcal{N}(B_\bullet))$$

for simplicial $B_\bullet$!!