Homotopy Type Theory MPIM-Bonn 2016

Dependent Type Theories

Lecture 3. Presheaf extensions of C-systems. B-sets of C-systems and C-subsystems theorem.

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In the last lecture we have outlined, for each category T, the construction of functions

$$LwCs : Lw(T) \to Cs_{\mathbf{N}}(T^{op})$$

$$CsLw : Cs_{\mathbf{N}}(T^{op}) \to Lw(T)$$

and stated the lemma that they are mutually inverse bijections. From these functions we derived the functors

$$LC : LW(U) \to CS_{\mathbf{N}}(U)$$

$$CL : CS_{\mathbf{N}}(U) \to LW(U)$$

and, using the lemma, proves the theorem stating that these functors a mutually inverse isomorphisms of categories for any universe U.

This provides a description, in terms familiar to categorical logic, of the simplest class of C-systems - the l-bijective C-systems, i.e., the C-systems for which the length function $l : CC \to \mathbf{N}$ is a bijection.

There is a larger class of C-systems that can be described in similar terms.

Definition 1 *A C-system is called 1-generated if it coincides with its smallest subsystem that contains all objects of length* 1*.*

I expect to be able to construct, for any set S, an isomorphism between the category of S-sorted Lawvere theories and pairs of the form (CC, Φ) where CC is 1-generated C-system and $\Phi : Ob_1(CC) \to S$ a bijection between the set of objects of CC of length 1 and S.

This will provide a classical description for the class of 1-generated C-systems.

Intuitively, 1-generated C-systems correspond to type theories without dependent types. So their connection with more classical objects of categorical logic is not entirely unexpected.

We now proceed to the description of a construction that generates C-systems that are not 1-generated and takes us out of the realm of classical categorical logic. It is called *the presheaf extension of a C-system*.

Let CC be a C-system and $F : CC^{op} \to Sets$ a presheaf on the category underlying CC. We will construct a new C-system $CC[F]$ which we call the F-extension of CC.

We will first construct a C0-system $CC[F]$ and then show that it is a C-system.

Problem 2 *Given a C-system CC and a presheaf*

$$F : CC^{op} \to Sets$$

to construct a C0-system that will be denoted $CC[F]$ and called the F-extension of CC.

Construction 3

We set

$$Ob(CC[F]) = \amalg_{X \in CC} F(ft^{l(X)}(X)) \times \ldots \times F(ft^2(X)) \times F(ft(X)) \quad (1)$$

where the product of the empty sequence of factors is the one element set.

We will write elements of $Ob(CC[F])$ as (X, Γ) where $X \in CC$ and $\Gamma = (T_0, \ldots, T_{l(X)-1})$.

Note that $ft^{l(X)}(X) = pt$ for any X and therefore all the products in (1) start with $F(pt)$.

We set

$$Mor(CC[F]) = \amalg_{(X,\Gamma),(Y,\Gamma')} Mor_{CC}(X,Y)$$

We will write elements of $Mor(CC[F])$ as $((X,\Gamma),(Y,\Gamma'),f)$. When the domain and the codomain of a morphism are clear from the context we may write f instead of $((X,\Gamma),(Y,\Gamma'),f)$.

We define the composition function by the rule

$$((X,\Gamma),(Y,\Gamma'),f)) \circ ((Y,\Gamma'),(Z,\Gamma''),g) = ((X,\Gamma),(Z,\Gamma''),f \circ g)$$

and the identity morphisms by the rule

$$Id_{CC[F],(X,\Gamma)} = ((X,\Gamma),(X,\Gamma),Id_{CC,X})$$

The associativity and the identity conditions of a category follow easily from the corresponding properties of CC. This completes the construction of a category $CC[F]$.

We define the length function as

$$l((X,\Gamma)) = l(X)$$

If $l((X,\Gamma)) = 0$ then $X = pt_{CC}$ and $\Gamma = ()$ where $()$ is the unique element of the one point set that is the product of the empty sequence, i.e., $pt_{CC[F]} = ((pt_{CC},()))$.

We define the ft-function on (X,Γ) such that $l(X) > 0$ as

$$ft((X,(T_0,\ldots,T_{l(X)-1})) = (ft(X),(T_0,\ldots,T_{l(X)-2}))$$

which is well defined because $l(ft(X)) = l(X) - 1$. We will write $ft(\Gamma)$ for $(T_0,\ldots,T_{l(X)-2})$ so that $ft((X,\Gamma)) = (ft(X), ft(\Gamma))$.

We define the p-morphisms as

$$p_{(X,\Gamma)} = ((X,\Gamma), ft(X,\Gamma), p_X)$$

For (Y,Γ') such that $l((Y,\Gamma'))>0$ and $f:(X,\Gamma)\to ft(Y,\Gamma')$ where $\Gamma=(T_0,\dots,T_{l(X)-1})$ and $\Gamma'=(T'_0,\dots,T'_{l(Y)-1})$ we set

$$f^*((Y,\Gamma'))=(f^*(Y),(T_0,\dots,T_{l(X)-1},F(f)(T'_{l(Y)-1}))). \tag{2}$$

In the same context as above we define the q-morphism as

$$q(f,(Y,\Gamma'))=(f^*((Y,\Gamma')),(Y,\Gamma'),q(f,Y))$$

This completes the construction of the elements of the structure of a C0-system.

For the proof that they satisfy the axioms of a C0-structure see "C-system of a module over a Jf-relative monad."

Lemma 4 *The functions*

$$Ob(CC[F]) \to Ob(F)$$

$$Mor(CC(F)) \to Mor(CC)$$

given by

$$(X, \Gamma) \mapsto X$$

and

$$((X, \Gamma), (Y, \Gamma'), f) \mapsto f$$

form a functor $tr_F : CC[F] \to CC$ *and this functor is fully faithful.*

Proof: Straightforward from the construction.

Lemma 5 *The C0-system of Construction 3 is a C-system.*

Proof: By Proposition 3 from the first lecture it is sufficient to prove that the canonical squares of $CC[F]$, i.e., the squares

$$\begin{array}{ccc} f^*((Y,\Gamma')) & \xrightarrow{q(f,(Y,\Gamma'))} & (Y,\Gamma') \\ {\scriptstyle p_{f^*((Y,\Gamma'))}}\downarrow & & {\scriptstyle p_{(Y,\Gamma')}}\downarrow \\ (X,\Gamma) & \xrightarrow{f} & ft((Y,\Gamma')) \end{array}$$

are pull-back squares. The functor of Lemma 4 map these square to canonical squares of the C-system CC that are pull-back squares. Since this functor is fully faithful we conclude that the canonical squares in $CC[F]$ are pull-back squares. The lemma is proved.

This completes the construction of the presheaf extension of a C-system.

For any two objects of $CC[F]$ of the form $(X, \Gamma), (X, \Gamma')$ the formula

$$can_{X,\Gamma,\Gamma'} = ((X, \Gamma), (X, \Gamma'), Id_X)$$

defines a morphism which is clearly an isomorphism with $can_{X,\Gamma',\Gamma}$ being a canonical inverse. Therefore, all objects of $CC[F]$ with the same image in CC are "canonically isomorphic".

If $F(pt_{CC}) = \emptyset$ then $CC[F] = \{pt_{CC[F]}\}$. On the other hand, the choice of an element y in $F(pt_{CC})$ defines distinguished elements

$$y_X = F(\pi_X)(y)$$

in all sets $F(X)$ and therefore distinguished objects

$$(X, \Gamma_{X,y}) = (X, (y, \ldots, y_{ft^2(X)}, y_{ft(X)}))$$

in the fibers of the object component of tr_F over all X.

Mapping X to $(X, \Gamma_{X,y})$ and $f : X \to Y$ to $((X, \Gamma_{X,y}), (Y, \Gamma_{Y,y}), f)$ defines, as one can immediately prove from the definitions, a functor $tr^!_{F,y} : CC \to CC[F]$.

This functor clearly satisfies the conditions $tr^!_{F,y} \circ tr_F = Id_{CC}$.

One verifies easily that the morphisms

$$can_{X,\Gamma,\Gamma_{(X,y)}} : (X, \Gamma) \to tr^!_{F,y}(X, \Gamma)$$

form a natural transformation. We conclude that tr_F and $tr^!_{F,y}$ is a pair of mutually inverse equivalences of categories.

However these equivalences are not isomorphisms unless $F(X)$ is a one element set for all X and as a C-system $CC[F]$ is often very different from CC, for example, it may have many more C-subsystems.

The proofs of the following two lemmas are straightforward:

Lemma 6 *The functor* $tr : CC[F] \to CC$ *is a homomorphism of C-systems.*

Lemma 7 *For any* $y \in F(pt)$, *the functor* $tr_{F,y} : CC[F] \to CC$ *is a homomorphism of C-systems.*

Next we will explain a method for constructing subsystems of C-systems that leads us to a very important area of exploration - the theory of B-systems. A similar method exists for constructing sub-quotients but we will restrict our attention to the case to subsystems and refer to "Subsystems and regular quotients of C-systems" for the sub-quotients.

Let CC be a C-system. Define $B(CC)$ as $Ob(CC)$ and $\widetilde{B}(CC)$ as the subset in $Mor(CC)$ of the form:

$$\widetilde{B}(CC) =$$

$$\{s \in Mor(CC) \,|\, dom(s) = ft(codom(s))\, and\, s \circ p_{codom(s)} = Id_{dom(s)}\}$$

that is, elements of $\widetilde{B}(CC)$ are sections of the p-morphisms of CC.

The sets $B(CC)$ and $\widetilde{B}(CC)$ are called the B-sets of CC.

Note that $B(CC)$ is another notation for $Ob(CC)$ that we also abbreviate sometimes to CC. In some of my papers I write $\widetilde{Ob}(CC)$ instead of $\widetilde{B}(CC)$.

We let $\partial : \widetilde{B}(CC) \to B(CC)$ denote the function $s \mapsto codom(s)$ such that

$$s : ft(\partial(s)) \to \partial(s)$$

Define the relation $\geq$ on CC by the condition that $Y \geq X$ if and only if $l(Y) \geq l(X)$ and

$$ft^{l(Y)-l(X)}(Y) = X.$$

Define the relation $>$ on CC by the condition that $Y > X$ if and only if $Y \geq X$ and $l(Y) > l(X)$.

Lemma 8 *For any C-system CC one has*

1. *the relation $\geq$ is a partial order relation, i.e., it is reflexive, transitive and antisymmetric,*
2. *the relation $>$ is a strict partial order relation, i.e., it is transitive and asymmetric.*

An object Y is said to be an object over X if $Y \geq X$. In this case the composition of the canonical projections $Y \stackrel{p_Y}{\to} ft(Y) \stackrel{p_{ft(Y)}}{\to} \ldots \to X$ is denoted by $p_{Y,X}$.

For a morphism $f : X' \to X$ one defines $f^*(Y)$ by induction using the f^* structure of the C-system. One also defines by induction a morphism $q(f, Y) : f^*(Y) \to Y$.

For $Y, Y' \geq X$ a morphism $g : Y \to Y'$ is said to be a morphism over X if $p_{Y,X} = g \circ p_{Y',X}$. For such a morphism g and a morphism $f : X' \to X$ there is a unique morphism $f^*(g) : f^*(Y) \to f^*(Y')$ over X' such that the square

$$\begin{array}{ccc} f^*(Y) & \xrightarrow{q(f,Y)} & Y \\ f^*(g)\downarrow & & \downarrow g \\ f^*(Y') & \xrightarrow{q(f,Y')} & Y' \end{array}$$

commutes.

Consider the following sets where we write B and $\widetilde{B}$ instead of $B(CC)$ and $\widetilde{B}(CC)$:

$$T_{dom} \subset B \times B \qquad T_{dom} = \{X, Y \in B, l(X) > 0, Y > ft(X)\})$$

$$\widetilde{T}_{dom} \subset B \times \widetilde{B} \qquad \widetilde{T}_{dom} = \{X \in B, s \in \widetilde{B}, (X, \partial(s)) \in T_{dom}\}$$

$$S_{dom} \subset \widetilde{B} \times B \qquad S_{dom} = \{r \in \widetilde{B}, Y \in B, Y > \partial(r)\}$$

$$\widetilde{S}_{dom} \subset \widetilde{B} \times \widetilde{B} \qquad \widetilde{S}_{dom} = \{r, s \in \widetilde{B}, (r, \partial(s)) \in S_{dom}\}$$

$$\delta_{dom} \subset B \qquad \delta_{dom} = \{X \in B, l(X) > 0\}$$

Consider now the following operations defined on these sets

$$T : T_{dom} \to B \qquad T(X, Y) = p_X^*(Y)$$

$$\widetilde{T} : \widetilde{T}_{dom} \to \widetilde{B} \qquad \widetilde{T}(X, s) = p_X^*(s)$$

$$S : S_{dom} \to B \qquad S(r, Y) = r^*(Y)$$

$$\widetilde{S} : \widetilde{S}_{dom} \to \widetilde{B} \qquad \widetilde{S}(r, s) = r^*(s)$$

$$\delta : \delta_{dom} \to \widetilde{B} \qquad \delta(X) = s_{Id_X}$$

Operation T is well defined because for $(X, Y) \in T_{dom}$ we have $Y > ft(X)$ and therefore Y is over $ft(X)$.

Operation $\widetilde{T}$ is well defined because

$$s : ft(\partial(s)) \to \partial(s)$$

is a section of $p_{\partial(s)}$ and therefore a morphism over $ft(\partial(s))$. On the other hand for $(s, X) \in \widetilde{T}_{dom}$, one has $\partial(s) > ft(X)$ which implies that $ft(\partial(s)) \geq ft(X)$ and therefore $ft(\partial(s))$ is an object over $ft(X)$ and so the morphism s is a morphism over $ft(X)$.

Similar arguments show that S, $\widetilde{S}$ and δ are well defined. For more detail see the upcoming updated version of "B-systems".

Given a C-subsystem CC' of CC let

$$B(CC') = Ob(CC')$$

and

$$\widetilde{B}(CC') = Mor(CC') \cap \widetilde{B}(CC).$$

Theorem 9 *The mapping $CC' \mapsto (B(CC'), \widetilde{B}(CC'))$ defines a bijection between C-subsystems of CC and pairs of subsets $(B', \widetilde{B}')$ in the B-sets of CC that are closed under operations $ft, \partial, T, \widetilde{T}, S, \widetilde{S}, \delta$ and such that B' contains pt_{CC}.*

For the proof see “Subsystems and regular quotients of C-systems”.