Coming up

Homotopy Theory of Algebraic Varieties

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Homotopy theory of algebraic varieties or $\mathbb{A}^1$-homotopy theory

- Algebraic singular homology
- $\mathbb{A}^1$-homotopy category over a field
- Algebraic singular homology and etale (or Galois) cohomology

Milnor - Bloch - Kato - Beilinson - Lichtenbaum Conjecture

- $\mathrm{Aff}/k = (\mathrm{Alg}^{ft}/k)^{op}$

- $$\begin{array}{ccc} \mathrm{Alg}^{ft}/_k & \longrightarrow & \mathrm{Aff}/_k \\ \Psi & & \Psi \\ R & \longmapsto & \mathrm{Spec}(R) \end{array}$$

- $R = k[x_1, \dots x_n] \big/ (f_1, \dots f_m)$

- $\mathbb{A}^n = \mathrm{Spec}\ k[x_1, \dots x_n]$

- $\mathrm{Spec}\, R \hookrightarrow \mathbb{A}^n$

- $\operatorname{Spec} k$ – the final object, the "point"

- $\operatorname{Spec} k[x_1, \dots x_n] = \mathbb{A}_k^n$ is called the affine space over k

- $\operatorname{Spec} k[x] = \mathbb{A}_k^1$ is called the affine line over k.

• if $k = \mathbb{C}$ we have the functor of $\mathbb{C}$-points

$$\begin{array}{ccc} \mathrm{Aff}^{ft}/\mathbb{C} & \longrightarrow & \mathrm{Top} \\ \Psi & & \Psi \\ V & \longmapsto & V(\mathbb{C}) \end{array}$$

• $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$

if $f_1(t_1, \ldots t_n), \ldots f_m(t_1, \ldots, t_n)$ is a collection of polynomials and

$$R = k[t_1, \ldots t_n] / (f_1, \ldots f_m)$$

then $(\mathrm{Spec}\, R)(\mathbb{C})$ is the space of solutions of the system of eq.

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_m = 0 \end{cases} \quad \text{over } \mathbb{C}.$$

Cohomology problem:

given a finitely generated algebra A over $\mathbb{C}$ find a purely algebraic descr. of the cohomology groups $H^i((\mathrm{Spec}\, A)(\mathbb{C}), \mathbb{Z}/n)$

Solved by A. Grothendieck in the 60-ies in the theory of etale cohomology.

<u>Main drawback</u>:

gives meaningless answers for rational coefficients

$$H^i_{et}(V, \mathbb{Q}) = 0$$

for $i > 0$ and any smooth algebraic variety V.

Etale cohomology -

- algebro-geometrical version of Cech cohomology for topological spaces

Algebraic singular homology -

- algebro geometrical version of "Dold-Thom homology" for topological spaces

- T – topological space
- $\mathrm{Symm}^n T = T^n / \Sigma_n$
- $\mathrm{Hom}(X, \coprod_{n \geq 0} \mathrm{Symm}^n T)$

abelian monoid for any X. Denote by $\mathrm{Hom}(X, \coprod_{n \geq 0} \mathrm{Symm}^n T)^+$ associated abelian group

- $\Delta^i_{top} = \{(x_0, \dots x_n) : \sum x_i = 1, \; x_i \geq 0\}$

standard simplex

$$\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
\mathrm{Hom}\left(\Delta^{i}_{top}, \coprod_{n\geq 0} \mathrm{Symm}^n T\right)^{+} & = & C_i^{DT}(T) \\
\downarrow & & \downarrow \\
\mathrm{Hom}\left(\Delta^{i-1}_{top}, \coprod_{n\geq 0} \mathrm{Symm}^n T\right) & = & C_{i-1}^{DT}(T) \\
\downarrow & & \\
\vdots & &
\end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{C_*^{DT}(T)}$$

<u>Theorem</u>: $H_i(C_*^{DT}(T)) =$

$$= H_i^{sing}(T, \mathbb{Z})$$

• $V = \operatorname{Spec} A$

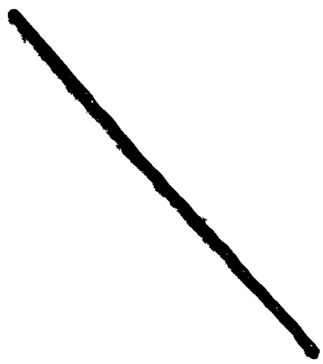
$$A = \mathbb{C}[t_1, \dots t_K] \Big/ \begin{matrix} f_1 = 0 \\ \vdots \\ f_m = 0 \end{matrix}$$

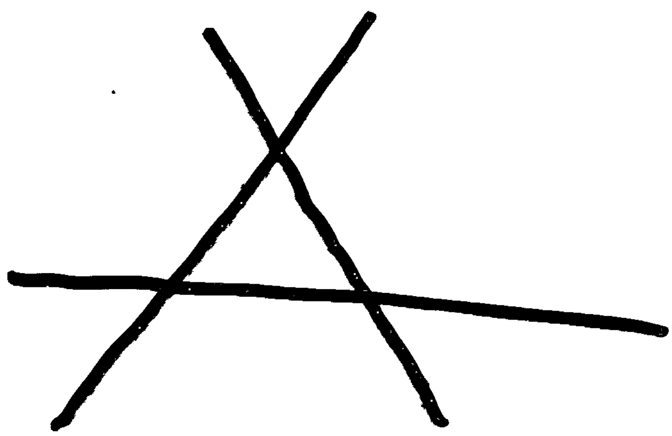
- affine algebraic variety over $\mathbb{C}$ given by the equations $\begin{cases} f_1(t_1, \dots t_K) = 0 \\ \vdots \\ f_m(t_1, \dots t_K) = 0 \end{cases}$

• $\operatorname{Symm}^n V = \operatorname{Spec} (A^{\otimes n})^{\Sigma_n}$

• $\Delta^i_{alg} = \operatorname{Spec} \mathbb{C}[x_0, \dots x_n] \Big/ \sum x_i = 1$

$$\Delta^0_{alg} = \bullet$$

$$\Delta^1_{alg} =$$ 

$$\Delta^2_{alg} =$$ 

$$\delta^i_j : \Delta^i_{alg} \hookrightarrow \Delta^{i+1}_{alg}$$

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Hom}\big(\Delta^i_{alg}, \coprod_{n\geq 0} \mathrm{Symm}^n V\big)^+ \arrow[d] \arrow[r, phantom, "="] & C_i(V) \arrow[d] \\
\mathrm{Hom}\big(\Delta^{i-1}_{alg}, \coprod_{n\geq 0} \mathrm{Symm}^n V\big)^+ \arrow[d] \arrow[r, phantom, "="] & C_{i-1}(V) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
$$

$$C_*^{alg}(V)$$

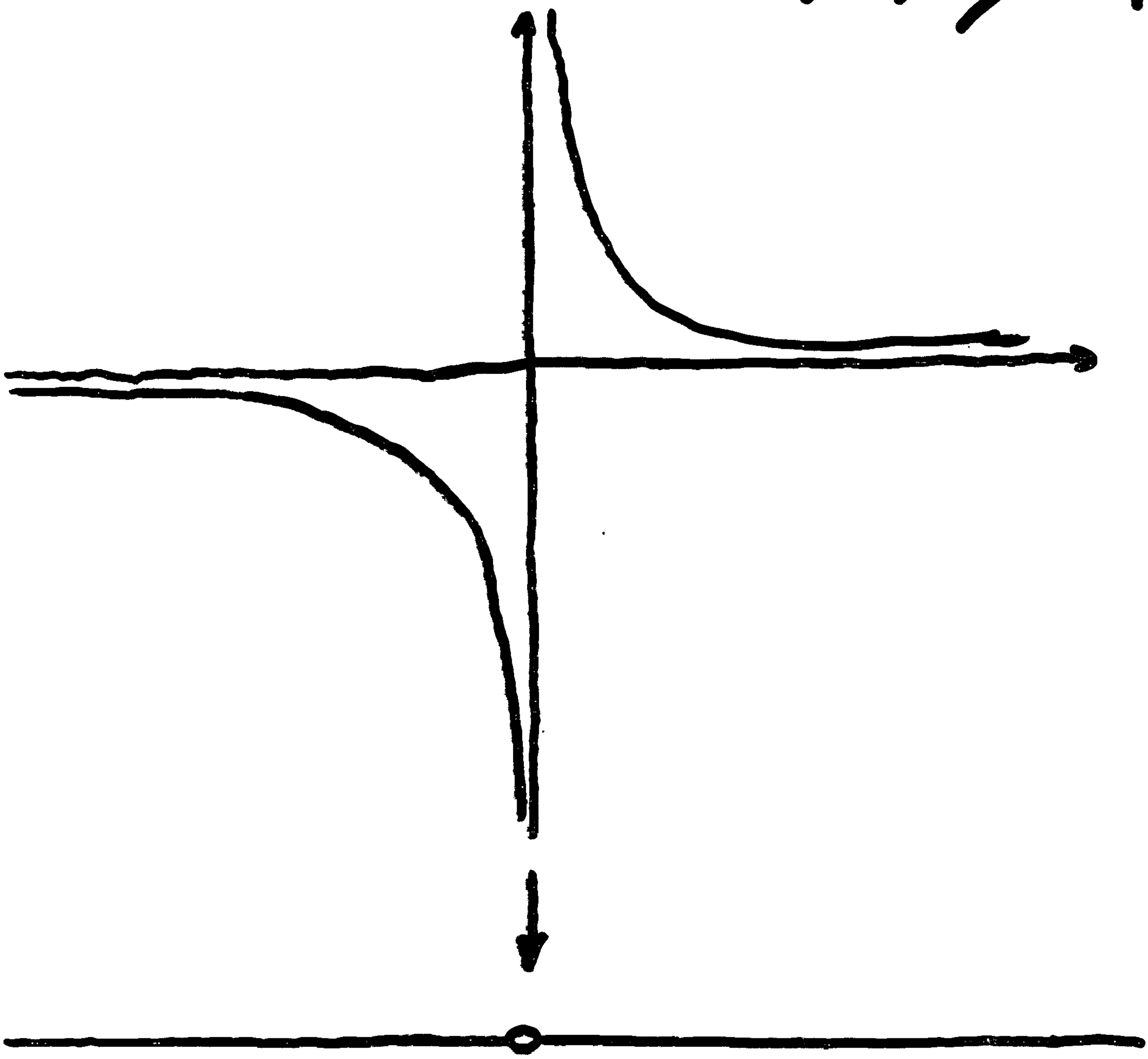
Def: $H_i^{alg}(V, \mathbb{Z}) = H_i(C_*^{alg}(V))$

$H_i^{alg}(V, A) = H_i(C_*^{alg}(V) \otimes A)$

Suslin homology of V

A computation:

$$V = \operatorname{Spec} \mathbb{C}[x,y]/_{xy=1}$$



$$V = \mathbb{A}^1 - \{0\}$$

$$\operatorname{Symm}^n V = \mathbb{A}^{n-1} \times (\mathbb{A}^1 - \{0\})$$

$$= \operatorname{Spec} \mathbb{C}[u_1, \ldots u_n, v_n]/u_n v_n = 1$$

Explicit computation gives:

$$H_i^{alg}(V, \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{C}^* & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Topologicaly

$$V(\mathbb{C}) = \mathbb{C}^* \approx S^1$$

$$H_i^{top}(V(\mathbb{C}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = 1 \\ 0 & i \neq 0, 1 \end{cases}$$

Trouble?

Universal coefficients long exact sequence

$$\to H_i(-,\mathbb{Z}) \to H_i(-,\mathbb{Z}) \to$$

$$\to H_i(-,\mathbb{Z}/\ell) \to H_{i-1}(-,\mathbb{Z}) \to \dots$$

gives:

$$H_i^{top}(V,\mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & i=0 \\ \mathbb{Z}/n & i=1 \\ 0 & i \neq 0,1 \end{cases}$$

$$H_i^{alg}(V,\mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & i=0 \\ \ker(\mathbb{C}^* \xrightarrow{\wedge n} \mathbb{C}^*) & i=1 \\ 0 & i \neq 0,1 \end{cases}$$

since $\ker = \mathbb{Z}/n$ they <u>agree</u>.

• $H_i^{alg}(V \times V, \mathbb{Z}) = ?$

<u>Th</u> (Suslin, Voevodsky)

$$H_0(V \times V) = \mathbb{Z} \oplus 2k^* \oplus K_2^M(k)$$

where:

$$K_2^M(k) = k^* \otimes_{\mathbb{Z}} k^* \Big/ \{x \otimes y : x + y = 1\}$$

~~Cor (Suslin)~~

$\cancel{H_0(V \times V, \mathbb{Z}/n) = \mathbb{Z}/n}$

<u>Conjecture</u>: $H_i(V \times V, \mathbb{Z}) = 0$ for $i \geq 2$.

Theorem (Suslin & Voevodsky)

For any algebraic variety V over $\mathbb{C}$ and any $n>0$ $i \geq 0$ there is a canonical isomorphism

$$H_i^{top}(V(\mathbb{C}), \mathbb{Z}/n) = H_i^{alg}(V, \mathbb{Z}/n)$$

See: A. Suslin, V. Voevodsky
Singular homology of abstract algebraic varieties
Inv. Math. v. 123 no. 1
pp. 61 - 94 1992.

- $H_i(T,\mathbb{Z}) = \pi_i\left(\left(\coprod_{n\geq 0} \mathrm{Symm}^n T\right)^+\right)$

$$= \mathrm{Hom}_{H_\bullet}\left(S^i, \left(\coprod_{n\geq 0} \mathrm{Symm}^n T\right)^+\right)$$

- $H^i(T,\mathbb{Z}) = \mathrm{Hom}_H\left(T, K(\mathbb{Z},i)\right)$

$$= \mathrm{Hom}_{H_\bullet}\left(T_+, \left(\coprod_{n\geq 0} \mathrm{Symm}^n S^i\right)^+\right)$$

Homotopy theory of algebraic varieties or $\mathbb{A}^1$-homotopy theory provides:

- a category $Spc(k)$ of "spaces" over k and a functor $Var/k \longrightarrow Spc(k)$
- a class of morphisms in $Spc(k)$ called $\mathbb{A}^1$-weak equivalences
- $H = Spc / W_{\mathbb{A}^1}$

Main properties of spaces:

- any diagram has limit and colimit

- for any X, Y there is a function space $\underline{Hom}(X, Y)$ such that for any V one has
$$Hom(V, \underline{Hom}(X, Y)) =$$
$$= Hom(V \times X, Y)$$

- for smooth alg. var X, Y
$$Hom_{Var}(X, Y) = Hom_{Spc}(X, Y)$$

Standard constructions:

- for $A \hookrightarrow X$ define X/A as the colimit of

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \\ pt & \end{tikzcd}$$

- for $A, B \hookrightarrow X$ define $A \cup B$ as the colimit of

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] & A \\ B & \end{tikzcd}$$

- for $(X,x), (Y,y) \in \mathrm{Spc}_*$

$$(X,x) \wedge (Y,y) = X \times Y \big/ (x \times Y) \cup (X \times y)$$

Properties of weak equivalences:

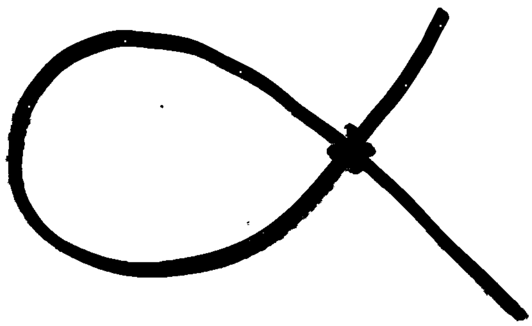
- $\mathrm{Spec}\, k \xrightarrow{\delta_0^0} \Delta^1_{alg}$ is a weak equivalence
- if $f: X \to Y$ is a weak equivalence then $f \times Id : X \times V \to Y \times V$ is a weak equivalence for any V
- given

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

if f - w.eq. and i - mono then g - w. eq.

Two circles:

- $S^1_s = \mathbb{A}^1 / \{0\} \cup \{1\} = \Delta^1_{alg} / \delta^0_0 \cup \delta^0_1$



- $S^1_t = \operatorname{Spec} k[x,y]/_{xy=1} = \mathbb{A}^1 - \{0\}$

- $S^n_t = (S^1_t)^{\wedge n} \qquad S^n_s = (S^1_s)^{\wedge n}$

Examples of weak equivalences:

- $SL_2 \cong S^1_s \wedge S^2_t$

- $\mathbb{P}^n / \mathbb{P}^{n-1} \cong S^n_s \wedge S^n_t$

- $\mathbb{A}^n - \{0\} \cong S^{n-1}_s \wedge S^n_t$

<u>Th.</u> $H_i^{alg}(V, \mathbb{Z}) =$

$$= \mathrm{Hom}_{H_0}\left(S^i_s, \left(\coprod_{n \geq 0} \mathrm{Symm}^n V\right)^+\right)$$

- $L(V) = \left(\coprod_{n\geq 0} \mathrm{Symm}^n V\right)^+$

- for $X = V/(\cup Z_i)$

set

$$L(X) = \left(L(V) / \sum L(Z_i) \right)_{ab}$$

<u>Def</u>: $H_i^{alg}(X, Z) =$

$$= \mathrm{Hom}_{H_0}\left(S_s^i, L(X)\right)$$

For any field k of characteristic zero there are canonical homomorphisms

$$H_i^{alg}(S_t^n, \mathbb{Z}/\ell) \to H_{et}^{n-i}(k, \mu_\ell^{\otimes n})$$

<u>Conjecture</u> (Beilinson - Lichtenbaum)

These homomorphisms are isomorphisms for all $i \geq 0$.

<u>Theorem</u> (Voevodsky)

Beilinson - Lichtenbaum conjecture holds for $\ell = 2$.

Th $H_0^{alg}(S_t^n, \mathbb{Z}/\ell) = K_n^M(k)/\ell$

where

$$K_n^M(k) = k^{\otimes n} / \{x_1 \otimes \dots \otimes x_n \text{ s.t. } x_i + x_j = 1 \text{ for some } i \neq j\}$$

Cor: $H^n_{et}(k, \mathbb{Z}/2) = K_n^M(k)/2$

Cor: For any k, char $k \neq 2$ the ring $H^*_{et}(k, \mathbb{Z}/2)$ is quadratique i.e. generated by H^1 with relations in H^2.