ÉTALE TOPOLOGIES OF SCHEMES OVER FIELDS OF FINITE TYPE OVER $\mathbf{Q}$

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ABSTRACT. The author proves a conjecture of Grothendieck concerning the possibility of recovering a normal scheme over a field of finite type over $\mathbf{Q}$ from its étale site.

Introduction

This work is devoted to proving a conjecture of Grothendieck which he made in a letter to Gerd Faltings (1983) and in an unpublished paper entitled "Esquisse d'un programme" (1984). Roughly speaking, he claims that in the case of normal schemes of finite type over finitely generated fields of characteristic zero, all of the information about a scheme is contained in its étale topology.

With some modifications, our proof seems to apply to schemes over finitely generated fields of characteristic $p > 0$ which have transcendence degree ≥ 1, but the proof does not go through for finite fields.

I would like to thank G B. Shabat for directing my attention to the remarkable paper "Esquisse d'un programme," and I would like to thank the participants in I. R. Shafarevich's seminar for their interest in my work.

§1. The étale site of a scheme

In this section we give the basic definitions which will be needed later, and we prove some elementary properties of the étale topology of a scheme.

The basic notion I shall be working with is that of a *site*. A completely elementary exposition of the concepts connected with sites can be found in [1] (see also [2] and [3]).

DEFINITION 1.1. A *site* is a category with fibered products in which families $\{U_i \to U\}$ of morphisms have been distinguished, each family having the same target object; these families are called *coverings*. Here the following conditions must be fulfilled:

1) Any isomorphism is a covering.
2) The coverings are invariant under base change.
3) If $\{U_i \to U\}$ is a covering and if $\{V_{ij} \to U_i\}$ is a covering for each i, then $\{V_{ij} \to U\}$ is a covering.

The objects of the underlying category of a site T are called the *open sets* of T.

DEFINITION 1.2. Let X and Y be two sites. A *morphism* (or *continuous map*) $\varphi: X \to Y$ is a functor φ^{-1} from the underlying category of Y to the underlying category of X which preserves fibered products and takes coverings to coverings.

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A *2-morphism* from a continuous map $\varphi: X \to Y$ to a continuous map $\psi: X \to Y$ is a morphism of functors $\varphi^{-1} \to \psi^{-1}$.

The sites, the morphisms of sites and the 2-morphisms of sites obviously form a 2-category, which we denote GTop.

Let X be a scheme. We let $X_{\text{ét}}$ denote the étale site of X. The open set of $X_{\text{ét}}$ are the étale schemes which are separated and of finite type over X, and the morphisms are the morphisms of schemes over X. A family $\{U_i \to U\}$ is a covering if the images of the U_i cover U. A morphism of schemes $\varphi: X \to Y$ determines a morphism $\varphi_{\text{ét}}: X_{\text{ét}} \to Y_{\text{ét}}$ according to the rule

$$\varphi_{\text{ét}}^{-1}(U \to Y) = (U \times_{\varphi} X \to X).$$

This construction obviously gives a functor ét from the category of schemes to the 2-category of sites.

If T is a site and U is open in T, then one can define the the site $U_{\text{ét}}$ which is the "restriction" of T to U. The open sets of $U_{\text{ét}}$ are the morphisms $V \to U$ in T, the morphisms of $U_{\text{ét}}$ are the commutative triangles

$$\begin{array}{ccc} V & \to & V' \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

and a family $\{V_i \to V\}$ is a covering in $U_{\text{ét}}$ if and only if it is a covering in T.

The rule $V \to (V \times U \to U)$ determines a continuous map $U_{\text{ét}} \to T$. If $\varphi: U_1 \to U_2$ is a morphism in T, then one can analogously define a continuous map $\varphi_{\text{ét}}: (U_1)_{\text{ét}} \to (U_2)_{\text{ét}}$. Given a scheme X, we let $T(X)$ denote the underlying topological space.

PROPOSITION 1.1. *Let X and Y be schemes, and let $\varphi: X_{\text{ét}} \to Y_{\text{ét}}$ be a morphism of étale sites. There exist a unique continuous map $T(\varphi): T(X) \to T(Y)$ such that, if $U \subset T(Y)$ is an open set, then*

$$T(\varphi)^{-1}(U) = \operatorname{Im} \varphi^{-1}(U). \tag{1.1}$$

(*On the right U with the induced scheme structure is regarded as an open set $Y_{\text{ét}}$.*)

PROOF. We shall only give the construction of the map $T(\varphi)$. The verification that (1.1) holds, and that $T(\varphi)$ is unique, is a simple exercise in general topology. The basic role in constructing $T(\varphi)$ is played by the following well-known fact: for any scheme X, the map $x \to \{\overline{x}\}$ gives a one-to-one correspondence between points and irreducible closed subsets of $T(X)$.

Given a point $x \in T(X)$, we set

$$T(\varphi)(x) = \left(\text{generic point } T(Y) - \bigcup_{\varphi^{-1}(U) \subset T(X) - \{\overline{x}\}} U \right)$$

(of course, one must check that the set on the right is irreducible). If X and Y are sites and $\varphi: X \to Y$ is a continuous map, then for every open set U in Y the rule

$$(V \to U) \to (\varphi^{-1}(V) \to \varphi^{-1}(U))$$

determines a continuous map $\varphi_U: \varphi^{-1}(U)_{\text{ét}} \to U_{\text{ét}}$, the "restriction" of φ to $\varphi^{-1}(U)$. This along with Proposition 1.1 shows that a continuous map φ of étale sites of schemes determines a compatible system of continuous maps of topological spaces $T(\varphi^{-1}(U)) \to T(U)$ for all open U in $Y_{\text{ét}}$.

In what follows we shall be primarily interested in the relative case, i.e., the 2-category of sites over a certain base. The definition here is not completely trivial:

DEFINITION 1.3. Let T be a site. The *2-category of sites over* T is the 2-category GTop/T of the following form:

(a) The objects of GTop/T are the morphisms $(X \to T)$ in GTop.

(b) The morphisms from $p_X\colon X \to T$ to $p_Y\colon Y \to T$ are pairs $\varphi = (\varphi^{-1}, \alpha)$, where φ^{-1} is a functor from the underlying category of Y to the underlying category of X corresponding to a morphism $X \to T$ in GTop, and $\alpha\colon p_X^{-1} \to \varphi^{-1} \circ p_Y^{-1}$ is an isomorphism of functors;

(c) A 2-morphism $(\varphi_1^{-1}, \alpha_1) \to (\varphi_2^{-1}, \alpha_2)$ is a morphism of functors $\varphi_1^{-1} \to \varphi_2^{-1}$ such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{\alpha_1}{\nearrow} & \varphi_1^{-1} \circ p_Y^{-1} \\ p_X^{-1} & & \downarrow \\ & \underset{\alpha_2}{\searrow} & \varphi_2^{-1} \circ p_Y^{-1} \end{array}$$

If $X \to S$ is a scheme over S, then in the obvious way one associates to it the étale site over $S_{\text{ét}}$:

$$(p_X)_{\text{ét}}\colon X_{\text{ét}} \to S_{\text{ét}}.$$

This correspondence extends to a functor from the category S_{sch}/S of schemes over S to the 2-category $\mathrm{GTop}/S_{\text{ét}}$ of sites over $S_{\text{ét}}$.

If $\varphi \in \mathrm{Mor}_S(X, Y)$ and $\varphi_{\text{ét}} = (\varphi^{-1}, \alpha)$, then α is determined as follows. For V an open set in $S_{\text{ét}}$ we have

$$p_X^{-1}(V) = (V \times X \to X), \qquad \varphi^{-1} \circ p_Y^{-1}(V) = ((V \times Y) \times_\varphi X \to X).$$

The isomorphism $\alpha_V\colon V \times X \to (V \times Y) \times_\varphi X$ is $((\mathrm{pr}_V \times (\varphi \circ \mathrm{pr}_X)) \times \mathrm{pr}_X)$. From now on we will be working over a field K, which we assume to be perfect. Given schemes X and Y over K, we let $\mathrm{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$ denote the category $\mathrm{Mor}_{\mathrm{GTop}/(\mathrm{Spec}\,K)_{\text{ét}}} \times (X_{\text{ét}}, Y_{\text{ét}})$.

A morphism $\varphi \in \mathrm{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$ is said to be *admissible* if $T(\varphi)$ takes closed points of $T(X)$ to closed points of $T(Y)$. We let $\mathrm{Mor}_K^0(X_{\text{ét}}, Y_{\text{ét}})$ denote the set of isomorphism classes of admissible morphisms.

REMARK 1.1. This is a good definition of admissible morphisms only in the case of schemes over a field. In general, it seems that one needs to require that $T(\varphi)$ satisfy the conditions in Lemma 1.2.13 of [4].

PROPOSITION 1.2. *Let X and Y be schemes over K, and let Y be of finite type. For any extension E of K (which one can assume is an algebraic field extension) one has a functor*

$${}_E\colon \mathrm{Mor}_K(X_{\text{ét}}, Y_{\text{ét}}) \to \mathrm{Mor}_E((X_E)_{\text{ét}}, (Y_E)_{\text{ét}}),$$

where $X_E = X \times_K \mathrm{Spec}\,E$ (and similarly for Y_E). This functor is natural relative to morphisms of $X_{\text{ét}}$ and $Y_{\text{ét}}$ and the following diagram is commutative for any

$\varphi \in \mathrm{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$:

$$\begin{array}{ccc} (X_E)_{\text{ét}} & \xrightarrow{\varphi_E} & (Y_E)_{\text{ét}} \\ \downarrow & & \downarrow \\ X_{\text{ét}} & \longrightarrow & Y_{\text{ét}} \end{array} \tag{1.2}$$

PROOF. Again we shall only give the construction, and only in the case when E is finite over K. The general case reduces to this one, since the assumption that Y is of finite type means that

$$\mathrm{Mor}_E(X_{\text{ét}}, Y_{\text{ét}}) = \varinjlim_{L\subset E,[L:K]<\infty} \mathrm{Mor}_L((X_L)_{\text{ét}}, (Y_L)_{\text{ét}})$$

in the appropriate category-theoretic sense.

Let $\varphi = (\varphi^{-1}, \alpha) \in \mathrm{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$. We define $\varphi_E = (\varphi_E^{-1}, \alpha_E)$ as follows:

(a) Let V be the open set in $(\operatorname{Spec} K)_{\text{ét}}$ corresponding to $\operatorname{Spec} E \to \operatorname{Spec} K$. Then $(X_E)_{\text{ét}} = p_X^{-1}(V)_{\text{ét}}$ and $(Y_E)_{\text{ét}} = p_Y^{-1}(V)_{\text{ét}}$. We set $\varphi_E^{-1} = \alpha_V^{-1} \circ \varphi_{Y_E}^{-1}$.

(b) Given an open set U in $\operatorname{Spec} E$, we must construct an isomorphism $(\alpha_E)_U$: $p_{X_E}^{-1}(U) \to \varphi_E^{-1} p_{Y_E}^{-1}(U)$. It suffices to note that

$$p_{X_E}^{-1}(U) = p_X^{-1}(U \times_K \operatorname{Spec} E),$$
$$\varphi_E^{-1} p_{Y_E}^{-1}(U) = \varphi^{-1} p_Y^{-1}(U \times_K \operatorname{Spec} E) \times \alpha_V X_E.$$

We set $(\alpha_E)_U = \alpha_{V\times U} \times (\mathrm{pr}\colon p_{X_E}^{-1}(U) \to X_E)$. The commutativity of (1.2) is obvious. It is also simple to verify that this construction is unique for given $X_{\text{ét}}$ and $Y_{\text{ét}}$. □

PROPOSITION 1.3. *Suppose that X and Y are schemes over an algebraically closed field K, Y is of finite type, and $\varphi = (\varphi^{-1}, \alpha) \in \mathrm{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$ is an admissible morphism. Then for any open U in Y and any closed point $x \in T(X)$, the map $T(\varphi_U)$ maps the fiber of $\varphi^{-1}(U)$ over x bijectively onto the fiber of U over $T(\varphi_{\text{ét}})(x)$.*

PROOF. (a) *Surjectivity.* We let y denote $T(\varphi)(x)$. Let z be a point in $T(U)$ over y, and suppose that it does not lie in the image of $T(\varphi_U)$. Let U_z denote the open subset of U obtained by removing all of the points in the fiber over y except for z. Then $T(\varphi^{-1}(U_z))$ has no points over x. But U_z is a covering in some neighborhood of y; consequently, $\varphi^{-1}(U_z)$ must be a covering in some neighborhood of $T(\varphi)^{-1}(y)$. We have obtained a contradiction.

(b) *Injectivity.* Again let z be a point of $T(U)$ over y. We pass to U_z just as in the proof of surjectivity. We consider $\mathrm{pr}_1\colon U_z \times U_z \to U_z$. Since K is algebraically closed, Y is of finite type, and z is closed, it follows that there is a unique point (z, z) in the fiber of this projection over z. Thus, the diagonal is a covering in some neighborhood of $\mathrm{pr}_1^{-1}(z)$. The same must be the case for the diagonal $\varphi^{-1}(U_z) \to \varphi^{-1}(U_z) \times \varphi^{-1}(U_z)$ and the projection $\mathrm{pr}_1\colon \varphi^{-1}(U_z) \times \varphi^{-1}(U_z) \to \varphi^{-1}(U_z)$. This implies that there is a unique point in the fiber over x in $\varphi^{-1}(U_z)$. □

REMARK 1.2. From the proof it is clear that surjectivity holds for any field K and point x.

§2. The topological meaning of points

Let X be a scheme over K. If E is an extension of K, then an E-point of X is a morphism which completes the diagram

$$\begin{array}{ccc} \operatorname{Spec} E & \dashrightarrow & X \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} K & \end{array}$$

In topological language this means that the point corresponds to a local section of the natural projection $X_{\text{ét}} \to (\operatorname{Spec} K)_{\text{ét}}$. The basic task in this section is to prove the converse, i.e., that any local section which is an admissible morphism is induced by some point of X.

It seems that one has a more general fact.

Conjecture. *Let S be a normal Noetherian scheme, let $X \xrightarrow{p} S$ be a morphism of finite type, and let U be étale over S. Then any section $p_{\text{ét}}\colon X_{\text{ét}} \to S_{\text{ét}}$ over U which is admissible (see Remark 1.1) is induced by some morphism of schemes.*

Proposition 2.1. *Let K be an algebraically closed field, and let X be a scheme of finite type over K. Then the natural map*

$$X(K) \to \operatorname{Mor}^0_K((\operatorname{Spec} K)_{\text{ét}}, X_{\text{ét}})$$

is injective.

Proof. The injectivity of the map is obvious, since over an algebraically closed field a K-point is determined by its image in $T(X)$. To prove surjectivity we first note that in our case the existence of the isomorphism $\varphi_1^{-1} \to \varphi_2^{-1}$ for $(\varphi_1^{-1}, \alpha_1)$, $(\varphi_2^{-1}, \alpha_2) \in \operatorname{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$ implies the existence of an isomorphism $(\varphi_1^{-1}, \alpha_1) \to (\varphi_2^{-1}, \alpha_2)$. In fact, let $\xi\colon \varphi_1^{-1} \to \varphi_2^{-1}$ be such an isomorphism, and let V be open in $(\operatorname{Spec} K)_{\text{ét}}$. Consider the diagram

$$\begin{array}{ccc} & \xrightarrow{\alpha_{1,V}} & \varphi_1^{-1}(p_X^{-1}(V)) \\ V & & \downarrow \xi * p_X^{-1} \\ & \xrightarrow{\alpha_{2,V}} & \varphi_2^{-1}(p_X^{-1}(V)) \end{array}$$

We must show that this diagram commutes. Here V is an étale scheme over $\operatorname{Spec} K$, i.e., it consists simply of several copies of $\operatorname{Spec} K$. We may suppose that there are $n > 2$ copies. The automorphism

$$\alpha_{2,V}^{-1} \circ (\xi * p_X^{-1}) \circ \alpha_{1,V}\colon V \to V$$

commutes with the action of the group $\operatorname{Aut}(V/\operatorname{Spec} K)$ (since α_1, α_2, and ξ are functorial isomorphisms), and so it is the identity.

Now let $(\varphi^{-1}, \alpha) \in \operatorname{Mor}_K((\operatorname{Spec} K)_{\text{ét}}, X_{\text{ét}})$ be an admissible morphism. We must prove that there exists an element in $\operatorname{Mor}_K(\operatorname{Spec} K, X)$, which induces φ. Obviously, for such an element we can only take the point corresponding to $\operatorname{Im}(T(\varphi))$. That this point really induces φ follows from Proposition 1.3. □

Proposition 2.2. *Let X be a scheme of finite type over K. Then the map*

$$\operatorname{Mor}_{\overline{K}}(\operatorname{Spec} \overline{K}, \overline{X}) \to \operatorname{Mor}^0_K((\operatorname{Spec} \overline{K})_{\text{ét}}, X_{\text{ét}})$$

is bijective.

PROOF. We let a denote this map. We shall construct a^{-1}. Let

$$\varphi \in \mathrm{Mor}_K((\operatorname{Spec}\overline{K})_{\text{ét}}, X_{\text{ét}})$$

be an admissible morphism. Using Proposition 1.2, we construct

$$\varphi_{\overline{K}} \in \mathrm{Mor}_{\overline{K}}((\operatorname{Spec}\overline{K} \times_K \operatorname{Spec}\overline{K})_{\text{ét}}, \overline{X}_{\text{ét}}).$$

We have the commutative diagram

$$\begin{array}{ccc} (\operatorname{Spec}\overline{K} \times_K \operatorname{Spec}\overline{K})_{\text{ét}} & \xrightarrow{\varphi_K} & \overline{X}_{\text{ét}} \\ \downarrow & & \downarrow \\ (\operatorname{Spec}\overline{K})_{\text{ét}} & \xrightarrow{\varphi} & X_{\text{ét}} \end{array}$$

We set $a^{-1}(\varphi) =$ (the morphism corresponding to $\varphi_{\overline{K}} \circ \Delta_{\text{ét}}$ by Proposition 2.2), where $\Delta\colon \operatorname{Spec}\overline{K} \to \operatorname{Spec}\overline{K} \times_K \operatorname{Spec}\overline{K}$ is the diagonal. Obviously, $a \circ a^{-1} = 1$. It remains to verify that a is injective, i.e., if x, $x'\colon \operatorname{Spec}\overline{K} \to X$ are geometric points of X and $x_{\text{ét}} \cong x_{\text{ét}}$, then $x = x'$. Let L be a Galois extension of K which contains the residue field $\operatorname{Im} x = \operatorname{Im} x'$. Using the same construction as above for a^{-1}, we can lift x and x' to points of X_L, where the two liftings coincide as continuous maps. The residue field of their image is L, and they are morphisms over L. Hence, they coincide. □

COROLLARY 2.1. *Let X and Y be schemes of finite type over K, with X reduced. Then the map*

$$\mathrm{Mor}_K(X, Y) \to \mathrm{Mor}_K^0(X_{\text{ét}}, Y_{\text{ét}}) \tag{2.1}$$

is injective.

Let $\varphi \in \mathrm{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$. We say that φ is realized by a morphism of schemes if its isomorphism class lies in the image of the map (2.1).

PROPOSITION 2.3. *Let X and Y be schemes of finite type over K, with X reduced. In order for an admissible morphism $\varphi = (\varphi^{-1}, \alpha) \in \mathrm{Mor}_K(X_{\text{ét}} Y_{\text{ét}})$ to be realized by a morphism of schemes it is sufficient that, for any U which is open in $Y_{\text{ét}}$, there exist a morphism $\tilde{\varphi}_U \in \mathrm{Mor}_K(\varphi^{-1}(U), U)$ which coincides with φ_U on $\varphi^{-1}(U)(\overline{K})$.*

PROOF. We shall show that $\tilde{\varphi} = \tilde{\varphi}_Y$ induces φ. Consider the diagram

$$\begin{array}{ccccc} \varphi^{-1}(U) & & \searrow & & \\ & \searrow & X \times_{\tilde{\varphi}} U & \longrightarrow & U \\ & & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

It is commutative, since it is commutative on geometric points and X is reduced. Thus, there exist a morphism $a_U\colon \varphi^{-1}(U) \to \tilde{\varphi}^{-1}(U)$, which completes the diagram. We have obviously obtained a morphism $a\colon \varphi^{-1} \to \tilde{\varphi}^{-1}$. We show that it is an isomorphism. We lift our diagram to a diagram over $\overline{K}$. Clearly, $\overline{\tilde{\varphi}}_U = (\varphi_U)_{\overline{K}}$ and $\overline{\tilde{\varphi}} = \tilde{\varphi}_{\overline{K}}$ on the geometric points of $\overline{\varphi^{-1}(U)}$ and $\overline{X}$, respectively. From Proposition 1.3 we immediately conclude that $\overline{a}_U$ is an isomorphism. Consequently, so is a_U. It remains to check that a is an isomorphism in $\mathrm{GTop}/(\operatorname{Spec} K)_{\text{ét}}$, and not only in GTop. It is easy to see that this amounts to commutativity of the diagram

$$\begin{array}{ccc} & \varphi^{-1}(Y_E) & \\ {}^{\alpha_E}\nearrow & & \searrow^{a_{Y_E}} \\ X_E & \longrightarrow & \tilde{\varphi}^{-1}(Y_E) \end{array}$$

Since all of the rows are isomorphisms over X, it is sufficient to verify that the following diagram commutes:

$$\begin{array}{ccccc} & \overset{\alpha_E}{\nearrow} & \varphi^{-1}(Y_E) \xrightarrow{\tilde{\varphi}_{Y_E}} Y_E & \searrow & \\ X_E & & & & \operatorname{Spec} E \\ & \searrow & \tilde{\varphi}^{-1}(Y_E) \longrightarrow Y_E & \nearrow & \end{array}$$

The composition of the lower arrows is simply the natural projection $X_E \to \operatorname{Spec} E$. Hence, we must show that $\tilde{\varphi}_{Y_E} \circ \alpha_E$ is a morphism over Spec E. But this is the case for the continuous map $(\varphi_{Y_E})_{\text{ét}} \circ (\alpha_E)_{\text{ét}}$ (see the proof of Proposition 1.2), and hence also for $\tilde{\varphi}_{Y_E} \circ \alpha_E$.

Proposition 2.4. *Let X and Y be schemes of finite type over K, with X reduced. For an admissible $\varphi \in \operatorname{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$, if there exist a morphism $\tilde{\varphi}_E \colon X_E \to Y_E$ which coincides with φ_E on $X_E(\overline{K})$, then there exists $\tilde{\varphi} \colon X \to Y$ which coincides with φ on $X(\overline{K})$ (E is an extension of K).*

Proof. This follows from the fact that $X_E \to X$ is a strict epimorphism in the category of schemes over K for any scheme of finite type X and any extension E.

Proposition 2.5. *Let X be a scheme of finite type over the field K, and let E be an extension of K. Then the natural map*

$$X(E) = \operatorname{Mor}_K(\operatorname{Spec} E, X) \to \operatorname{Mor}_A((\operatorname{Spec} E)_{\text{ét}}, X_{\text{ét}})$$

is bijective.

Proof. We already know injectivity. Suppose that $\varphi \in \operatorname{Mor}_K((\operatorname{Spec} E)_{\text{ét}}, X_{\text{ét}})$ is an admissible morphism. We consider its lifting (by Proposition 1.2):

$$\begin{array}{ccc} (\operatorname{Spec} E \times_K \operatorname{Spec} \overline{K})_{\text{ét}} & \xrightarrow{\varphi_{\overline{K}}} & \overline{X}_{\text{ét}} \\ \downarrow & & \downarrow \\ (\operatorname{Spec} E)_{\text{ét}} & \xrightarrow{\quad\varphi\quad} & X_{\text{ét}} \end{array}$$

Since $\operatorname{Spec} E \times_K \operatorname{Spec} \overline{K}$ is a union of several copies of $\operatorname{Spec} \overline{K}$, $\varphi_{\overline{K}}$ can be realized by a morphism on geometric points. Hence, by Proposition 2.4, the same is true for φ. It is obvious that this construction enables us to realize all of the φ_U on geometric points. It then follows, by Proposition 2.3, that φ can be realized by a morphism of schemes. □

We shall later need one more construction, which gives a topological (or rather, a homotopic) interpretation of the points. That is, for any geometrically connected scheme X of finite type over K and any point $x \in X(K)$ we shall construct a map

$$X(\overline{K}) \ \varinjlim_{E} H^1(\Gamma_E, \pi_1(\overline{X}, \overline{x})),$$

where Γ_E denotes the Galois group $\operatorname{Gal}(\overline{E}/E)$ for any field E. It will be shown that these maps are natural relative to morphisms of étale sites of schemes. (For the definition of the fundamental group of a scheme see, for example, [5].)

For the rest of this section X will denote a geometrically connected scheme of finite type over K.

PROPOSITION 2.6. *The sequence of morphisms of schemes* $\overline{X} \to X \to \operatorname{Spec} K$ *induces an exact sequence of fundamental groups*

$$1 \to \pi_1(\overline{X}, \overline{x}) \to \pi_1(X, x) \to \Gamma_K \to 1. \tag{2.2}$$

For the proof, see [5].

PROPOSITION 2.7. *Suppose that* X *and* Y *are geometrically connected, and* $\varphi \in \operatorname{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$ *is an admissible morphism. Then one has a morphism of exact sequences*

$$\begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ \pi_1(\overline{X}, \overline{x}) & \to & \pi_1(\overline{Y}, \varphi(\overline{x})) \\ \downarrow & & \downarrow \\ \pi_1(X, \overline{x}) & \to & \pi_1(Y, \varphi(\overline{x})) \\ \searrow & & \swarrow \\ & \Gamma_K & \\ & \downarrow & \\ & 1 & \end{array}$$

PROOF. It suffices to prove that, given a connected X, $\overline{x} \in X(\overline{K})$, and an admissible $\varphi \in \operatorname{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$, one can define a homomorphism $\varphi_*\colon \pi_1(X, \overline{x}) \to \pi_1(Y, \varphi(\overline{x}))$, and this correspondence is functorial.

By assumption, φ takes geometric points of X to geometric points of Y; hence, it is sufficient to prove that if $U \to Y$ is an étale covering, so is $\varphi^{-1}(U) \to X$. This follows from the following criterion [5]: an étale morphism $U \to Y$ is an étale covering if and only if its fibers over all of the closed geometric points contains the same number of elements. □

Now let $x \in X(K)$. This point determines a section $x_*\colon \Gamma_K \to \pi_1(X, \overline{x})$ of the sequence (2.2) (here $\overline{x}$ is the geometric point corresponding to x).

If $y \in X(E)$, then this point gives a section of (2.2) over $\Gamma_E \subset \Gamma_K$, $\Gamma_E \to \pi_1(X, \overline{y})$. If we choose an isomorphism $\pi_1(X, \overline{x}) \cong \pi_1(X, \overline{y})$, we obtain a section $\Gamma_E \to \pi_1(X, \overline{x})$, and hence a cocycle in $C^1(\Gamma_E, \pi_1(X, \overline{x}))$. (The action of Γ_K on $\pi_1(\overline{X}, \overline{x})$, and hence also the action of Γ_E, are determined by the sequence (2.2) and the section x_*.) It is easy to see that the nonuniqueness in the choice of isomorphism does not affect the cohomology class of this cocycle. We have thereby obtained a map

$$\iota_{X,E}\colon X(E) \to H^1(\Gamma_E, \pi_1(\overline{X}, \overline{x})).$$

These maps are clearly compatible with one another for different E, and they give a map

$$\iota_x\colon X(\overline{K}) \to \varinjlim_E H^1(\Gamma_E, \pi_1(\overline{X}, \overline{x})).$$

PROPOSITION 2.8. *Suppose that* X *and* U *are geometrically connected schemes of finite type over* K, $x \in X(K)$, *and* $\varphi \in \operatorname{Mor}_K(X_{\text{ét}}, Y_{\text{ét}})$ *is an admissible morphism. Then the following diagram commutes:*

$$\begin{array}{ccc} X(\overline{K}) & \xrightarrow{\varphi} & Y(\overline{K}) \\ {\scriptstyle \iota_x}\downarrow & & \downarrow{\scriptstyle \iota_{\varphi(x)}} \\ \varinjlim_E H^1(\Gamma_E, \pi_1(\overline{X})) & \to & \varinjlim_E H^1(\Gamma_E, \pi_1(\overline{Y})) \end{array}$$

where the bottom map is induced by the homomorphism $\overline{\varphi}_*$.

PROOF. This is a direct consequence of the construction and Proposition 2.5. □

There are several interesting questions concerning the map ι. In many cases one can prove that it is injective—for example, in the case of an arbitrary subvariety of an abelian variety over a field of finite type $\mathbf{Q}$. It seems to be that a more intriguing question is the image of this map. The image will be computed below in one special case. In the general case I think it is reasonable to conjecture that the image is everywhere dense in the projective limit topology

$$(H^1(\Gamma_K, \pi_1(\overline{X})) = \varprojlim H^1(\Gamma_K, \pi_1(\overline{X})/H),$$

where $[\pi_1(\overline{X}) : H] < \infty$ for schemes X of type $K(\pi, 1)$ over $\mathbf{Q}$.

§3. The main theorem

In this section K denotes a field of finite type over $\mathbf{Q}$.

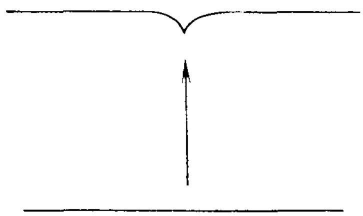
THEOREM 3.1. *Let X and Y be schemes of finite type over K, where X is reduced and nonsingular in codimension 1. Then the map*

$$\mathrm{Mor}_K(X, Y) \to \mathrm{Mor}^0_K(X_{\text{ét}}, Y_{\text{ét}})$$

is bijective.

COROLLARY 3.1. *Let X and Y be normal schemes of finite type over K. If there exists a homeomorphism $X_{\text{ét}} \to Y_{\text{ét}}$ over $(\operatorname{Spec} K)_{\text{ét}}$, then $X \cong Y$.*

REMARK 3.1. If one does not impose any restrictions on the singularity of X, then the theorem is false. For example, the normalization of the cubic curve with cusp



is a homeomorphism, but the inverse homeomorphism is obviously not realized by a morphism.

On the other hand, it will be clear from the proof that the theorem remains true for a scheme X with the property that $\mathrm{Pic}(\overline{X})$ does not contain subgroups isomorphic to $\mathbf{G}_a$; for example, it is true for stable curves.

The plan of proof is as follows. In the first step we show that it suffices to consider affine, geometrically connected X with a nonempty set $X(K)$ of K-rational points, and to prove that, for any admissible φ in $\mathrm{Mor}_K(X_{\text{ét}}, (\mathbf{A}^1_K - \{0\})_{\text{ét}})$, there exists a morphism $X \to \mathbf{A}^1_K - \{0\}$ which coincides with φ on $X(\overline{K})$. We then construct maps of

$$\mathrm{Mor}^0_K(X_{\text{ét}}, (\mathbf{A}^1_K - \{0\})_{\text{ét}}) \quad \text{and} \quad \widehat{\mathscr{O}}^*(\overline{X}) = \varprojlim \mathscr{O}^*(\overline{X})/\mathscr{O}^*(\overline{X})^n$$

to $H^1(\overline{X}, \widehat{\mathbf{Z}}(1)) = \varprojlim H^1(\overline{X}, \mu_n)$, and we show that it suffices for the image of $\mathrm{Mor}^0_K(X_{\text{ét}}(\mathbf{A}^1_K - \{0\})_{\text{ét}}$ to line in the image of $\widehat{\mathscr{O}}^*(\overline{X})$. Finally, we show that the absence of infinitely divisible elements in $\mathrm{Pic}(\overline{X})^{\Gamma_K}$ for affine schemes nonsingular in codimension 1 implies that the map $\widehat{\mathscr{O}}^*(\overline{X}) \to H^1(\overline{X}, \widehat{\mathbf{Z}}(1))^{\Gamma_K}$ is surjective for such schemes.

PROPOSITION 3.1. *Let X and Y be schemes of finite type over K, with X reduced. In order for the map*

$$\mathrm{Mor}_K(X, Y) \to \mathrm{Mor}_K^0(X_{\text{ét}}, Y_{\text{ét}})$$

to be surjective (and hence bijective) it is sufficient that, given any affine, geometrically connected scheme U which is étale over X and has nonempty $U(E)$ for some finite E over K, and given an admissible $\varphi \in \mathrm{Mor}_E((U_E)_{\text{ét}}, (\mathbf{A}_E^1 - \{0\})_{\text{ét}}$, there exists a morphism $\tilde{\varphi}\colon U_E \to \mathbf{A}_E^1 - \{0\}$ which coincides with φ on $U_E(\overline{K})$.

PROOF. By Proposition 2.3, it suffices to prove that, for any admissible φ and étale U over Y, the map $\varphi_U\colon (\varphi^{-1}(U))_{\text{ét}} \to U_{\text{ét}}$ can be realized by a morphism of schemes on $\varphi^{-1}(U)(\overline{K})$. Both sides of the problem are clearly local, and we may suppose that we are considering maps $\varphi\colon V_{\text{ét}} \to Y_{\text{ét}}$, where V and Y are affine, and V is geometrically connected and affine over X. Proposition 2.4 enables one to pass to finite E over K for which $V(E) \neq \varnothing$. Let $V_E = \mathrm{Spec}\,A$ and $Y_E = \mathrm{Spec}\,B$.

We construct a map $\varphi^*\colon B \to A$. If $(\xi\colon Y_E \to \mathbf{A}_E^1) \in B$, then let V_0 be open in $T(V_E)$, with $V_0 = T(V_E) - (\xi \circ T(\varphi_E))^{-1}$, and let V_1 be open in $T(V_E)$, with $V_1 = T(V_E) - (\xi \circ T(\varphi_E))^{-1}$. Then $V_0 \cup V_1 = V_E$, and, by our assumption, $\xi \circ \varphi_E$ is realized by morphisms of schemes on $V_0(\overline{K})$ and $V_1(\overline{K})$, where these morphisms obviously coincide on $V_0 \cap V_1$. Thus, there exists a morphism $\psi_\xi\colon V_E \to \mathbf{A}^1$ which coincides with $\xi \circ \varphi_E$ on $V_E(\overline{K})$.

We set $\varphi^*(\xi) = \psi_\xi$. It is trivial to check that this is a homomorphism. Let $\tilde{\varphi}_E\colon V_E \to Y_E$ be the corresponding morphism of schemes. It is again easy to verify that this morphism coincides with φ_E on $V_E(\overline{K})$.

We introduce the notation

$$\mathscr{O}_{\text{top}}^*(X) = \mathrm{Mor}_K^0(X_{\text{ét}}, (\mathbf{A}_K^1 - \{0\})_{\text{ét}}),$$
$$H^1(X, \widehat{\mathbf{Z}}(1)) = \varprojlim_n H^1(X, \boldsymbol{\mu}_n),$$

where $\boldsymbol{\mu}_n$ is the sheaf of nth roots of 1 in $\mathscr{O}^*$, and the limit is taken over the projective system $(\boldsymbol{\mu}_n, f_{mn,n})$, $f_{mn,n}\colon \boldsymbol{\mu}_{mn} \to \boldsymbol{\mu}_n$ is raising to the mth power.

If K is an algebraically closed field (of course, $\mathrm{char}\,K = 0$), then $\boldsymbol{\mu}_n$ is a constant sheaf:

$$\boldsymbol{\mu}_n(U) = \{\text{the set } \sqrt[n]{1} \text{ in } K\}.$$

In this case we have a canonical isomorphism

$$\boldsymbol{\mu}_n(U) = \pi_1(\mathbf{A}_K^1 - \{0\}, 1)/\pi_1(\mathbf{A}_K^1 - \{0\}, 1)^n,$$

and hence

$$H^1(X, \widehat{\mathbf{Z}}(1)) \cong \mathrm{Hom}(\pi_1(X, \overline{x}), \pi_1(\mathbf{A}^1 - \{0\}, 1)) \tag{3.1}$$

for geometrically connected X.

From now on X will denote an affine, geometrically connected scheme of finite type over K, which is reduced and for which $X(K)$ is nonempty.

The identification (3.1) obviously enables us to construct a map

$$\chi_{\mathrm{top}}: \mathcal{O}^*_{\mathrm{top}}(X) \to H^1(X, \widehat{\mathbf{Z}}(1))$$

(where we use the fact, from §2.7, that π_1 is natural).

Γ_K acts on the group on the right (for example, using the identification (3.1)), and clearly $\operatorname{Im}\chi_{\mathrm{top}} \subset H^1(\overline{X}, \widehat{\mathbf{Z}}(1))^{\Gamma_K}$.

We now consider the Kummer sequence for $\overline{X}$

$$1 \to \mathcal{O}^*(\overline{X})/\mathcal{O}^*(\overline{X})^n \xrightarrow{\chi_n} H^1(\overline{X}, \boldsymbol{\mu}_n) \to \operatorname{Pic}(\overline{X})_n \to 1$$

(see, for example, [5]). Passing to the limit over n, we obtain

$$1 \to \widehat{\mathcal{O}}^*(\overline{X}) \xrightarrow{\chi} H^1(\overline{X}, \widehat{\mathbf{Z}}(1)) \to \widehat{T}(\operatorname{Pic}(\overline{X})) \to 1, \qquad (*)$$

where the term on the right is the Tate module of $\operatorname{Pic}(\overline{X})$.

PROPOSITION 3.2. *The diagram*

$$\begin{array}{ccccc} & & \widehat{\mathcal{O}}^*(\overline{X}) & & \\ & \nearrow & & \searrow & \\ \mathcal{O}^*(X) & & & & H^1(\overline{X}, \widehat{\mathbf{Z}}(1)) \\ & \searrow & & \nearrow & \\ & & \mathcal{O}^*_{\mathrm{top}}(X) & & \end{array}$$

commutes.

PROOF. This can be verified directly by comparing the maps χ_n and the identification (3.1.). □

PROPOSITION 3.3. *Let $x \in X(K)$. Then each class $\xi \in H^1(\overline{X}, \widehat{\mathbf{Z}}(1))^{\Gamma_K}$ determines a map*

$$\varphi_\xi: X(\overline{K}) \to \varinjlim_{E\supset K} \varprojlim_n E^*/(E^*)^n. \qquad (3.2)$$

If

$$i: \overline{K} \to \varinjlim \varprojlim E^*/(E^*)^n$$

is the natural map, then for any $\varphi \in \mathcal{O}^*_{\mathrm{top}}(X)$ for which $\varphi(x) = 1$ one has

$$i(\varphi(\overline{y}) = \varphi_{\chi_{\mathrm{top}}}(\overline{y}), \qquad \overline{y} \in X(\overline{K}). \qquad (3.3)$$

PROOF. By (3.1), the class ξ corresponds to a homomorphism $\pi_1(\overline{X}, \overline{x}) \to \pi_1(\mathbf{A}^1_{\overline{K}} - \{0\}, 1)$ which is compatible with the action of Γ_K on these groups (since $\xi \in H^1(\overline{X}, \widehat{\mathbf{Z}})(1))^{\Gamma_K}$). The homomorphism determines a map

$$H^1(\Gamma_K, \pi_1(\overline{X}, \overline{x}) \to H^1(\Gamma_K, \pi_1(\mathbf{A}^1_{\overline{K}} - \{0\}, 1)).$$

The group on the right is $\varprojlim K^*/(K^*)^n$. If we take the composite with the map

$$\iota_x: X(K) \to H^1(\Gamma_K, \pi_1(\overline{X}, \overline{x})),$$

we obtain $\varphi_{\xi, K}: X(K) \to \varprojlim K^*/(K^*)^n$. Finally, passing to the limit over finite extensions of K, we obtain φ_ξ. Relation (3.3) follows because the maps ι_x are natural (Proposition 2.8). □

PROPOSITION 3.4. *If* $\xi \in H^1(\overline{X}, \mathbf{Z}(1))^{\Gamma_K}$ *lies in the image of* $\widehat{\mathscr{O}}^*(\overline{X})$ *and* $\operatorname{Im}\varphi_\xi \subset \operatorname{Im} i$, *then* $\xi \in \operatorname{Im}\mathscr{O}^*(X)$.

PROOF. We let $\mathscr{O}_1^*(\overline{X})$ denote the subgroup of $\mathscr{O}^*(\overline{X})$ consisting of maps which take the value 1 at x. Obviously, $\operatorname{Im}\mathscr{O}_1^* = \operatorname{Im}\mathscr{O}^*$ in $H^1(\overline{X}, \widehat{\mathbf{Z}}(1))$. The group $\mathscr{O}_1^*(\overline{X})$ is a finitely generated free abelian group. Let $f_1, \dots, f_n$ be its generators. Then $\widehat{\mathscr{O}}^*(\overline{X})$ can be identified with the free profinite abelian group generated by $f_1, \dots, f_n$. The "pairing" $\mathscr{O}_1^*(\overline{X}) \times X(\overline{K}) \to \overline{K}^*$ extends to a "pairing"

$$\widehat{\mathscr{O}}_*(\overline{X}) \times X(\overline{K}) \to \varinjlim \varprojlim E^*/(E^*)^n.$$

Thus, we must prove that, if $\varphi = f_1^{\varepsilon_1} \cdots f_n^{\varepsilon_n}$, $\varepsilon_i \in \widehat{\mathbf{Z}}$, has the property that $\varphi(\overline{x}) \in i(\overline{K})$ for all $\overline{x} \in X(\overline{K})$, then $\varepsilon_i \in \mathbf{Z}$. (The fact that here we have $\varphi \in \operatorname{Im}\mathscr{O}^*(X)$, and not only $\mathscr{O}^*(\overline{X})$, follows from the Γ_K-invariance of its cohomology class.)

We need the following elementary fact.

LEMMA 3.1. *Let* $\varepsilon \in \widehat{\mathbf{Z}}$. *If* $n\varepsilon \in \mathbf{Z}$ *for some* $n \in \mathbf{Z}$, $n \neq 0$, *then* $\varepsilon \in \mathbf{Z}$.

The proof is obvious. □

Let E be a finite extension of K, and let $\kappa\colon E^* \to \mathbf{Z}$ be a valuation. The valuation extends to a homomorphism $\hat{\kappa}\colon \varprojlim E^*/(E^*)^n \to \widehat{\mathbf{Z}}$. We may assume that E is sufficiently large. Let $x_1, \dots, x_n \in X(\overline{E})$. We consider the system of equations

$$\sum \varepsilon_i \hat{\kappa}(f_i(x_j)) = \hat{\kappa}(\varphi(x_j)).$$

In trying to solve this system for the ε_i, we obtain

$$\Delta\varepsilon_i = p_i(\hat{\kappa}(f_i(x_j)), \hat{\kappa}(\varphi(x_j))),$$

where $\Delta = \det(\hat{\kappa}(f_i(x_j)))$, and p_i is some polynomial. By assumption, the right side and Δ are in $\mathbf{Z}$; hence, using the lemma, we see that, if $\Delta \neq 0$, then $\varepsilon_i \in \mathbf{Z}$. It remains to show that κ, E, and $x_1, \dots, x_n$ can be chosen in such a way that $\Delta \neq 0$. We use induction on n. If $n = 1$, then $\Delta = \hat{\kappa}(f_1(x_1))$. Since $f_1 \neq \text{const}$, it follows that f_1 is open as a map. Consequently, there exists x_1 such that $\kappa(f_1(x_1)) \neq 0$. This completes the case $n = 1$.

Suppose that our claim is proved for $n-1$. We consider the matrix

$$A = \begin{pmatrix} \hat{\kappa}(f_1(x_1)) \cdots \hat{\kappa}(f_n(x_1)) \\ \\ \hat{\kappa}(f_1(x_n)) \cdots \hat{\kappa}(f_n'(x_n)) \end{pmatrix}.$$

We expand Δ in minors:

$$\Delta = \sum \pm\hat{\kappa}(f_i(x_1))M_i,$$

where M_i is the determinant of the matrix obtained by removing the top row and the ith column. By the induction assumption, there exist $x_2, \dots, x_n \in X(E)$ such that $M_i \neq 0$ for some i. Consider the function

$$f' = f_1^{\pm M_1} \cdots f_n^{\pm M_n}.$$

Since the f_i are "linearly independent" in $\mathscr{O}_1^*(\overline{X})$ and $M_i \neq 0$, it follows that $f' \neq \text{const}$. Hence, there exists a point x_1 such that $\kappa(f'(x_1)) \neq 0$; then $\Delta \neq 0$.

PROOF OF THE THEOREM. By Proposition 3.1, it suffices to prove that, given a scheme X which satisfies all of our conditions and is nonsingular in codimension 1, for every $\varphi \in \mathscr{O}^*_{\text{top}}(X)$ there exists $\tilde{\varphi}$ which coincides with φ on $X(\overline{K})$. We may obviously assume that $\varphi(x) = 1$ $(x \in X(K))$. Then by Propositions 3.3 and 3.4 it is sufficient that we have $\chi_{\text{top}}(\varphi) \in \operatorname{Im}\chi$ (note that $i\colon K^* \to \varinjlim \varprojlim E^*/(E^*)^n$ is injective in the present situation). The theorem now follows from the next proposition and the exact sequence $(*)$. □

PROPOSITION 3.5. *Let* X *be an affine reduced scheme which is nonsingular in codimension* 1. *Then*

$$\widehat{T}(\operatorname{Pic}(\overline{X}))^{\Gamma_K} = 0. \tag{3.4}$$

PROOF. Since $\widehat{T}(\operatorname{Pic}(\overline{X})) = \varprojlim_n \operatorname{Pic}_n(\overline{X})$, where $\operatorname{Pic}_n(\overline{X})$ is the group of points of order n on $\operatorname{Pic}(\overline{X})$, it follows that (3.4) says simply that there are no infinitely divisible elements in $\operatorname{Pic}(\overline{X})^{\Gamma_K}$. But under our assumptions this group is finitely generated (by the Mordell-Weil theorem and an obvious argument about passing from a projective scheme to an affine scheme). □

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