Talk in Bonn (2010)

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The original title of this talk was 'Univalent Fibrations", however, after some thought I have decided to give a more general talk. What I want to talk about are new foundations of mathematics which I am working on. Here are the main points which I will try to say something about. Of course the whole topic is very broad and very much "in the works" so this is not an attempt to give a detailed picture:

1. New foundations of mathematics which can be used both for constructive and for non-constructive mathematics.
2. Naturally include "axiomatization" of the categorical and higher categorical thinking.
3. Can be conveniently formalized using the well known in computer science class of languages called dependent type systems.
4. Based on direct axiomatization of the "world" of homotopy types instead of the world of sets.

How homotopy theory relates to the foundations of mathematics? Let me start with the following definitions which take place in the usual homotopy category:

A (homotopy) type T is said to be of level 0 if it is contractible,

A (homotopy) type T is said to be of level 1 if for all t1, t2 in T the paths space "paths T t1 t2" is contractible,

A (homotopy) type T is said to be of level n+1 if for all t1, t2 in T the paths space "paths T t1 t2" is of level n.

Let's see what homotopy types of small levels are there:

There is only one (up to a homotopy equivalence) type of level 0 - the one point type *pt*.

There are exactly two types of level 1, *pt* and $\emptyset$ i.e. types of level 1 correspond to truth values.

Types of level 2 are types such that for any two points t1 t2 the space of paths from t1 to t2 is either empty or contractible. Such a type is a disjoint union of contractible components i.e. (up to an equivalence) types of level 2 are sets.

Types of level 3 are (homotopy types of nerves of) groupoids.

More generally, types of level n+2 can be seen as equivalence classes of n-groupoids.

We can now reason as follows:

Set-level mathematics works with structures on sets i.e. on types of level 2,

Category-level mathematics in fact works with structures on groupoids i.e. on types of level 3. It is easy to see that a category is a groupoid level analog of a partially ordered set.

Higher categorical levels can be seen as working with structures on types of higher levels.

We conclude that mathematics of all levels deals with structures on homotopy types. This picture was not very useful until now because there was no way to axiomatize the world of homotopy types directly without reducing everything to sets. The key point which makes the present day situation different is the following:

It is possible to directly formalize the world of homotopy types using the class of languages called dependent type systems and in particular Martin-Lof type systems.

As far as I know this was first understood by myself and, independently and from a somewhat different perspective by Steve Awodey, in around 2005.

One of the flavors of Martin-Lof type systems has been implemented in a proof assistant called Coq. It's development was mostly influenced by its role as a tool for the verification of programs. At the moment it is probably the leader among the the general purpose proof assistants and is being taught in the context of the program verification to a large number of computer science students every year.

In the second part of my talk I will show how to actually reason about homotopy types in Coq (run through emacs with the help of Proof General).

See the next file.