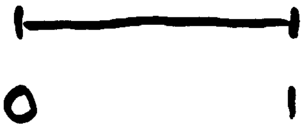
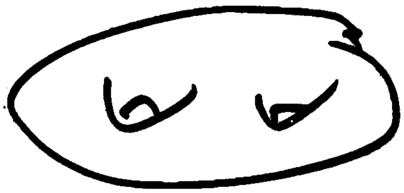
Spaces:

a. •

b. 

$[0,1]$

c. 

d. $GL_n(\mathbb{C})$ - matrices

$n \times n$ with nonzero

determinant

Different axiomatizations

- topological, metric :

does not matter.

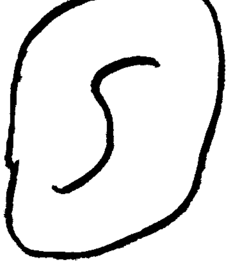
To distinguish spaces use "invariants"

<u>Ex. 1</u> Dimention

$\dim(\bullet) = 0 \qquad \dim(\vdash\!\!\dashv) = 1$

<u>Ex. 2</u> The number of "pieces"

$\bullet$ has 1 piece

 has 2 pieces

Homotopy theory studies fancy versions of Ex. 2 called π_n - homotopy sets. π_0 is the set of pieces.

M, N - spaces

$f: M \longrightarrow N$ continuous map is any way to assign to a point of M a point of N such that close $\Rightarrow$ close.

Notation: $C(M, N)$ - continuous maps $M \to N$

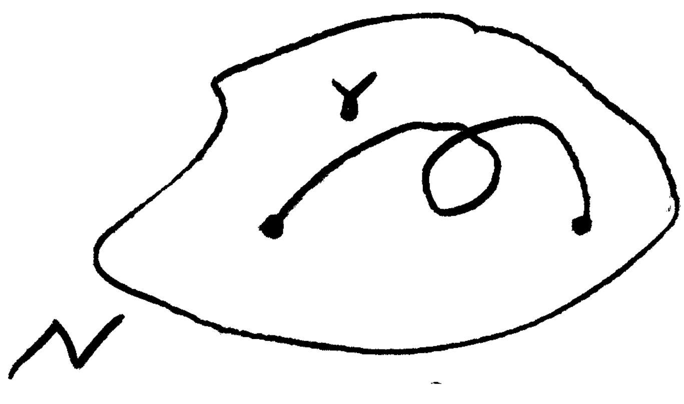
Ex. 1 $C(pt, N)$ -

- the set of points of N

Ex. 2 $C(N, pt)$ - always one element.

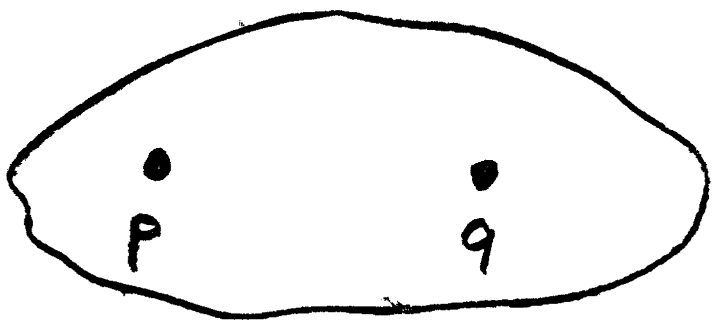
Ex. 3 $C([0,1], N)$

$\gamma : [0,1] \to N$

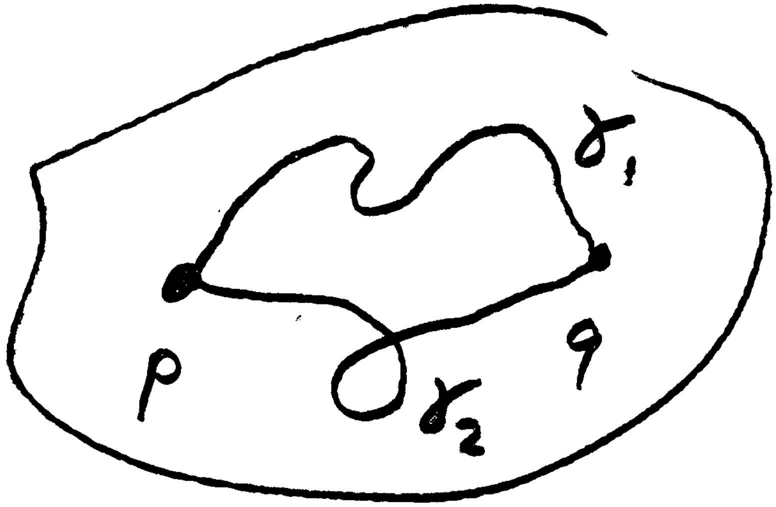


Two points p, q of N lie on the same piece $\iff$ there is a path which starts at p and ends at q.
Hence we can use paths to define $\pi_0(N)$

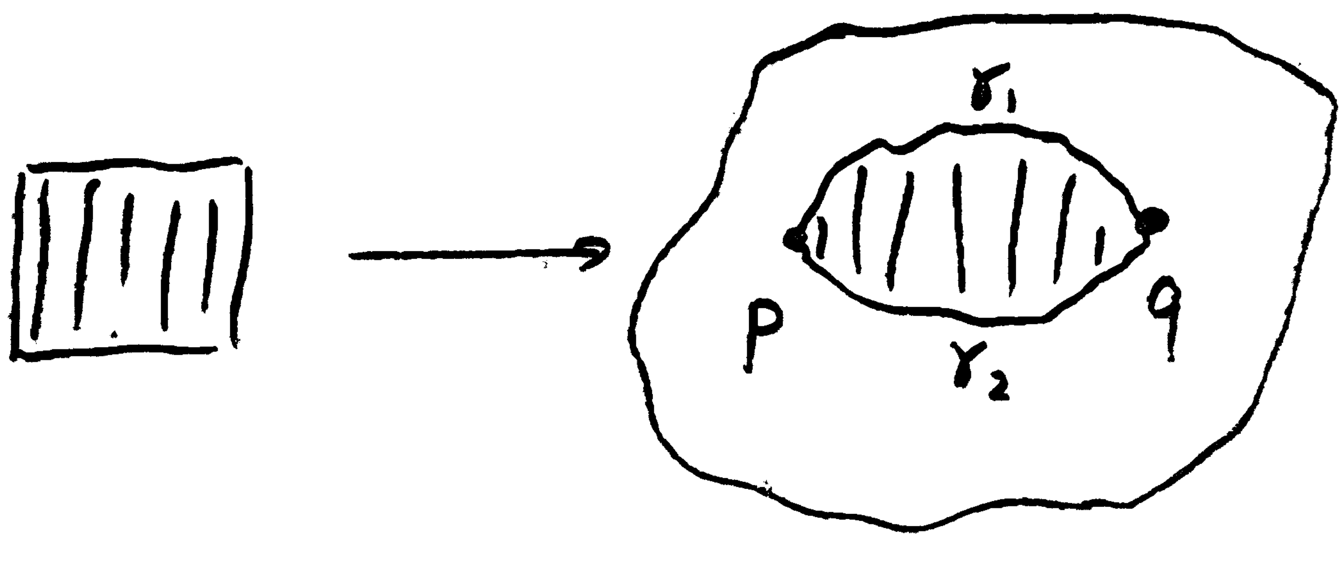
Now choose p, q in the same component



Consider all paths from p to q



and all maps from the square $\square$ to N



Say that γ_1 and γ_2 lie on the same "piece" of the space of paths if there is a map like above. Set $\pi_1(N; p, q)$ to be the set of all such pieces. Use cubes etc. to define:

$\pi_n(N; p, q)$ - <u>the homotopy sets of N</u>

One example:

$\pi_0 (GL_n (\mathbb{C}); 1, 1) = *$

$\pi_1 (GL_n (\mathbb{C}); 1, 1) =$ integral numbers

$\pi_2 (\qquad) = *$

$\pi_3 (\qquad) =$ integral numbers

$\vdots$ goes like that for π_i when n is much bigger than i

Bott periodicity Theorem

Categories – one of the most important ideas of the 20-th century mathematics.

A category consists of three things: objects, morphisms and compositions.

Examples:

$$\begin{tikzcd}
 & B \arrow[dr, bend left, "z"] & \\
A \arrow[ur, bend left, "x"] \arrow[ur, "y"'] \arrow[rr, bend right, "t"'] & & C
\end{tikzcd}$$

A, B, C - objects

x, y, z, t - morphisms

$$\left.\begin{array}{l} xz = t \\ yz = t \end{array}\right\} \text{composition rules.}$$

Example:

Spaces are objects

continuous maps are morphisms

obvious compositions

Formally a category X is:

- a set of objects of X
- for any objects N, M a set of morphisms from N to M $Mor(N, M)$
- A rule giving for a morphism $N \xrightarrow{f} M$ and a morphism $M \xrightarrow{g} K$ __composition__ $N \xrightarrow{fg} K$

Let N be a space what can we learn from its place among other objects of the category of spaces?

- When is N a point?

  $N = pt \iff Mor(M, N)$ has one element for all M.

- When N is connected (consists of one piece)?

More generally can we compute $\pi_0(N)$ in categorical terms?

- $\mathrm{Mor}(pt, N)$ - the set of points of N
- $pt \xrightarrow{a} [0,1]$
  $pt \xrightarrow{b} [0,1]$
- $\mathrm{Mor}([0,1], N)$ - the set of paths of N
- for $\gamma : [0,1] \to N$
  γa - the beginning
  γb - the end

Similarly to compute π_1 we need to know pt, $\{0,1\}$, I^2 - the square and the maps $\{0,1\} \to I^2$ corresponding to the sides of the square.

A collection $I^0, I^1, \ldots, I^n, \ldots$ of objects of a category with morphisms $I^n \to I^{n+1}$ which model cubes and their faces is called

A cubical object in a category.

<u>Observation</u>: Given any category and a cubical object in it we can define analogs of the homotopy sets π_n for all objects of this category.

A system of algebraic equations consists of:

- a list of letters called variables $x_1, \dots x_n$
- a list of polynomials in $x_1, \dots x_n$: $f_1, \dots f_m$ called equations.

<u>Ex</u>: $(x \,;\, x^2+1)$

$(x_1, \dots x_n \,;\,)$ $\qquad \mathbb{A}^n$

$GL_n \quad (x_{11}, \dots x_{nn}, t \,;\, \det(x_{ij}) \cdot t - 1)$

Choose a system of coefficients k with addition, substraction and multiplication (e.g. k - integers, reals etc.)

Define a category:

- objects all systems with coefficients in k
- morphisms changes of variables

$\underline{Ex}$: $Mor(\mathbb{A}^0, X) =$

$=$ the set of solutions of X

$\underline{Ex}$ $Mor(X, \mathbb{A}^0) = *$

The objects $\mathbb{A}^n$ form a cubical object and we may define π_n

$\underline{Ex}$: $\pi_n(GL_N) = K^{alg}_{n+1}(k)$

– the algebraic K-theory of our system of coefficients.

Geometry:

A construction from homotopy theory e.g. π_n

$\downarrow$

Categorical formulation e.g. through cubical objects

$\downarrow$

Specialization for the category of algebraic equations

Algebra