Voevodsky's lectures on cross functors
Fall 2001

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Contents

1 General theory

1 Motivation: the etale case

1.1 In l-adic cohomology, the formalism of the 6 operations ($Rf_*, Lf^*, Rf_!, Rf^!$ for $f : X \to Y$ and $\overset{\mathbf{L}}{\otimes}$, $R\underline{Hom}$) is a useful and compact way to look at the functoriality of homology and cohomology and at Poincare duality. We want to have a similar formalism for the *motivic stable homotopy theory.* There are a number of dificulties. Among them: the lack of finiteness results parallel to the preservation of constructibility by Rf_* and $Rf^!$, and the lack of an analogue to the biduality theorem. This will force us to consider only the first four operations which we will abbreviate as $f_*, f^*, f_!, f^!$.

We begin by recalling the corresponding part of the l-adic formalism. As this is only a template, we will gloss over some difficulties: can one use unbounded complexes, can one really use an l-adic derived category or only the $\mathbf{Z}/l^n$-derived categories from which it is obtained by a limiting process?

What is said should at least be true for the category of separated schemes X over a field k, and the $D^b(X, \mathbf{Z}/l^n)$. This is the bounded derived category of the abelian category of sheaves of $\mathbf{Z}/l^n$-modules over the (small) etale site X_{et} of X. Variants: replace b by $+$ or $-$ or drop it. Restrict to the case where cohomology sheaves are constructible. Restrict to the finite Tor-dimension case D^b_{ft}.

1.2 We fix a noetherian scheme S and consider only schemes X of finite type over S. For each X, we have the corresponding derived category $D(X)$. For $f : X \to Y$ we have four functors

$$f_*, f_! : D(X) \to D(Y)$$

$$f^*, f^! : D(Y) \to D(X).$$

we will separate properties they have in three groups: "basic", "identities" and "localization".

A. Basic (i) For a composite morphism

$$gf : X \to Y \to Z$$

we have $(gf)_* = g_* f_*$ and $(gf)^* = f^* g^*$ and similarly for "!". The "=" sign is an abuse of language, explained below.

(ii) (f^*, f_*) and $(f_!, f^!)$ are pairs of adjoint functors

(iii) For a cartesian square of schemes over S

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

one has change of base isomorphisms

$$g^* f_! = f_! g^* \;\text{ and }\; g^! f_* = f_* g^!,$$

exchanged by adjunction.

Here are two convenient ways to correctly state (i), (ii).

1^{st} **way:** For "!": the $D(X)$ organize as the fibers of a fibered and cofibered category (SGA VI) over the category of schemes of finite type over S. For "$*$": they organize as well as the fibers of fibered and cofibered category over the opposite of the category of schemes of finite type over S.

2^{d} **way:** (i) The data $(X \mapsto D(X), f \mapsto f_*, \dots)$ and $(X \mapsto D(X), f \mapsto f_!, \dots)$ are 2-functors from the category of schemes to the 2-category of categories. The data $(X \mapsto D(X), f \mapsto f^*, \dots)$ and $(X \mapsto D(X), f \mapsto f^!, \dots)$ are similarly 2-functors from the opposite category of the category of schemes of finite type over S.

(ii) Adjunctions are the data of morphisms of functors

$$Id \to f_* f^* \quad Id \to f^! f_!$$

$$f^* f_* \to Id \quad f_! f^! \to Id$$

satisfying suitable identities.

The first point of view avoids mentioning 2-categories and 2-functors. For "$*$", and for the case of categories of sheaves, rather than derived categories, morphisms from G to F, above a morphism of schemes $f : X \to Y$, have a simple description: a morphism is the data, for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U etale over X and V etale over Y, of a map $G(V) \to F(U)$, with a compatibility for $U' \to U$ over X or $V \to V'$ over Y.

The second point of view allows replacing the 2-category of categories by other 2-categories. This makes it possible , starting from symmetries in the formulas, to use duality arguments. It will be our point of view.

B. Identities (i) For f separated, one has a morphism of functors

$$f_! \to f_*.$$

It is an isomorphism when f is proper.

(ii) If $f : X \to Y$ is smooth, there is a self equivalence

$$\Sigma_f : D(X) \to D(X)$$

and an isomorphism of functors

$$f^! \to \Sigma_f f^*.$$

If X/Y is everywhere (on X) of dimension d, the functor Σ_f is $K \mapsto K(d)[2d]$.

C. Localization The categories $D(X)$ are triangulated. For $i : Y \to X$ a closed embedding, and j the open embedding of the complement of Y in X, one has maps

$$i_* i^* K \to (j_! j^! K)[1]$$

$$j_* j^* K \to (i_! i^! K)[1]$$

functorial in K which together with the adjunction maps give distinguished triangles

$$j_! j^! K \to K \to i_* i^* K \to (j_! j^! K)[1]$$

$$i_! i^! K \to K \to j_* j^* K \to (i_! i^! K)[1].$$

1.3 Let us assume that $S = Spec(k)$ with k algebraically closed. The 4 operations formalism explained above suffices to define cohomology, plain or with compact support, as well as homology, plain or Borel-Moore, with coefficients in $K \in D(S)$, and contains Poincare duality. The category $D(S)$ is here very explicit; it contains an object 1, the unit for the tensor product. For p the projection from X to S, one defines:

$$H^n(X, K) = Hom(1, p_* p^* K[n]) \qquad H_n(X, K) = Hom(1, p_! p^! K[-n])$$

$$H^n_c(X, K) = Hom(1, p_! p^* K[n]) \quad H^{BM}_n(X, K) = Hom(1, p_* p^! K[-n])$$

with the Hom computed in $D(S)$.

If X is smooth purely of dimension d, one has

$$H^*(X, K) = H^{BM}_*(X, K(d)[2d])$$

$$H_c^*(X, K) = H_*(X, K(d)[2d])$$

For $\mathbf{Z}/l^n$ or $\mathbf{Q}_l$ coefficients, one has on S that $Hom(1, K)$ and $Hom(K, 1)$ are in duality (when K is of finite type), and for $K = 1$ this gives, still for X smooth

$$H^n(X, 1) = Hom(1, p_* p^*(1)[n]) = Hom(p^*(1), p^*(1)[n]) =$$

$$= Hom(p^!(1), p^!(1)[n]) = Hom(p_! p^!(1), 1[n]) = Hom(p_! p^!(1)[-n], 1) =$$
$$= \text{ dual of } H_n(X, 1).$$

2 Generalities on 2-categories

2.1 The notion of 2-category comes in two flavors: strict and lax. We will use the strict version. There are objects, 1-morphisms $f : X \to Y$ between objects, and 2-morphisms between 1-morphisms with the same source and target.

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

The 2-morphisms can be composed, turning each $Hom(X, Y)$ into a category and composition of 1-morphisms is a functor

$$g, f \mapsto g \circ f : Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z)$$

The composition is associative with units ("on the nose"). If $\alpha : f \to f'$ and $\beta : g \to g'$ are 2-morphisms, we write $\beta * \alpha$ for the image of (β, α) by the composition functor. It is a 2-morphism from gf to $g'f'$. We write $\beta * f$ for $\beta * Id_f$ and similarly for $g * \alpha$.
Example: the 2-category of small categories: objects are small categories, 1-morphisms are functors and 2-morphisms are natural transformations betwen functors.

Definition 2.2 *A 2-functor (also called: pseudo-functor) from a category C to a 2-category D is:*

(a) a map $F : Ob(C) \to Ob(D)$;

(b) for X, Y in C, a map from $Hom(X,Y)$ to the set of 1-morphisms from $F(X)$ to $F(Y)$;

(c) for $X \xrightarrow{f} Y \xrightarrow{g} Z$ in C, an invertible 2-morphism

$$c(g,f) : F(gf) \to F(g)F(f); \tag{2.2.1}$$

it is called the composition isomorphism.

The data should satisfy:

(i) for a triple composite hgf in C, the diagram of isomorphisms deduced from the isomorphisms (c)

$$\begin{array}{ccc} F(hgf) & \longrightarrow & F(hg)F(f) \\ \big\downarrow & & \big\downarrow \\ F(h)F(gf) & \longrightarrow & F(h)F(g)F(f) \end{array}$$

is commutative.

(ii) for X in C, $F(Id_X)$ is an equivalence, that is, there exists $u : F(X) \to F(X)$ such that $u \circ F(Id_X)$ and $F(Id_X) \circ u$ are isomorphic to the identity of $F(X)$.

Remark 2.3 This definition of a 2-functor is "lax"; equalities are only assumed between 2-morphisms. As a consequence, if for each f in C we give a 1-morphism $F(f)' : F(X) \to F(Y)$ and an isomorphism $\alpha : F(f) \to F(f)'$, one can uniquely organize the $F(f)'$ into a 2-functor F', in such a way that α is a strict morphism of 2-functors $F \to F'$: one transports the composition isomorphisms (2.2.1) from the $F(f)$ to te $F(f)'$ using α.

Remark 2.4 A category can be viewed as a 2-category in which the only 2-morphisms are identities, and from this point of view 2.2 is a special case of the notion of 2-functor between 2-categories. The definition of such a 2-functor is obtained by modifying 2.2 as follows: in (b), F is a functor from $Hom(X,Y)$ to $Hom(F(X),F(Y))$, and in (c), $c(g,f)$ is assumed to be functorial in f and g. One composes 2-functors in the obvious way.

Lemma 2.5 *(i) If F is a 2-functor, then for each $X \in Ob(C)$ there is a unique isomorphism $\alpha : Id_{F(X)} \to F(Id_X)$ such that the diagram*

$$\begin{array}{ccc} Id_{F(X)} & \xrightarrow{=} & Id_{F(X)}Id_{F(X)} \\ \downarrow & & \downarrow \\ F(Id_X) = F(Id_X Id_X) & \xrightarrow{(2.2.1)} & F(Id_X)F(Id_X) \end{array} \tag{2.5.1}$$

commutes.

(ii) If, using the isomorphisms $\alpha^{-1} : F(Id_X) \to Id_{F(X)}$ we replace as in 2.3 each $F(Id_X)$ by $Id_{F(X)}$ keeping unchanged the other $F(f)$, then, for the 2-functor F' so obtained, and for any $f : X \to Y$, the morphisms

$$F'(f) = F'(f \circ Id_X) \overset{(2.2.1)}{\longrightarrow} F'(f)F'(Id_X) = F'(f)$$

$$F'(f) = F'(Id_Y \circ f) \overset{(2.2.1)}{\longrightarrow} F'(Id_Y)F'(f) = F'(f)$$

are the identity of $F'(f)$.

Proof:

(i) If α, β are isomorphisms from $Id_{F(X)}$ to $F(Id_X)$, the diagram of isomorphisms

$$\begin{array}{ccc} Id_{F(X)} & \xrightarrow{=} & Id_{F(X)}Id_{F(X)} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta * \alpha} \\ F(Id_X) & \xrightarrow{F(Id_X)*\alpha} & F(Id_X)F(Id_X) \end{array}$$

is commutative. For $\beta = \alpha$, this shows that the commutativity of (2.5.1) is equivalent to the equality $c(Id_X, Id_X) = F(Id_X) * \alpha$.

By the assumption ii, $F(Id_X)$ is an equivalence, hence for any Y in D the composition with $F(Id_X)$ is an equivalence of categories:

$$F(Id_X)\circ : Hom(Y, F(X)) \to Hom(Y, F(X))$$

Let us take $Y = F(X)$. The isomorphism

$$c(Id_X, Id_X) : F(Id_X)Id_{F(X)} = F(Id_X) = F(Id_X Id_X) \longrightarrow F(Id_X)F(Id_X)$$

comes from a unique isomorphism $\alpha : Id_{F_X} \to F(Id_X)$; it is the unique isomorphism for which (2.5.1) commutes.

(ii) By construction, the functor F' is such that $F'(Id_X) = Id_{F'(X)}$ and that $c(Id_X, Id_X)$ is the identity. We have to show that for such a functor the conclusions of (ii) hold. Let us apply i to f, Id_X, Id_X. We obtain the commutativity of the square

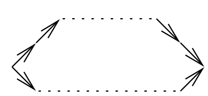
$$\begin{tikzcd}
F'(f) = F'(fId_XId_X) \arrow[r] \arrow[d] & F'(fId_X)Id_{F(X)} \arrow[d] \\
F'(f)F(Id_XId_X) \arrow[r, "="] & F'(f)Id_{F(X)}Id_{F(X)} = F'(f)
\end{tikzcd}$$

i.e. that the isomorphism $c(f, Id_X)$ is idempotent, hence the identity. Considering i for Id_Y, Id_Y, f, one similarly sees that $c(Id_Y, f)$ is the identity.

Remark 2.6 This lemma shows the essential equivalence of our "lax" setting with the "strictly unital" setting where 2-functors are assummed such that $F(Id_X) = Id_{F(X)}$ and that $c(Id_X, f)$, $c(f, Id_Y)$ are always the identity.

Remark 2.7 It is convenient to use 3-dimensional cell complexes to picture compatibilities in a 2-category.

For ordinary categories, a 1-dimensional CW complex with oriented edges is used to picture a system of arrows, and if one attaches to

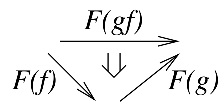


a 2-cell with boundary, this 2-cell is said to be commutative if the top composite equals the bottom composite.
Example: Two composable maps define a commutative triangle

$$\begin{tikzcd}
{} \arrow[rr, "gf"] \arrow[dr, "f"'] & & {} \\
& {} \arrow[ur, "g"'] &
\end{tikzcd} \tag{2.7.1}$$

When we work in a 2-category, such a 2-cell will be given with a top and a bottom, and will correspond to a 2-morphism from the top to the bottom composite.

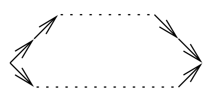
Example: the datum 2.2.c of a 2-functor $F : C \to D$ attaches to the commutative triangle 2.7.1 in C the triangle



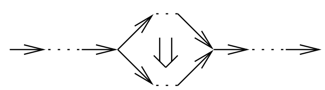
(2.7.2)

The 2-morphism 2.7.2 being a 2-isomorphism, it could be lablled $\Leftrightarrow$.

A decomposition of the disc with boundary

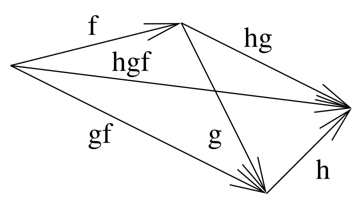


with a 2-morphism given for each 2-cell into which the disc is decomposed, will define a *unique* morphism from the top to the bottom composite if one can go from top to bottom by a sequence of moves

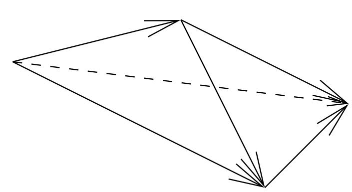


Suppose that two such decompositions are given. They define decomposition of S^2, to which one can attach B^3, This 3-ball will be said to be commutative if the two 2-morphisms corresponding to the decompositions are equal.

Example: In a category C, a sequence of 3 composable maps defines a tetrahedron with commutative faces



If we apply to it a 2-functor, we obtain using 2.7.1, the 2-dimensional picture (boundary of the 3-simplex)



obtained by gluing

$$\begin{tikzcd} \cdot \arrow[rrr, "hgf"] \arrow[d, "f"'] \arrow[drr] \arrow[dr, phantom, "\Downarrow"] & & \arrow[d, phantom, "gf \Downarrow"] & \cdot \\ \cdot \arrow[rr, "g"'] & & \cdot \arrow[ur, "h"'] & \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd} \cdot \arrow[rrr, "hgf"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & & & \cdot \\ \cdot \arrow[rr, "g"'] \arrow[urrr, "gh"] & & \cdot \arrow[ur, "h"'] \arrow[u, phantom, "\Downarrow"] & \end{tikzcd}$$

. The axiom 2.2.i in the definition of a 2-functor can be expressed by saying that the 3-simplex having this as a boundary is commutative.

Remark 2.8 To a 2-category D, one can attach another D^{1-opp} (resp. D^{2-opp}) by reversing the direction of the 1-maps (resp. 2-maps). One can also reverse both, obtaining D^{12-opp}.

2.9 We will use the results of A. J. Power (see [3]).

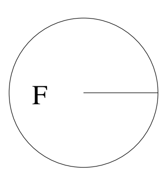
Let Γ be a planar directed graph, given with an embeding in an oriented plane. The complememnt of Γ in the plane has one unbounded connected component whose boundary is called the boundary of Γ, and a number of bounded connected components, the *faces*.

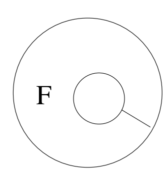
Power says that Γ is a *pasting scheme* if:

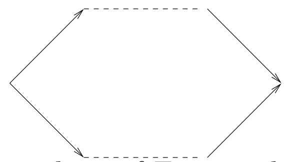
(i) There are on the boundary of Γ distinct vertices s and t called source and target, such that for any vertex x there are paths from s to x and from x to t. "Path" means "directed path". It results from this condition that Γ is connected.

(ii) If one goes clockwise around a face F, starting from some vertex v, one obtains a sequence of oriented edges. One requires that there are on the boundary of each F distinct vertices $s(F)$, $t(F)$ such that, starting from $s(F)$, this sequence is a non-empty directed path $\sigma(F)$ from $s(F)$ to $t(F)$ followed by the inverse of a directed path $\tau(F)$ from $s(F)$ to $\tau(F)$.

It follows from (i) and (ii) that the boundary of each face is S^1, with no identification:

no 

nor 

It looks like 

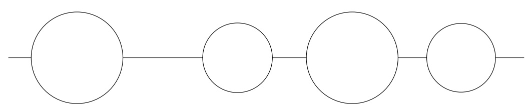
The edges on the boundary of Γ are either

(a) adjacent to no face;

(b) adjacent to one face, and going clockwise around it;

(c) adjacent to one face, and going counter-clockwise around it.

Under the assumptions (i),(ii), the edges of type (a) or (b) (resp. (a) or (c)) form a path from s to t. It is called $\sigma(\Gamma)$ (resp. $\tau(\Gamma)$): the boundary looks like:



The assumptions (i),(ii) also imply that Γ has no cycle.

If γ is any path from s to t, it cuts Γ into a part between $\sigma(\Gamma)$ and γ and a part between γ and $\tau(\Gamma)$. Those two halves are still pasting schemes. For one, Γ_1, one has

$$\sigma(\Gamma_1) = \sigma(\Gamma), \quad \tau(\Gamma_1) = \gamma$$

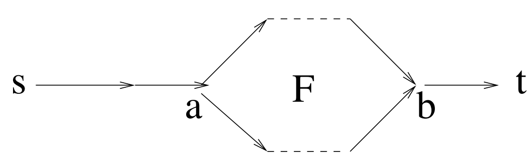
for the other, Γ_2, one has

$$\sigma(\Gamma_1) = \gamma, \quad \tau(\Gamma_1) = \tau(\Gamma).$$

If γ is distinct from $\sigma(\Gamma)$ and $\tau(\Gamma)$, both Γ_1 and Γ_2 have faces, hence have each less faces than Γ. This gives a reduction process to Γ's with no faces:

(2.9.1)

or just one face:



(2.9.2)

2.10 Let T be a 2-category. A labelling of Γ with values in T, or a 2-diagram of shape Γ in T, is an assignment D:

vertex $x \mapsto$ object $D(x)$ of T

edge e from x to $y \mapsto$ 1-morphism $D(e)$ from $D(x)$ to $D(y)$

face $F \mapsto$ 2-morphism $D(F)$ from the composite of the $D(e)$, e in $\sigma(F)$ to the composite of the $D(e)$, e in $\tau(F)$.

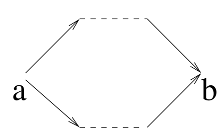
Such a labeling defines a composite of D, which is a 2-morphism between two 1-morphisms from $D(s)$ to $D(t)$. The source (resp. target) is the composite of the $D(e)$ for e in $\sigma(\Gamma)$ (resp. in $\tau(\Gamma)$). For Γ with no faces (2.9.1), it is the identity. For Γ with just one face, as in (2.9.2), it is

(composite 1-morphism from $D(b)$ to $D(t)$)$*D(F)*$(composite 1-morphism from $D(s)$ to $D(a)$)

If γ is a path from s to t, distinct from $\sigma(\Gamma)$ and from $\tau(\Gamma)$, which as explained above cuts Γ into Γ_1 and Γ_2, D restricts to labellings D_1, D_2 of Γ_1, Γ_2, and

$$m(D) = m(D_1)m(D_2).$$

2.11 If in Γ we have two paths from a to b, meeting only at a and b



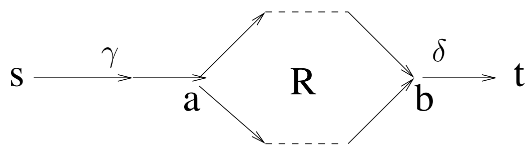
(2.11.1)

they limit a region R in the plane. The part of Γ in $\bar{R}$ is a pasting scheme, with source a and target b, denoted $\Gamma \cap \bar{R}$. If we remove from Γ the part in R, we also get a pasting scheme, noted $P(\Gamma, R)$, of which R is a face. It has the same σ and τ as Γ.

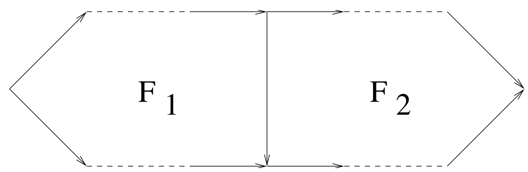
A labelling D of Γ induces a labelling $D_{\bar{R}}$ of $\Gamma \cap \bar{R}$. It also induces a labelling $D_{P(\Gamma,R)}$ of $P(\Gamma, R)$: the only new face is R, and $D(R)$ is defined to be $m(D_{\bar{R}})$. One has:

$$m(D) = m(D_{P(\Gamma,R)}) \tag{2.11.2}$$

To check this, one chooses a path γ from s to a, a path δ from b to t and one computes $m(D)$ by cutting Γ along $\delta\sigma(R)\gamma$ and $\delta\tau(R)\gamma$:



Special case: suppose that an edge e is in the boundary of two faces F_1 and F_2, and that it is the last edge of $\sigma(F_1)$ and the first of $\tau(F_2)$. The faces F_1 and F_2 must then look as



(2.11.3)

and if we remove e from Γ, the resulting graph $\Gamma - e$ is still a pasting scheme. The two faces F_1 and F_2 join to form a single face of $\Gamma - e$.

If in addition e is not the whole of $\sigma(F_1)$ nor the whole of $\tau(F_2)$, then chamging the direction of e also results in a pasting scheme.

Other cases when reversing the direction of e gives again a pasting scheme are when

(a) e is the last edge of $\sigma(\Gamma)$, not the whole of $\sigma(\Gamma)$, and on at least one face

(b) e is the first edge of $\tau(\Gamma)$, not the whole of $\tau(\Gamma)$, and on at least one face.

Notation:

(i) It is convenient to sometimes use the standard orientation of the sheet of paper on which the diagramm is drawn, sometimes the opposite. This can be indicated by specifying $\sigma(F)$ and $\tau(F)$ for any face F.

(ii) In a diagram of shape Γ, we will represent as $\underline{\longrightarrow}$ identity 1-morphisms.

3 Adjunctions in 2-categories

3.1 An adjunction between functors $f : A \to B$ and $g : B \to A$ a functorial bijection

$$Hom(fA, B) = Hom(A, gB). \tag{3.1.1}$$

The data of an adjunction amounts to that of morphisms of functors

$$\epsilon : Id_A \to gf \quad \text{and} \quad \eta : fg \to Id_B$$

such that the resulting compositions

$$f \to fgf \to f \quad \text{and} \quad g \to gfg \to g$$

are identities. The morphisms ϵ and η are deduced from (3.1.1) as follows: $\epsilon(A) : A \to gf(A)$ corresponds by (3.1.1) to the identity of $f(A)$, and $\eta(B) : fg(B) \to B$ to the identity of B. See [2].

The second description of adjunction keeps making sense for functors replaced by 1-morphisms in any 2-category. Terminology (in any 2-category): g, given with ϵ and η, is a right adjoint to f; f, given with ϵ and η, is a left adjoint to g, (ϵ, η) is an adjunction between f and g (the left adjoint being mentioned first).

In the language of pasting schemes, the axioms of an adjunction mean that the composites of the pasting schemes

$$\begin{tikzcd} A \arrow[r, Rightarrow] \arrow[d, "f"'] \arrow[dr, phantom, near start, "\varepsilon"] \arrow[dr, phantom, near end, "\eta"] & A \arrow[d, "f"] \\ B \arrow[ur] \arrow[r, Rightarrow] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, Rightarrow] \arrow[d, "g"'] \arrow[dr, phantom, near start, "\eta"] \arrow[dr, phantom, near end, "\varepsilon"] & B \arrow[d, "g"] \\ A \arrow[ur] \arrow[r, Rightarrow] & A \end{tikzcd}$$

are the identity of f and g resp.

Lemma 3.2 *If (g, ϵ, η) and (g', ϵ', η') are two right adjoints to f, there exists a unique isomorphism α from g to g' which transports (ϵ, η) into (ϵ', η').*

Proof: Suppose that $\alpha : g \xrightarrow{\sim} g'$ transforms ϵ into ϵ'. Applied to the target of $Id : g \to g$, α transforms Id_g into $\alpha : g \to g'$. Expressing Id_g in terms of $(\epsilon, \eta,)$ we see that α is the composite of

$$\begin{tikzcd} B \arrow[r, Rightarrow] \arrow[d, "g"'] \arrow[dr, phantom, near start, "\eta"] \arrow[dr, phantom, near end, "\epsilon'"] & B \arrow[d, "g'"] \\ A \arrow[ur] \arrow[r, Rightarrow] & A \end{tikzcd}$$

This gives the unicity of α, and a formula for it. Let us check that this formula gives indeed an ismorphism from g to g' and that this isomorphism transposes transports (ϵ, η) into (ϵ', η').

The inverse is obtained by permuting the roles of g and g'. The composite

$$\begin{tikzcd} B \arrow[r, Rightarrow] \arrow[d, "g"'] \arrow[dr, phantom, near start, "\eta"] \arrow[dr, phantom, near end, "\epsilon'"] & B \arrow[r, Rightarrow] \arrow[d, "g'"] \arrow[dr, phantom, near start, "\eta'"] \arrow[dr, phantom, near end, "\epsilon"] & B \arrow[d, "g"] \\ A \arrow[ur] \arrow[r, Rightarrow] & A \arrow[ur] \arrow[r, Rightarrow] & A \end{tikzcd}$$

is the identity of g (compose first the inner parallelogram), and similarly with g replaced by g'. The isomorphism obtained transposrts ϵ into ϵ'

$$\begin{tikzcd}
 & B \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\eta"] & B \arrow[d, "g'"] \\
A \arrow[ur, "f"] \arrow[r, equal] \arrow[ur, phantom, "\epsilon", near end, swap] & A \arrow[ur] \arrow[r, equal] \arrow[ur, phantom, "\epsilon'", near end, swap] & A
\end{tikzcd}$$

is indeed ϵ' (compose firts the left parallelogram). It also transports η into η'. The composite

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "g'"'] \arrow[dr, phantom, "\eta'"] & B \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\eta"] & B \\
A \arrow[ur] \arrow[r, equal] \arrow[ur, phantom, "\epsilon", near end, swap] & A \arrow[ur, "f"'] &
\end{tikzcd}$$

is η'.

Remark 3.3 (i) This lemma is a justification for speaking of *the* right adjoint of f, when such an adjoint exists: it is unique up to a unique isomorphism.

(ii) The formula we obtained in the proof of 3.2 for the isomorphism of g with g' used only η and ϵ'. It follows that η' is uniquely determined by ϵ', and that in any adjunction, ϵ uniquely determines η. Dually, η determines ϵ. This justifies describing an adjunction by giving only ϵ, or only η.

Example 3.4 (i) Suppose that f is an equivalence: that it has right and left inverses u, v (fu and vf are isomorphic to the identities). The inverses u and v are necessarily isomorphic (same proof as the unicity of the inverse in a group: consider vfu), and an isomorphism $fu \to Id$ makes the inverse u right adjoint to f. An equivalence and its inverse form an adjoint pair for which ϵ and η are isomorphisms.

(ii) Suppose that composable morphisms f and g have righta adjoints f_r, g_r. Then $g_r f_r$ is righta adjoint to fg, with the adjunctions

$$\epsilon : Id \to g_r g \to g_r f_r f g$$

$$\eta : f g g_r f_r \to f f_r \to Id.$$

3.5 If $f, g : A \rightleftarrows B$ form an adjoint pair in a 2-category, with $f : A \to B$ the left adjoint, for any C,

$$f\circ, g\circ : Hom(C, A) \rightleftarrows Hom(C, B)$$

form an adjoint pair of functors, and similarly for

$$\circ g, \circ f : Hom(C, B) \rightleftarrows Hom(C, A)$$

Going back to the first description of adjunction, for functors, this means that one has

(a) bijections

$$Hom(fx, y) = Hom(x, gy) \tag{3.5.1}$$

functorial in $x \in Hom(C, A)$ and $y \in Hom(C, B)$;

(b) bijections

$$Hom(xg, y) = Hom(x, yf) \tag{3.5.2}$$

functorial in $x \in Hom(B, C)$ and $y \in Hom(A, C)$.

Unravelling the definition of (3.5.1) one checks that $u : fx \to y$ maps to the composite $(Id_g * u) \circ (\epsilon * Id_x) : x \to gfx \to gy$. The inverse bijection maps $v : x \to gy$ to $(\eta * Id_y) \circ (Id_f * v) : fx \to fgy \to y$.

In the language of pasting schemes, (3.5.2) maps

$$\begin{tikzcd} & A \arrow[dd, "f"] \\ C \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[r, phantom, "\phi"] & \\ & B \end{tikzcd} \quad \text{to the composite of} \quad \begin{tikzcd} & A \arrow[r, Rightarrow] \arrow[dd] \arrow[dr, phantom, "\varepsilon"] & A \\ C \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[r, phantom, "\phi"] & & \\ & B \arrow[uur, "g"'] & \end{tikzcd}$$

$$\begin{tikzcd} & A \\ C \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[r, phantom, "\psi"] & \\ & B \arrow[uu, "g"'] \end{tikzcd} \quad \text{to the composite of} \quad \begin{tikzcd} & A \arrow[ddr, "f"] & \\ C \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[r, phantom, "\psi"] & \arrow[r, phantom, "\eta"] & \\ & B \arrow[uu] \arrow[r, Rightarrow] & B \end{tikzcd}$$

Dual diagrams hold for (3.5.2). For instance

$$\begin{tikzcd} A \arrow[dr, "x"] \arrow[d, phantom, shift left=3ex, "\phi"] & \\ & C \\ B \arrow[uu, "g"] \arrow[ur, "y"'] & \end{tikzcd} \quad \text{maps to the composite of} \quad \begin{tikzcd} A \arrow[r, Rightarrow] \arrow[ddr, "f"'] & A \arrow[dr, "x"] & \\ & \arrow[l, phantom, "\varepsilon"] \arrow[r, phantom, "\phi"] & C \\ & B \arrow[uu] \arrow[ur, "y"'] & \end{tikzcd}$$

In algebraic notations, (3.5.2) maps $xg \to y$ to $x \to xgf \to yf$ and the inverse maps $x \to yf$ to $xg \to yfg \to y$.

Notations We will write

- (3.5.1) as $\phi \mapsto \phi^g$ (from $fx \to y$ to $x \to gy$),
- its inverse as $\psi \mapsto {}^f\psi$ (from $x \to gy$ to $fx \to y$),
- (3.5.2) as $\phi \mapsto \phi^f$ (from $xg \to y$ to $x \to yf$),
- its inverse as $\psi \mapsto {}^g\psi$ (from $x \to yf$ to $xg \to y$).

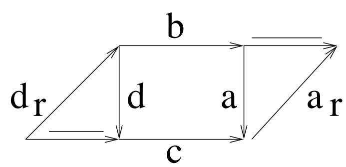
Given a 2-morphism

$$f_1 \ldots f_r \to g_1 \ldots g_s,$$

we can use a sequence of those bijections to transmute it into a 2-morphism between other composites, obtained by moving to the right some initial f's, replacing them by right adjoints, and moving to the left some final g's, replacing them by right adjoints. Principle: the order in which we do these moves is immaterial. Basic case: assuming the required adjoints exist, the following diagram is commutative. A right adjoint is denoted by an index r.

$$\begin{tikzcd} Hom(ab, cd) \arrow[r, "(3.5.1)"] \arrow[d, "(3.5.2)"'] & Hom(b, a_r cd) \arrow[d, "(3.5.2)"] \\ Hom(abd_r, c) \arrow[r, "(3.5.1)"] & Hom(bd_r, a_r c). \end{tikzcd} \tag{3.5.3}$$

This is checked by composing



in two different ways. By 3.5.3 the notation ${}^{d_r}\phi^{a_r}$ for ${}^{d_r}(\phi^{a_r}) = ({}^{d_r}\phi)^{a_r}$ is unambiguous. Similarly for left adjoints.

Special case: if 1-morphisms f, g have righta adjoints f_r, g_r, a morphism $\phi : f \to g$ defines a morphism $\phi_r := {}^{g_r}\phi^{f_r}$ from g_r to f_r.

3.6 We now list computation rules. We mainly consider right adjoints; duality gives similar statements for left adjoints.

3.6.1 Let D be a diagram of shape Γ, where Γ is a pasting scheme. Suppose we have in Γ

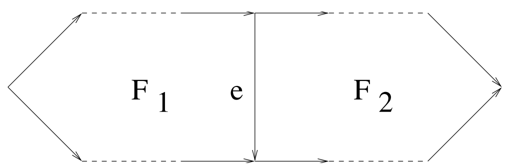
$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$$

and there are no other edges adjacent to the middle vertex. Let Γ' be obtained from Γ by substituting for this part of Γ

$$\bullet \xrightarrow{c} \bullet$$

We deduce from D a diagram D' of shape Γ', by defining $D(c) = D(b)D(a)$ and keeping the rest unchanged. The composites of D and D' are equal. In what follows, $D(a)$ or $D(b)$ will often be an identity.

3.6.2 Suppose that in a pasting scheme Γ an edge e is in the boundary of faces F_1 and F_2, that it is the last edge in $\sigma(F_1)$, the first in $\tau(F_2)$ and that neither $\sigma(F_1)$ nor $\tau(F_2)$ is reduced to e:



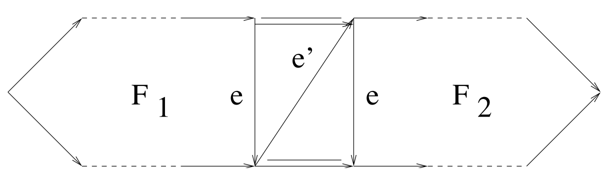
(*)

Let Γ' be deduced from Γ by reversing the orientation of e. If D is a diagram of shape Γ, and $D(e)_r$ a righta adjoint to D, we obtain a diagram D' of shape Γ' by changing $D(e)$, $D(F_1)$ and $D(F_2)$ as follows:

$$D'(e) = D(e)_r; \quad D'(F_1) = D(F_1)^{D(e)_r}; \quad D'(F_2) = {}^{D(e)_r}D(F_2).$$

The composites of D and D' are equal.

Proof: the story takes place withoin (*) and one considers



3.6.3 If e is the last edge of $\sigma(\Gamma)$, on at least one face F and is not the whole of $\sigma(\Gamma)$, we similarly can reverse the direction of e and define D' using a right adjoint $D(e)_r$ to $D(e)$. The conclusion is now that

$$(\text{composite of } D') = (\text{composite of } D)^{D(e)_r}.$$

Proof:

$$\begin{tikzcd}
{} \arrow[r, dashed] & \bullet \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "e"'] & t(\Gamma') \\
{} \arrow[r, dashed] & t(\Gamma) \arrow[ur, "e'"']
\end{tikzcd}$$

The proof also shows that the statement can be modified when $e = \sigma(\Gamma)$ by inserting an identity.

3.6.4 Similarly for e the first edge of $\tau(\Gamma)$ one has

$$(\text{composite of } D') = {}^{D(e)_r}(\text{composite of } D).$$

3.6.5 If f, g, h have right adjoints f_r, g_r, h_r, for 2-morphisms $f \xrightarrow{\phi} g \xrightarrow{\psi} h$, one has

$$(\psi\phi)_r = \phi_r\psi_r$$

Proof: in

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "h"'] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] \arrow[u, phantom, "\psi" near start, shift left=6] & \bullet \arrow[r] \arrow[u, phantom, "\phi" near start, shift left=6] & \bullet
\end{tikzcd}$$

change the direction of vertical arrows starting from the right.

Remark 3.7 Let D' be the following sub-2-category of a 2-category D:

- objects: same as in D,
- 1-morphisms: the 1-morphisms of D admitting a right adjoint,
- 2-morphisms: all 2-morphisms between such 1-morphisms.

That D' is a sub-2-category follows from the stability by compositions 3.4.ii, and from the fact that equivalences - in particular identities - have adjoints.

The unicity of adjoints defines a natural isomorphism from $(gf)_r$ to $f_r g_r$. If we take it as the composition isomorphism, the assignment $X \mapsto X$, $f \mapsto f_r$, $\phi \mapsto \phi_r$ is a 2-functor from $(D')^{12-opp}$ to D.

If $F : C \to D$ is a 2-functor from a category to a 2-category, such that each $F(f)$ has a right adjoint, composing F with this 2-functor we obtain a new 2-functor from C^{op} to D.

4 e-functors

4.1 Let C be a category provided with a class A of commutative squares. This class is assumed to be stable by vertical as well as horizontal compositions, and to contain the squares for which vertical (resp. horizontal) maps are isomorphisms.

We will define 4 kinds of "e-functors" from C to a 2-category D. The letter "e" is for "exchange".

A. Lower e-functors The data are:

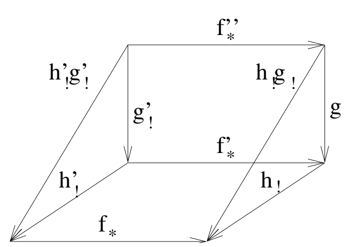
(a) two 2-functors H_* and $H_!$ from C to D, with $H_*(X) = H_!(X)$ for any X in C. We will write simply $H(X)$ for $H_*(X) = H_!(X)$, f_* for $H_*(f)$ and $f_!$ for $H_!(f)$.

(b) For each square $Q = \begin{array}{ccc} \bullet & \xrightarrow{f'} & \bullet \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow{f} & \bullet \end{array}$ in A, a 2-morphism

$$e_Q : g_! f'_* \to f_* g'_! \qquad \begin{tikzcd} \bullet \arrow[r, "f'_*"] \arrow[d, "g'_!"'] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d, "g_!"] \\ \bullet \arrow[r, "f_*"'] & \bullet \end{tikzcd}$$

Axioms:

(i) compatibility for vertical and horizontal composition of squares. For "vertical" this means commutativity of the solid



where the squares bear the 2-morphisms b and the triangles the 2-morphisms 2.2.c for $H_!$.

(ii) If both vertical or both horizontal morphisms are identities then the exchange morphism is an isomorphism.

B. Upper e-functors: lower e-functors with values in D^{1-opp}. The direction of the structural 2-morphisms remains the same. Each square Q in A gives a 2-morphism

$$e_Q : f'^* g^! \to g'^! f^* \qquad \begin{tikzcd} \cdot & \cdot \arrow[l, "f'^*"'] \\ \cdot \arrow[u, "g'^!"] & \cdot \arrow[l, "f^*"] \arrow[u, "g^!"'] \arrow[ul, phantom, "\Swarrow"] \end{tikzcd}$$

C. e* and e! contradirectional functors : the data is that of a covariant and a contravariant 2-functors $H_\bullet$, $H^\bullet$, and for each Q in A,

$$\text{for } *: \quad e_Q : f^\bullet g_\bullet \to g'_\bullet f'^\bullet \qquad \begin{tikzcd} \cdot \arrow[d, "g'_\bullet"'] & \cdot \arrow[l, "f'^\bullet"'] \arrow[d, "g_\bullet"] \\ \cdot & \cdot \arrow[l, "f^\bullet"] \arrow[ul, phantom, "\Nwarrow"] \end{tikzcd}$$

$$\text{for } !: \quad e_Q : g'_\bullet f'^\bullet \to f^\bullet g_\bullet \qquad \begin{tikzcd} \cdot \arrow[d, "g'_\bullet"'] & \cdot \arrow[l, "f'^\bullet"'] \arrow[d, "g_\bullet"] \\ \cdot & \cdot \arrow[l, "f^\bullet"] \arrow[ul, phantom, "\Searrow"] \end{tikzcd}$$

Axioms: same as for codirectional (upper and lower).

Remark 4.2 If in a lower e-functor, $H_!$ transforms identities to identities and the composition isomorphisms $c(Id, f)$, $c(f, Id)$ are identities (cf. 2.5.ii), and if Q is a square in A such that g and g' are identities, then $e_Q = Id$. If we compose vertically Q with itself, one obtains from the vertical composition axiom that $e_Q^2 = e_Q$. Since e_Q is an ismorphism it follows that $e_Q = Id$. Similar statement holds for H_* and squares where f and f' are identities and for upper and contradirectional e-functors.

Remark 4.3 There is a number of "duality" operations around:

replace squares in A by their transposed

replace D by D^{1-opp} or D^{2-opp}

exchange $*$ and !, or $H^\bullet$ and $H_\bullet$

Replacing D by D^{1-opp} exchanges upper and lower e-functors. Replacing D by D^{2-opp} exchanges e* and e! cases of 2.8.C. We have also symmetries:

For upper (lower) e-functors:

$$A \mapsto A^{tr}, \ * \text{ and } ! \text{ exchanged, and } D \mapsto D^{2-opp}$$

In the contradirectional case:

$$A \mapsto A^{tr}, \ {}_\bullet \mapsto {}^\bullet \ \text{ and } \mathrm{D} \mapsto \mathrm{D}^{1-\mathrm{opp}}$$

4.4 Example of a lower e-functor: take for A the commutative squares, for H_* a 2-functor, $H_! = H_*$ and the obvious structural 2-isomorphisms.

4.5 Example: we will prove later that in the l-adic situation, $(f_*, f_!)$ is a lower e-functor, $(f^*, f^!)$ is an upper e-functor, and $(f_*, f^!)$ as well as $(f_!, f^*)$ are e*-functors for which the exchange morphisms are isomorphisms. Inverting the exchange morphisms they can be viewed as e!-functors as well.

4.6 Suppose that a lower e-functor $(H_*, H_!)$ is such that each $f_!$ has a right adjoint, and let us choose one, $f^!$. By 3.7, the $f^!$ organize into a contravariant 2-functor $H^!$. For a square

$$Q \quad = \quad \begin{array}{ccc} \bullet & \xrightarrow{f'} & \bullet \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow{f} & \bullet \end{array}$$

in A, the exchange map e_Q transmutes into a map

$${}^{g'^!}(e_Q)^{g^!} : f'_* g'^! \to g^! f_*$$

Those maps turn $(H_*, H^!)$ into a e!-functor. Let us check for instance compatibility with vertical composition. For $(H_*, H_!)$ the compatibility with vertical composition asserts that the composite of exchange morphisms and composition isomorphisms or their inverse given by

$$
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[dd, "(g'h')_!"'] & \bullet \arrow[r, "f''_*"'] \arrow[d, "h'_!"'] & \bullet \arrow[r, equal] \arrow[d, "h_!"] & \bullet \arrow[dd, "(gh)_!"] \\
& \bullet \arrow[r, "f'_*"'] \arrow[d, "g'_!"'] & \bullet \arrow[d, "g_!"] & \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "f_*"'] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}
$$

is the exchange morphism for

$$
\begin{tikzcd}
\bullet \arrow[r, "f''"] \arrow[d, "g'h'"'] & \bullet \arrow[d, "gh"] \\
\bullet \arrow[r, "f"] & \bullet
\end{tikzcd}
$$

Reversing the direction of arrows, in the order h, g, gh, g', h', $g'h'$, we obtain the vertical composition axiom for $(H_*, H^!)$.

4.7 Similarly, if each f_* has a left adjoint f^*, and we choose one, we obtain a e*-functor $(H^*, H_!)$.

When each f_* has a left adjoint and each $f_!$ a righat adjoint $f^!$, the exchange morphisms for $(H^*, H_!)$ and $(H_*, H^!)$ are deduced from each other by adjunction. If one is an isomorphism so is another. If this holds one obtains new contradirectional functors by inverting the exchange morphisms, an from those we obtain in the same way an upper e-functor $(H^*, H^!)$. The system so obtained is called a *cross functor*.

4.8 Let A be a class of commutative squares in a category C , as in 4.1.

A *cross-functor* from C to a 2-category D, relative to A, is the following:

(a) an upper e-functor $(H^*, H^!)$ and a lower e-functor $(H_*, H_!)$ from C to D, rel A. One supposes that the 2-functors $H^*, H_*, H^!, H_!$ agree on objects. For $f : X \to Y$ in C we will write $f^*, f_*, f^!, f_!$ for $H^*(f), \ldots$;

(b) for any $f : X \to Y$ in C, an adjunction bewteen f^* and f_* as well as between $f_!$ and $f^!$. The left adjoints are f^* and $f_!$.

The following axioms should hold:

(c) compatibility of adjunction with composition: an iterated bijection (3.5.1), (3.5.2) transforms the isomorphism $(gf)_* \to g_*f_*$ into the inverse of the isomorphism $(gf)^* \to f^*g^*$ and similarly for !'s.

(d) given a square in A

$$\begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

the corresponding exchange morphisms

$$g_! f'_* \to f_* g'_! \quad \text{and} \quad f'{*}g^! \to g'^! f^*$$

transmute by (3.5.1), (3.5.2) into morphisms

$$f_* g_! \to g'_! f^* \quad \text{and} \quad g'_! f'^* \to f^* g_!$$

which are mutually inverse.

Example 4.9 In the l-adic case,

smooth $\Longrightarrow$ strongly upper transversal

proper $\Longrightarrow$ strongly lower transversal

unramified $\Longrightarrow$ strongly upper cotransversal

separated $\Longrightarrow$ strongly lower cotransversal

5 Transversal and cotransversal morphisms

5.1 Let C be a category with fiber products and $(H^!, H^*)$ an upper e-functor from C to D, rel. the class of Cartesian squares in C. It will be convenient to normalize the 2-functors $H^!$ and H^* to be strictly unital, as in **??**.

By application of the duality $D \mapsto D^{1-opp}$, our constructions give parallel constructions for lower e-functors. The duality $D \mapsto D^{2-opp}$ exchanges the role of $H^!$ and H^*.

Under the heading "etale example", we will from time to time explain how the notions introduced apply in the case of etale cohomology.

A monomorphism $f : X \to Y$ gives rise in C to a cartesian square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{5.1.1}$$

and in D to an exchange morphism in the square

$$\begin{tikzcd} H(X) \arrow[r, equal] \arrow[d, equal] & H(X) \arrow[d, "f^!"] \\ H(X) \arrow[r, "f^*"'] & Y \end{tikzcd} \tag{5.1.2}$$

that is to a 2-morphism

$$f^! \to f^* \quad \text{for } f \text{ a monomorphism} \tag{5.1.3}$$

This construction is self-dual (for $D \mapsto D^{2-opp}$, exchanging $*$ and $!$ and reversing 2-morphisms).

If f is an isomorphism, the corresponding morphism (5.1.3) is an isomorphism. Indeed, it is the exchange map rel a square in which the vertical maps are isomorphisms.

Etale example: open embeddings and closed embeddings are monomorphisms. For open embeddings, (5.1.3) is an isomorphism. For closed embeddings, (5.1.3) is derived from the map, defined for any sheaf of abelian groups on Y

$$(\text{sections with support on } Y) \to (\text{sections on } Y).$$

We now list some compatibilities. In the diagrams used to justify them, we will use the following convention to show in just one picture both a diagram in C and one (often a pasting scheme) in D: each object X in C is to be replaed by $H(X)$, a map f in C is to be replaced by f^* or by $f^!$, which one being indicated by a mark $*$ or !, and relevant exchange 2-morphisms are shown. Composition isomorphisms will often be omitted. For instance, (5.1.1) and (5.1.2) would be compressed as

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "!"] \\ X \arrow[r, "*"] & Y \end{tikzcd} \tag{5.1.4}$$

When this ... non ambiguity, parallel maps in a diagram will sometimes be denoted by the same letter.

Lemma 5.2 *The formation of (5.1.3) is compatible with the composition of monomorphisms.*

In other words, for a composite $X \xrightarrow{f} Y \xrightarrow{g} Z$, the composition isomorphisms $(gf)^* \to f^*g^*$ and $(gf)^! \to f^!g^!$ identify (5.1.3) for gf with ((5.1.3) for f)$*$((5.1.3) for g).

Proof: Consider

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] \arrow[d, "!"] & X \arrow[d, "!"] \\
X \arrow[r, "*"] \arrow[d, equal] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, "!"] \\
X \arrow[r, "*"'] & Y \arrow[r, "*"'] & Z
\end{tikzcd}$$

and use the compatibility with vertical and horizontal composition.

Lemma 5.3 *The formation of (5.1.3) is compatible with a base change $h : Y' \to Y$, using h^* or $h^!$*

For a cartesian square

$$\begin{tikzcd}
X' \arrow[r, "h'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

this means the commutativity of

$$\begin{tikzcd}
h'^*f^! \arrow[r, "exchange"] \arrow[d, "g'^{*}*(5.1.2)"'] & f'^!h^* \arrow[d, "(5.1.2)*g^*"] \\
h^*f^* \arrow[r] & f'^*h^*
\end{tikzcd}
\quad \text{and dually of} \quad
\begin{tikzcd}
h'^!f^* \arrow[d] & f'^*h^! \arrow[l] \arrow[d] \\
h^!f^! \arrow[r] & f'^!h^!
\end{tikzcd} \tag{5.3.1}$$

Proof: It suffice to consider the first diagram. It is a special case of the compatibility between exchange morphisms and change of base: .....

5.4 A morphism $f : X \to Y$ is said to be:

Transversal (rel H): if the exchange morphisms corresponding to the square

$$\begin{tikzcd}
X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{5.4.1}$$

is an isomorphism;

strongly transversal (rel H): if for any $g : Y' \to Y$ the morphism $pr_2 : X \times_Y Y' \to Y'$ deduced from f by the change of base g is transversal;

cotransversal (rel H): if the diagonal $X \to X \times_Y X$ is transversal;

strongly cotransversal (rel H): if the diagonal $X \to X \times_Y X$ is strongly transversal;

bitransversal (rel H): if it is transversal and cotransversal;

strongly bitransversal (rel H): if it is strongly transversal and strongly cotransversal;

A monomorphism f is transversal if and only if the 2-morphism (5.1.2): $f^! \to f^*$ is an isomorphism.

Etale example: open embeddings are transversal monomorphisms.

Etale example: unramified morphisms are strongly cotransversal.

Etale example: etale morphisms are bitransversal for $(f^*, f^!)$, proper morphisms are bitransversal for $(f_*, f_!)$.

If $f : X \to Y$ is cotransversal, composing the exchange map for (5.4.1) with the inverse of the isomorphism $\Delta^! \to \Delta^*$ we obtain a morphism

$$f^! \to f^* \tag{5.4.2}$$

which generalizes (5.1.3) from monomorphisms to cotransversal maps. We will prove for (5.4.2) compatibilities similar to lemmas 5.2 and 5.3. The following lemma gives a meaning to the question.

Lemma 5.5 *(i) We have the following stabilities by composition:*

(transversal monomorphism)$\circ$(transversal) is transversal

(cotransversal)$\circ$(strongly cotransversal) is cotransversal

(strongly cotransversal)$\circ$(strongly cotransversal) is strongly cotransversal

(ii) Strongly cotransversal map is universally so

Proof: The first stability is clear from

$$\begin{tikzcd}
\bullet \arrow[r, "*"] \arrow[d, "!"'] & \bullet \arrow[r, equal] \arrow[d] & \bullet \arrow[d, "g^!"] \\
\bullet \arrow[r, "*"] \arrow[d, equal] & \bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[d, "f^!"] \\
\bullet \arrow[r, "g^*"'] & \bullet \arrow[r, "f^*"'] & \bullet
\end{tikzcd}$$

For the second and third, one uses that the diagonal map $\Delta_{X/Z}$ for a composite $X \to Y \to Z$ is the composite first line of the diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta_{X/Y}"] & X_Y^2 \arrow[r] \arrow[d] & X_Z^2 \arrow[d] \\
& Y \arrow[r, "\Delta_{Y/Z}"] & Y_Z^2
\end{tikzcd}$$

in which the square is cartesian, and one applies the first stability. The statement (ii) is clear.

5.6 6.6 A morphism $p : X \to S$ and a section $x : X \to S$ define the following two 1-morphisms $H(S) \to H(S)$:

$$\Sigma_{X,x} := x^* p^! \quad \text{and} \quad \Omega_{X,x} := x^! p^*$$

they are exchanged by the duality $D \mapsto D^{2-opp}$.

As x is a monomorphism, it gives rise to a 2-morphism from $x^!$ to x^*. Applying to it $(-) * p^!$, we obtain

$$(x^! \to x^*) : Id \to \Sigma_{X,x}. \tag{5.6.1}$$

Etale example: if X/S is smooth purely of relative dimension d, $\Sigma_{X,x}$ is $K \mapsto K(d)[2d]$ and (5.6.1) is the cup product with the Euler class of the pullback by x of the relative tangent bundle. The dual $\Omega_{X,x}$ is $K \mapsto K(-d)[-2d]$.

For any $f : X \to Y$, the diagonal map $X \to X^2_Y$ is a section of $pr_1 : X^2_Y \to X$. We denote by $\Sigma_f : H(X) \to H(X)$ the corresponding 1-morphism. The diagram:

...

defines morphisms

$$f^! \xrightarrow{\Delta^* * (exchange)} \Sigma_f f^* \xleftarrow{(5.6.1) * f^*} f^* \tag{5.6.2}$$

If f is cotransversal, (5.6.1) is an isomorphism and (5.6.2) gives (5.4.2). To unravel compatibilities obeyed by (5.4.2), thisdecomposition will be useful.

We now list compatibilities obeyed by (5.6.1). The choice of x will be clear and we will write $\Sigma_{X/S}$ for $\Sigma_{X,x}$.

5.6.3 Fix $p : X \to S$ and $q : Y \to S$, with sections x and y, and let $f : X \to Y$ be an S-morphism such that $f(x) = y$. Define $K := S \times_Y X$. The diagram

$$\begin{tikzcd} S \arrow[r, "*"] & K \arrow[r, "*"] \arrow[d, "!"] & X \arrow[d, "!"] \\ & S \arrow[r, "*"] & Y \arrow[d, "!"] \\ & & S \end{tikzcd}$$

defines a morphism

$$\Sigma_{X/S} \to \Sigma_{K/S} \Sigma_{Y/S} \tag{5.6.4}$$

Lemma 5.7 *The diagram*

$$\begin{tikzcd} Id \arrow[r, equal] \arrow[d, "(5.6.1)"'] & IdId \arrow[d, "(5.6.1)*(5.6.1)"] \\ \Sigma_{X/S} \arrow[r, "(5.6.4)"] & \Sigma_{K/S}\Sigma_{Y/S} \end{tikzcd}$$

is commutative.

Proof: We will write ... for .... With this notation the upper composite is

...

while the lower composite is, by (5.2),

...

The compatibility to be checked in the square is (5.3) for the !-change of base.

5.8 6.9 Fix $p : X \to S$ pointed by a section x and let X', p', x' be deduced by a change of base $h : S' \to S$. The diagram

$$\begin{tikzcd} S' \arrow[r, "*"] \arrow[d, "*"'] & S \arrow[d, "x"', "*"] \\ X' \arrow[r, "*"] \arrow[d, "!"'] & X \arrow[d, "p"', "!"] \\ S' \arrow[r, "*"] & S \end{tikzcd}$$

defines a morphism

$$h^*\Sigma_{X/S} \to \Sigma_{X'/S'}h^* \tag{5.8.1}$$

Lemma 5.9 *(5.8.1) is compatible with (5.6.1):*

$$\begin{tikzcd} h^* \arrow[r, "(5.6.1)"] \arrow[d, equal] & h^*\Sigma_{X/S} \arrow[d, "(5.8.1)"] \\ h^* \arrow[r, "(5.6.1)"] & \Sigma_{X'/S'}h^*. \end{tikzcd}$$

Proof: Reduces to (5.3).

Proposition 5.10 *(cf. 5.2) The formation of (5.4.2) is compatible with the composition of strongly cotransversal morphisms.*

More generally, a composite $X \xrightarrow{f} Y \to gS$ is is cotransversal when g is cotransversal and f strongly cotransversal (by 5.5) and the compatibility 5.10 holds in that setting.**Proof**: Let us consider the following diagram in the category of 1-morphisms from $H(Z)$ to $H(X)$ (explanation follows)

$$\begin{tikzcd}
(gf)^! \arrow[r] \arrow[dddd] & \Sigma_{gf}(gf)^* \arrow[d, "\cong"] & (gf)^* \arrow[l, "(1)"'] \arrow[d, "\cong"] \\
& \Sigma_{gf} f^* g^* \arrow[d] & f^* g^* \arrow[l] \arrow[d, equal] \\
& \Sigma_f \Sigma_\Gamma f^* g^* & f^* g^* \arrow[l] \arrow[d, equal] \\
& & \\
f^! g^! \arrow[r] & \Sigma_f f^* \Sigma_g g^* \arrow[uu, "(2)"] & f^* g^* \arrow[l, "(1)"']
\end{tikzcd} \tag{5.10.1}$$

The morphisms (1) are isomorphisms, and we have to show the commutativity of the boundary square, after they are inverted. The morphism (2) will be shown to be an isomorphism (and Σ_Γ will be defined). Once this is shown, the required commutativity follows from that of each cell of (5.10.1). We first consider the right side $(5.10.1)_r$.

One defines Σ_Γ by $pr_1 : X \times_S Y \to X$ with the section (id_X, f), and the morphism $\Sigma_{gf} \to \Sigma_f \Sigma_\Gamma$ is defined by 5.6.3 applied to the morphism $X^2_S \to X \times_S Y$ over X which gives rise to

$$\begin{tikzcd}
X \arrow[r, "*"'] & X^2_Y \arrow[r, "*"'] \arrow[d, "!"] & X^2_S \arrow[d, "!"] \\
& X \arrow[r, "*"'] & X \times_S Y \arrow[d, "!"] \\
& & X
\end{tikzcd}$$

the commutativity of the middle square of $(5.10.1)_r$ being 5.7. The lower square is derived from 5.9 applied to

$$\begin{tikzcd}
X \arrow[r, "*"'] \arrow[d, "*"] & Y \arrow[d, "*", "\Delta"'] \\
X \times_S Y \arrow[r, "*"'] \arrow[d, "!"] & Y_S^2 \arrow[d, "!"] \\
X \arrow[r, "*"'] & Y.
\end{tikzcd}$$

As $Y \to S$ is strongly cotransversal, both horizontal arows are isomorphisms, and so is (2).

We now consider the left side $(5.10.1)_l$ of (5.10.1). We take the cartesian square of $X \to Y \to S$ over S and the pull-back of the left upper square by $\Delta : Y \to Y_S^2$, as well as the diagonal $X \to X_S^2$:

...

(unmarked arrows should be marked $*$). The starting point is $(gf)^!$, which we obtain, modulo composition isomorphisms, by going from S to X (by !) to X_S^2 to X. Its isomorphism with $f^! g^!$ - similarly obtained by going from S to Y (by !) to Y_S^2 to Y to X (by !) to X_Y^2 to X - is the exchange morphism for the face

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "!"] & Y \times_S X \arrow[r] \arrow[d, "!"] & X \arrow[d, "!"] \\
Y \arrow[r] & Y_S^2 \arrow[r] & Y
\end{tikzcd}$$

and the composition isomorphism for the back face of the cube. For these we go to $\Sigma_f f^* \Sigma_g g^*$ (S to Y to Y_S^2 (!) to Y to X to X_Y^2 (!) to X and then to $\Sigma_f \Sigma_\Gamma f^* g^*$. The identification with the other composite in $(5.10.1)_l$ is reduced to the compatibility .... for the cube.

Proposition 5.11 *(cf. 5.3) The formation of (5.4.2) for a strongly cotransversal $f : X \to Y$ is compatible with a base change $h : Y' \to Y$, using h^* or $h^!$.*

Proof: By duality, it suffices to consider h^*. One considers the product of the diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "*"] & \\
X \arrow[r, "!"] & X_Y^2 \arrow[r, "*"'] \arrow[d, "!"'] & X \arrow[d, "!"] \\
& X \arrow[r, "*"] & Y
\end{tikzcd}$$

which defines $f^! \to f^*$ and $Y' \xrightarrow{*} Y$ and uses compatibilities (5.3.1) to show the commutativity of the two cubes.

Lemma 5.12 *If in a cartesian square*

$$\begin{tikzcd}
\bullet \arrow[r, "g'", "*"'] \arrow[d, "f'"', "!"] & \bullet \arrow[d, "!"', "f"] \\
\bullet \arrow[r, "g", "*"'] & \bullet
\end{tikzcd}$$

f or g is strongly bitransversal then the exchange morphism is an isomorphism.

Proof: By duality, we may assume that it is f which is strongly bitransversal. By 5.11, the isomorphisms $f^! \to f^*$ and $f'^! \to f'^*$ transport th exchange morphism into the composition isomorphism $(fg')^* = (gf')^*$. The exchange morphism is hence an isomorphism.

6 The adjunction theorem

6.1 Let $x : S \to X$ be a section of $f : X \to S$. We will write Ω and Σ for $\Omega_X = x^! f^*$ and $\Sigma_X = x^* f^!$.

Suppose that f is transversal, that is the exchange morphism for the square

$$\begin{tikzcd}
X_S^2 \arrow[r, "pr_2", "*"'] \arrow[d, "!"', "pr_1"] & X \arrow[d, "!"'] \\
X \arrow[r, "*"'] & S
\end{tikzcd} \tag{6.1.1}$$

is an isomorphism. If one inverts this 2-morphism, the cartesian diagram

$$
\begin{tikzcd}
S \arrow[r, "!"'] \arrow[d, "*"] & X \arrow[r, "*"'] \arrow[d, "1"', "*"] & S \arrow[d, "*"] \\
X \arrow[r, "\Delta", "!"'] & X_S^2 \arrow[r, "pr_2", "*"'] \arrow[d, "!"', "pr_1"] & X \arrow[d, "!"'] \\
 & X \arrow[r, "*"'] & S
\end{tikzcd}
\tag{6.1.2}
$$

defines a morphism δ from $x^*\Delta^! pr_1^! f^* = x^* f^* = Id$ to $\Omega\Sigma$. Dually, one defines $\epsilon : \Sigma\Omega \to Id$ by exchanging the $*$ and $!$. In (6.1.2), the morphism 1 is $Id_X \times (xf)$.

Here is a way to picture the objects and morphisms in (6.1.2) which will be useful in more complicated cartesian diagrams. One first supposes that one is in $(Sets)$, and that S is one point. For each object in the diagram all maximal sequences of monomorphisms starting at this object terminate at the same object and hence we may consider each object as a subobject in one of the "maximal" objects which are not starting points for any monomorphisms.

Each object and in particular each maximal object is the product of some of the copies of X (in the case of (6.1.2) empty set of copies, the first copy, the second copy or both). We denote a generic element in the first copy of X by a and in the second copy of X by b. This gives us notations for the generic elements of all maximal objects. All morphisms between the maximal objects are projections and can be specified in the usual way by their action on generic elements e.g. $(a, b) \mapsto (a)$. The subobjects of the maximal objects can be specified by the image of their generic elements in the the maximal objects e.g. $(a = b)$ for the diagonal in X^2. Since the morphisms between all objects are restrictions of projections between the maximal objects we get a way to specify all morphisms in the diagram. The diagram (6.1.2) is encoded as follows:

$$
\begin{tikzcd}
(x, x) \arrow[r] \arrow[d] & (a, x) \arrow[r] \arrow[d] & x \arrow[d] \\
(a = b) \arrow[r] & (a, b) \arrow[r] \arrow[d] & b \arrow[d] \\
 & (a) \arrow[r] & (\bullet)
\end{tikzcd}
$$

Theorem 6.2 *If f is universally transversl, (Σ, Ω) is an adjoint pair, for the adjunctions morphisms δ and ϵ.*

Proof: It suffice to prove that the composite

$$\Omega \to \Omega\Sigma\Omega \to \Omega \tag{6.2.1}$$

is identity. That $\Sigma \to \Sigma\Omega\Sigma \to \Sigma$ is identity is the dual statement.

The morphism (6.2.1) is the composite of the following pasting diagram:

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & X \arrow[d, "*"', "\Delta"] & \\
X \arrow[r, "\Delta", "!"'] & XX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & XX \arrow[r] \arrow[d] & X \arrow[d, "!"'] \\
& X \arrow[r, "*"'] & S \arrow[r, "!"'] & X \arrow[r, "*"'] & S
\end{tikzcd} \tag{6.2.2}$$

where $*$ and ! are assumed to be the same for parallel maps at the same level (resp. same column). If we fold (6.2.2) along the middle vertical , we obtain two sides of the three dimensional cartesian diagram, described below:

$$\ldots \tag{6.2.3}$$

The diagram is made out of four cubes which we number as follows

$$\begin{matrix} 1 & 2 & \\ & 3 & \\ & & 4 \end{matrix} \tag{6.2.4}$$

We declare (6.2.3) to be viewed from above: 1, 2, 3 are the top cubes, 2,3,4 are the right cubes, 3,4 the front cubes.

We now describe each cube. Cube 4 consists of the fiber products of some of the three copies of X, which we label the first copy (left of S), the second (back) and the third (above S). The cube 3 is obtained from the top face of 4 by pull back by x. The cube 2 from the back face of 3 by pull back by Δ. The cube 1 from the left side of 2 by pull-back by $(\Delta, Id) : X^2 \to X^3$.

With the notations explained to describe the maps in (6.1.2), each object in (6.2.3) is a subobject of the product of some of the three copies of X with the generic elements a, b and c respectively. The diagram (6.2.3) can be encoded as follows:

$$\dots \tag{6.2.5}$$

We label the three directions in (6.2.3) as

$$u :\rightarrow \quad v :\downarrow \quad w :\searrow,$$

and decorate each arrow with $*$ or !, the decoration being the same for maps in the same direction and at the same level. Pattern used:

$$u :\xrightarrow{!}\xrightarrow{*} \quad v : \begin{array}{c}\downarrow_{*}\\ \downarrow_{!}\end{array} \quad w : \begin{array}{cc}\overset{*}{\searrow} & \\ & \overset{!}{\searrow}\end{array}$$

with this assignement, each face of each cube carries either an exchange morphism (some of which are isomorphisms by the universal transversality assumption), or a composition isomorphism.

Parallel faces at the same level are of the same type, and those types, viewed in projection to the plane (v, w) (resp. (u, w), (u, v)) are as follows:

2 Cross functors on schemes

1 Complimentary pairs

1.0.6 The 2-category TR of triangulated categories is defined as follows:

Objects: triangulated categories

1-morphisms: additive functors $F : D_1 \to D_2$ given together with a functorial isomorphism $\alpha : FT_1 \to T_2F$ where T_i are the translation functors for D_i, and carring diatinguished triangles to distinguished triangles.

2-morphisms: morphisms of functors for which

$$\begin{array}{ccc} FT_1 & \longrightarrow & T_2F \\ \downarrow & & \downarrow \\ F'T_1 & \longrightarrow & T_2F' \end{array}$$

is commutative.

We assume that in the definition of a triangulated category the shift functor T is given together with an inverse T^{-1}. Then the opposite of a triangulated category is again a triangulated category. This defines an isomorphism $D \mapsto D^{op} : TR \to TR^{2-opp}$.

1.0.7 Suppose given a cross functor H from a category C to TR. The following formalizes expected properties of the decomposition of a scheme into a closed subscheme and its open complement. Consider in C:

$$U \xrightarrow{j} X \xleftarrow{i} Z$$

where j is an upper transversal monomorphism, i a lower transversal monomorphism and $H(U \times_X Z) = 0$. These assumptions are invariant by replacing each $H(Y)$ by the opposite triangulated category and exchanging ! and $*$.

By transversality we have strings of adjunctions

$$j_!, \quad j^! \cong j^*, \quad j_*$$

and

$$i^*, \quad i_* \cong i_!, i^!$$

For j, the composite $Id \to j^! j_! \to j^* j_!$ is the exchange isomorphism which implies that the adjunction morphism $Id \to j^! j_!$ is an isomorphism. Similarly, $j^* j_* \to Id$ is an isomorphism. It follows that $j_!$ and j_* are fully faithful, with left inverse $j^! \cong j^*$.

For i, the composite $Id \to i^! i_! \to i^! i_*$ is an exchange isomorphism which implies that $Id \to i^! i_!$ is an isomorphism. Similarly for $i^* i_* \to Id$. It follows that $i_* \cong i_!$ is fully faithful, with left inverses $i^!$ and i^*.

The cartesian square

$$\begin{array}{ccc} Z \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

provides (contradirectional) exchange isomorphisms showing that $i^* j_! = i^! j_* = 0$ and $j^* i_! = 0$. The latter can be rewritten using $j^! \cong j^*$ and $i_! \cong i_*$.

It follows that $i_! H(Z) = i_* H(Z)$ is left orthogonal to $j_* H(U)$:

$$Hom(i_! A, j_* B) = 0$$

and that $i_! H(Z) = i_* H(Z)$ is right orthogonal to $j_! H(U)$.

Lemma 1.1 *With the notations and assumptions as above, the following conditions are equivalent:*

(a) (j^*, i^*) *is conservative,*

(b) for any K *in* $H(X)$, $j_! j^! K \to K \to i_* i^*$ *can be extended to a distinguished triangle,*

(c) $H(X) = j_! H(U) * i_* H(Z)$,

(a)* $(j^!, i^!)$ *is conservative,*

(b) for any* K *in* $H(X)$, $i_! i^! K \to K \to j_* j^*$ *can be extended to a distinguished triangle,*

(c)* $H(X) = i_! H(Z) * j_* H(U)$.

The starred statements correspond to the unstarred ones by duality. We will prove the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ and $(c) \Rightarrow (a*)$ from which all equivalences follow by duality.

The statement (a) is that if $j^*(\phi)$ and $i^*(\phi)$ are isomorphisms, so is ϕ. In the triangulated setting we are in, this ammounts to K being zero as soon as $j_* K$ and $i^* K$ are.

In (b), the boundary map $i_* i^* K \to j_! j^! K[1]$ completing the distinguished triangle is necessarily unique, and natural in K. Indeed, if we choose one for K and L in

$$\begin{array}{ccccccc}
j_! j^! K & \longrightarrow & K & \xrightarrow{u} & i_* i^* K & \xrightarrow{v} & j_! j^! K[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
j_! j^! L & \longrightarrow & L & \longrightarrow & i_* i^* L & \longrightarrow & j_! j^! L[1]
\end{array}$$

the first two vertical maps on the left can be completed to a morphism of distinguished triangles with the third map being (natural map)$+w$. We have $w \circ u = 0$, hence w factors through a map from $j_! j^! K[1]$ to $i_* i^* L$ such a map vanishes by orthogonality $j_! H(U) \perp i^* H(Z)$ and hence $w = 0$.

In (c), the notation $A * B$ has the following meaning. For A, B sets of objects in a triangulated category, $A * B$ is the set of objects e for which there is a distinguished triangle

$$a \to e \to b \to a[1]$$

with a in A and b in B (e is an "extension" of b by a). The octahedral axiom implies that $(A * B) * C = A * (B * C)$.

Proof of Lemma 1.1: $(a) \Rightarrow (b)$: Let us complete $j_!j^!K \to K$ to a distinguished triangle

$$j_!j^!K \to K \to ? \to j_!j^!K[1]$$

the composite $j_!j^!K \to K \to i_*i^*K$ vanishes by orthogonality $j_!H(U) \perp i_*H(Z)$, hence $K \to i_*i^*K$ factors through a morphism $w : ? \to i_*i^*K$. We will deduce from (a) that w is an isomorphism.

If we apply $j^* = J^!$, the triangle

$$j^!j_!j^!K \xrightarrow{\cong} j^!K \to j^!? \to \dots$$

shows that $j^!? = 0$, while $j^!i_*i^* = 0$ as well. If we apply i^*, $j_!j^*K$ gets killed and we have isomorphisms $i^*K \xrightarrow{\cong} i^*?$, $i^*K \xrightarrow{\cong} i^*i_*i^*K$. Hence i^*w is an isomorphism.

$(b) \Rightarrow (c)$: Trivial.

$(c) \Rightarrow (a)$: We have to show that if j^*K and i^*K vanish, so does K. Choose a distinguished triangle

$$j_!A \to K \to i_*B \to j_!A[1]$$

Applying $j^! = j^*$, we kill i_*B and obtain $j^!j_!A = 0$, hence $A = 0$. Applying i^*, we kill $j_!A$, obtain that $i^*i_*B = 0$, and hence $B = 0$. It follows that $K = 0$.

$(c) \Rightarrow (a*)$: Suppose that $j^!K = i^!K = 0$. Let us show that for any L, $Hom(L, K) = 0$. By the long exact sequence of Hom's, it suffices to check it for L of the form $j_!A$ and $i_!B$, for which it is clear by adjunctions.

Remark 1.2 When the equivalent conditions of the lemma hold, we are in the situation considered in [1, 1.4.3], where such a formalism is used to construct a t-structure on $H(X)$ from t-structures on $H(U)$ and $H(Z)$.

1.2.1 Fix a noetherian scheme S and let Sch/S be the category of separated schemes of finite type over S. Let

$$(H, f^*) : X \mapsto H(X) : f \mapsto f^*$$

be a contravariant 2-functor from Sch/S to TR. We will impose the following conditions:

1. *(0)* $H(\emptyset) = 0$

2. *(right adjoint)* for any f, f^* has a right adjoint f_* and for a closed embedding i the adjunction $i^* i_* \to Id$ is an isomorphism

3. *(left adjoint)* if p is smooth, p^* has a left adjoint $p_\#$ and for any pull-back square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow[f]{} & Y \end{array}$$

the exchange morphism $p'_\# f'^* \to f^* p_\#$ is an isomorphism

4. *(locality)* for a pair $U \xrightarrow{j} X \xleftarrow{i} Z$ where j is an open embedding, i is a closed embedding and $X = j(U) \amalg i(Z)$, the pair (j^*, i^*) is conservative

5. *(homotopy invariance)* if p is the projection $\mathbf{A}^1_X \to X$, the adjunction morphism $Id \to p_* p^*$ is an isomorphism

6. *(stability)* if s is the zero section of $p : \mathbf{A}^1_X \to X$, $p_\# s_*$ is an equivalence of $H(X)$.

We will show that under these conditions f^* extends uniquely to a cross functor (uniquely up to unique isomorphism in the category of cross functors with the same $H(X)$). For this unique extension smooth morphisms are upper transversal and proper morphisms are lower transversal.

Later, this will be applied to motivic triangulated categories, at least for S of equal characteristic.

Etale example: In the l-adic setting conditions 1 and 2 are obvious and conditions 4 and 5 are well known. To see 3 note that for smooth maps p^* agrees with $p^!$ up to a shift and twist and hence $p_\#$ is obtained from $p_!$ by the inverse shift and twist. The second half of 3 follows from the proper base change theorem. Property 6 holds because $p_\# s_*$ is $K \mapsto K(1)[2]$.

References

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