The four functors formalizm
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1 E-functors

1.1 First definitions

1.1.1 Let C be a category with fiber products and $\mathcal{D}$ a 2-category. A *covariant codirectional e-functor* $\mathcal{H}$ from C to $\mathcal{D}$ is given by:

Data

1. 2-functors $\mathcal{H}_*$ and $\mathcal{H}_!$ wich coincide on objects of C. The common value of $\mathcal{H}_*$ and $\mathcal{H}_!$ on $X \in ob(\mathsf{C})$ is denoted by $\mathsf{H}(X)$
2. for any pull back square $Q = \begin{pmatrix} X' & \stackrel{h_0}{\to} & X \\ {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle v_1} \\ Y' & \stackrel{h_1}{\to} & Y \end{pmatrix}$ a 2-morphism

$$e_Q : H_!(v_1)H_*(h_0) \to H_*(h_1)H_!(v_0)$$

called the exchange morphism or the e-morphism associated with Q.

Axioms

1. for a horizontally composable pair of of pull back squares

$$\begin{array}{ccccc} X'' & \stackrel{g_0}{\to} & X' & \stackrel{h_0}{\to} & X \\ {\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & \downarrow{\scriptstyle v_2} \\ Y'' & \stackrel{g_1}{\to} & Y' & \stackrel{h_1}{\to} & Y \end{array}$$

the diagram

$$\begin{tikzcd}
H(X'') \arrow[rr, "H_*(hg)"] \arrow[dr, "H_*(h)"] \arrow[dd, "H_!(v)"'] & & H(X) \arrow[dd, "H_!(v)"] \\
& H(X') \arrow[ur, "H_*(g)"'] \arrow[dd, "H_!(v)"] & \\
H(Y'') \arrow[rr, "H_*(hg)"] \arrow[dr, "H_*(h)"'] & & H(Y) \\
& H(Y') \arrow[ur, "H_*(g)"'] &
\end{tikzcd}$$

commutes in $\mathcal{D}$.

2. for a vertically composable pair of pull-back squares

$$\begin{array}{ccc} X' & \stackrel{h_0}{\to} & X \\ v_0 \downarrow & & \downarrow v_1 \\ Y' & \stackrel{h_1}{\to} & Y \\ w_0 \downarrow & & \downarrow w_1 \\ Z' & \stackrel{h_2}{\to} & Z \end{array}$$

the diagram

$$\begin{tikzcd}
H(X') \arrow[rr, "H_*(h)"] \arrow[dr, "H_!(v)"] \arrow[dd, "H_!(wv)"'] & & H(X) \arrow[dr, "H_!(v)"] \arrow[dd] & \\
 & H(Y') \arrow[rr, "H_*(h)"] \arrow[dl, "H_!(w)"] & & H(Y) \arrow[dl, "H_!(w)"] \\
H(Z') \arrow[rr, "H_*(h)"] & & H(Z) &
\end{tikzcd}$$

commutes in $\mathcal{D}$.

3. for a morphism $v : X \to Y$ the diagram

$$\begin{tikzcd}
H(X) \arrow[r, bend left, "H_*(id_X)"] \arrow[r, "Id_{H(X)}"'] \arrow[d, "H_!(v)"'] & H(X) \arrow[d, "H_!(v)"] \\
H(Y) \arrow[r, bend left, "H_*(id_Y)"] \arrow[r, "Id_{H(Y)}"'] & H(Y)
\end{tikzcd}$$

commutes in $\mathcal{D}$

4. for a morphism $h : X' \to X$ the diagram

$$\begin{tikzcd}
H(X') \arrow[r, "H_*(h)"] \arrow[d, "Id_{H(X')}"'] \arrow[d, bend left, "H_!(id_X)"] & H(X) \arrow[d, "Id_{H(X)}"'] \arrow[d, bend left, "H_!(id_{X'})"] \\
H(X') \arrow[r, "H_*(h)"] & H(X)
\end{tikzcd}$$

commutes in $\mathcal{D}$

1.1.2 If $\mathcal{H}$ is an e-functor with values in $\mathcal{D}$ and $\mathcal{D}^{2-op}$ is the 2-category obtained from $\mathcal{D}$ by the inversion of the direction of 2-morphisms then we have an e-functor $\mathcal{H}^{op} : \mathsf{C} \to \mathcal{D}^{op}$ such that $(\mathcal{H}^{op})^! = \mathcal{H}^*$ and $(\mathcal{H}^{op})^* = \mathcal{H}^!$ which we call the e-functor dual to $\mathcal{H}$.

1.1.3 Let $f : X \to Y$ be a monomorphism in C. Then the square $\begin{array}{ccc} X & \to & X \\ \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$ with the identity morphisms from X to X is a pull-back square and thus the exchange morphism $H^*(Id)H^!(f) \to H^!(Id)H^*(f)$ is defined. Together with the canonical isomorphisms $H^*(Id) = Id$ and $H^!(Id) = Id$ which are a part of the 2-functor structures on $\mathcal{H}^!$ and $\mathcal{H}^*$ it gives us a canonical 2-morphism $H^!(f) \to H^*(f)$ for any monomorphism f. The following properties of these morphisms can be easily seen from the e-functor axioms.

1.1.4 Lemma *For a composable pair of monomorphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ the square*

$$\begin{array}{ccc} H^!(gf) & \to & H^*(gf) \\ \downarrow & & \downarrow \\ H^!(f)H^!(g) & \to & H^*(f)H^*(g) \end{array}$$

commutes.

1.1.5 Lemma *For any object X the diagram*

$$\begin{array}{ccc} H^!(Id_X) & \to & H^*(Id_X) \\ \downarrow & & \downarrow \\ Id_{\mathsf{H}(X)} & = & Id_{\mathsf{H}(X)} \end{array}$$

commutes.

1.1.6 Lemma *Consider a pull-back square*

$$Q = \begin{pmatrix} X' & \xrightarrow{h_0} & X \\ v_0 \downarrow & & \downarrow v_1 \\ Y' & \xrightarrow{h_1} & Y \end{pmatrix}$$

If v_i are monomorphisms then the diagram

$$\begin{array}{ccc} H^*(h_0)H^!(v_0) & \to & H^*(h_0)H^*(v_0) \\ {\scriptstyle e_Q}\downarrow & & \downarrow\cong \\ H^!(v_1)H^*(h_1) & \to & H^*(v_1)H^*(h_1) \end{array}$$

commutes.
If h_i are monomorphisms then the diagram

$$\begin{array}{ccc} H^!(h_0)H^!(v_0) & \to & H^*(h_0)H^!(v_0) \\ {\scriptstyle cong}\downarrow & & \downarrow {\scriptstyle e_Q} \\ H^!(v_1)H^!(h_1) & \to & H^!(v_1)H^*(h_1) \end{array}$$

commutes.

1.1.7 Lemma *If in a pull back square* $Q = \begin{pmatrix} X' & \stackrel{h_0}{\to} & X \\ v_0\downarrow & & \downarrow v_1 \\ Y' & \stackrel{h_1}{\to} & Y \end{pmatrix}$ *the morphisms h_i or the morphisms v_i are isomorphisms then the exchange morphism e_Q is an isomorphism.*

1.1.8 Corollary *For any isomorphism $f : X \to Y$ in C the canonical morphism $H^!(f) \to H^*(f)$ is an isomorphism. These isomorphisms are compatible in the sense of Lemmas 1.1.4-1.1.6 with compositions of isomorphisms, identities and the exchange morphisms in the pull-back squares which consist of isomorphisms.*

1.2 Loops and suspensions

1.2.1 Let $f : X \to S$ be a morphism and $x : S \to X$ a section of p. Pairs (p, x) form the category $(\mathsf{C}/S)_\bullet$ of pointed objects over S. A morphism $f : (p_1, x_1) \to (p_2, x_2)$ in this category is a morphism $f : X_1 \to X_2$ such that $p_2 \circ f = p_1$ and $x_2 = f \circ x_1$. Below we use the notation (X, x) instead of (p, x) for the object given by a pair of morphisms $(p : X \to S, x : S \to X)$.

1.2.2 For any object (X, x) in $(\mathsf{C}/S)_\bullet$ define 1-endomorphisms of $\mathsf{H}(S)$

$$\Sigma^{\mathcal{H}}_{(X,x)} = H^*(x)H^!(p)$$

$$\Omega^{\mathcal{H}}_{(X,x)} = H^!(x)H^*(p).$$

1.2.3 Σ and Ω are dual in the sense that $\Sigma^{\mathcal{H}}_{(X,x)} = \Omega^{\mathcal{H}^{op}}_{(X,x)}$. Thus all the properties of Σ's proven below have immediate counterparts for $\Omega' s$ with the directions of all 2-morphisms inverted.

1.2.4 For any morphism $f : (X_1, x_1) \to (X_2, x_2)$ in $(\mathsf{C}/S)_\bullet$ and any pull-back square based on f of the form

$$\begin{array}{ccc} ker(f) & \to & X_1 \\ \downarrow & & \downarrow f \\ S & \stackrel{x_2}{\to} & X_2 \end{array}$$

the marked diagram

$$\begin{array}{ccccc} S & \stackrel{*}{\to} & ker(f) & \stackrel{*}{\to} & X_1 \\ & & \downarrow ! & & \downarrow ! \\ & & S & \stackrel{*}{\to} & X_2 \\ & & & & \downarrow ! \\ & & & & S \end{array}$$

defines a 2-morphism

$$\epsilon(f) : \Sigma_{(X_1,x_1)} \to \Sigma_{ker(f)} \Sigma_{(X_2,x_2)}$$

1.2.5 We have a canonical 2-isomorphism $\Sigma_{(S,id)} \to Id$ and for the identity morphism $Id_{(X,x)} : (X, x) \to (X, x)$ the diagram

$$\begin{array}{ccc} \Sigma_{(X,x)} & \stackrel{\epsilon(id)}{\to} & \Sigma_{(S,id)} \Sigma_{(X,x)} \\ \cong \downarrow & \swarrow & \\ Id\, \Sigma_{(X,x)} & & \end{array}$$

commutes.

1.2.6 Lemma *For a composable pair of morphisms*

$$(X_1, x_1) \stackrel{f}{\to} (X_2, x_2) \stackrel{g}{\to} (X_3, x_3)$$

in $(\mathsf{C}/S)_\bullet$ *and pull-back squares*

$$\begin{array}{ccc} ker(f) & \to & X_1 \\ \downarrow & & \downarrow f \\ S & \stackrel{x_2}{\to} & X_2 \end{array} \quad \begin{array}{ccc} ker(g) & \to & X_2 \\ \downarrow & & \downarrow g \\ S & \stackrel{x_3}{\to} & X_3 \end{array} \quad \begin{array}{ccc} ker(gf) & \to & X_1 \\ \downarrow & & \downarrow gf \\ S & \stackrel{x_3}{\to} & X_3 \end{array}$$

based on f, g and gf respectively the diagram

$$\begin{array}{ccc}
\Sigma_{(X_1,x_1)} & \stackrel{\epsilon(gf)}{\rightarrow} & \Sigma_{ker(gf)}\Sigma_{(X_3,x_3)} \\
{\scriptstyle \epsilon(f)}\downarrow & & \downarrow {\scriptstyle \epsilon(h)*Id} \\
\Sigma_{ker(f)}\Sigma_{(X_2,x_2)} & \stackrel{Id*\epsilon(g)}{\rightarrow} & \Sigma_{ker(f)}\Sigma_{ker(g)}\Sigma_{(X_3,x_3)}
\end{array}$$

where h is the canonical morphism $ker(gf) \to ker(g)$, commutes.

Proof: Follows from the axioms in view of the marked diagram

$$\begin{array}{ccccccc}
S & \stackrel{*}{\rightarrow} & ker(f) & \stackrel{*}{\rightarrow} & ker(gf) & \stackrel{*}{\rightarrow} & X_1 \\
 & & \downarrow ! & & \downarrow ! & & \downarrow ! \\
 & & S & \stackrel{*}{\rightarrow} & ker(g) & \stackrel{*}{\rightarrow} & X_2 \\
 & & & & \downarrow ! & & \downarrow ! \\
 & & & & S & \stackrel{*}{\rightarrow} & X_3 \\
 & & & & & & \downarrow ! \\
 & & & & & & S
\end{array}$$

1.2.7 For any object (X,x) the canonical morphism $(S,id) \to (X,x)$ defines by 1.2.4 and 1.2.5 a 2-morphism $\epsilon_{(X,x)} : Id \to \Sigma_{(X,x)}$ which coincides with the morphism given by the composition

$$Id \stackrel{\cong}{\rightarrow} H^!(x)H^!(p) \to H^*(x)H^!(p) = \Sigma_{(X,x)}$$

where the seond arrow is the 2-morphism of 1.1.3 associated with the monomorphism $x : S \to X$.

1.2.8 Lemma *In the notations of 1.2.4 and 1.2.7 the diagram*

$$\begin{array}{ccc}
Id & \stackrel{\epsilon_{(X_1,x_1)}}{\longrightarrow} & \Sigma_{(X_1,x_1)} \\
\downarrow & & \downarrow{\scriptstyle \epsilon(f)} \\
Id\,Id & \stackrel{\epsilon_{ker(f)}*\epsilon_{(X_2,x_2)}}{\longrightarrow} & \Sigma_{ker(f)}\Sigma_{(X_2,x_2)}
\end{array}$$

commutes.

1.2.9 For an object (X,x) of $(\mathsf{C}/S)_\bullet$, a morphism $f : S' \to S$ and any pull-back square

$$\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
S' & \stackrel{f}{\rightarrow} & S
\end{array}$$

the marked diagram

$$\begin{array}{ccc} S' & \xrightarrow{*} & S \\ \downarrow{*} & & \downarrow{*} \\ X' & \xrightarrow{*} & X \\ \downarrow{!} & & \downarrow{!} \\ S' & \xrightarrow{*} & S \end{array}$$

defines a 2-morphism $H^*(f)\Sigma_{(X,x)} \to \Sigma_{(X',x')}H^*(f)$.

1.2.10 Lemma *For a morphism $f : (X_1, x_1) \to (X_2, x_2)$ in $(\mathsf{C}/S)_\bullet$, a morphism $S' \to S$ in C, a pull-back square based on f of the form*

$$\begin{array}{ccc} ker(f) & \to & X_1 \\ \downarrow & & \downarrow{f} \\ S & \xrightarrow{x_2} & X_2 \end{array}$$

and pull-back squares

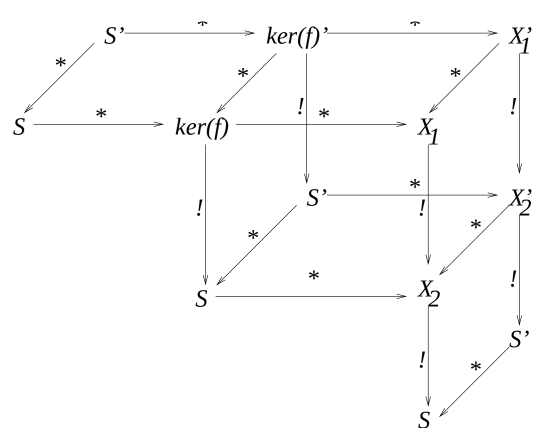
$$\begin{array}{ccc} X_1' & \to & X_1 \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array} \qquad \begin{array}{ccc} X_2' & \to & X_2 \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array} \qquad \begin{array}{ccc} ker(f)' & \to & ker(f) \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

the diagram

$$\begin{array}{ccccc} H^*(f)\Sigma_{(X_1,x_1)} & & \to & & \Sigma_{(X_1',x_1')}H^*(f) \\ \downarrow & & & & \downarrow \\ H^*(f)\Sigma_{(X_2,x_2)}\Sigma_{ker(f)} & \to & \Sigma_{(X_2',x_2')}H^*(f)\Sigma_{ker(f)} & \to & \Sigma_{(X_2',x_2')}\Sigma_{ker(f)'}H^*(f) \end{array}$$

commutes.

Proof:



1.2.11 Lemma *In the notations of 1.2.7 and 1.2.9 the diagram*

$$\begin{array}{ccccc} & & H^*(f)\,Id & \stackrel{Id*\epsilon_{(X,x)}}{\longrightarrow} & H^*(f)\Sigma_{(X,x)} \\ & \stackrel{\cong}{\nearrow} & & & \\ H^*(f) & & & & \downarrow \\ & \stackrel{\cong}{\searrow} & & & \\ & & Id\,H^*(f) & \stackrel{\epsilon_{(X',x')}*Id}{\longrightarrow} & \Sigma_{(X',x')}H^*(f) \end{array}$$

commutes.

1.2.12 For a morphism $f : X \to S$ and a pull-back square of the form

$$\begin{array}{ccc} X \times_S X & \stackrel{pr_2}{\to} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow {\scriptstyle f} \\ X & \stackrel{f}{\to} & S \end{array}$$

the pair $(pr_1 : X \times_S X \to X, \Delta : X \to X \times_S X)$ is an object of $(\mathsf{C}/X)_\bullet$. We denote the corresponding suspension and loop functors by Σ_{Δ_f} and Ω_{Δ_f} respectively. The marked diagram

$$\begin{array}{ccccc} X & \stackrel{\Delta *}{\to} & X \times_S X & \stackrel{*}{\to} & X \\ & & \downarrow ! & & \downarrow ! \\ & & X & \stackrel{*}{\to} & S \end{array}$$

defines a 2-morphism $H^!(f) \to \Sigma_{\Delta_f} H^*(f)$.

1.2.13 Lemma *For a composable pair of morphisms* $X \stackrel{f}{\to} Y \stackrel{g}{\to} S$ *and pull-back squares*

$$\begin{array}{ccc} X_S^2 & \to & X \\ \downarrow & & \downarrow {\scriptstyle gf} \\ X & \stackrel{gf}{\to} & S \end{array} \quad \begin{array}{ccc} Y_S^2 & \to & Y \\ \downarrow & & \downarrow {\scriptstyle g} \\ Y & \stackrel{g}{\to} & S \end{array} \quad \begin{array}{ccc} X_Y^2 & \to & X \\ \downarrow & & \downarrow {\scriptstyle f} \\ X & \stackrel{f}{\to} & Y \end{array} \quad \begin{array}{ccc} XY & \stackrel{pr_Y}{\to} & Y \\ {\scriptstyle pr_X}\downarrow & & \downarrow {\scriptstyle g} \\ X & \stackrel{gf}{\to} & S \end{array}$$

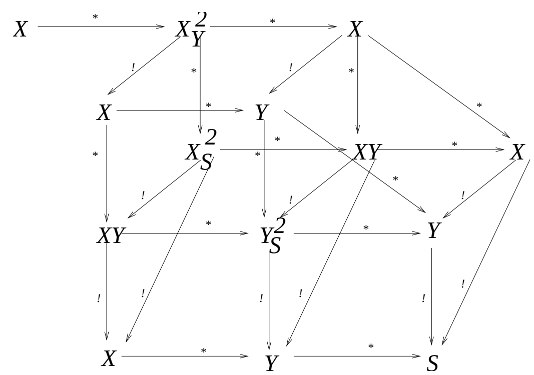
the diagram

$$\begin{array}{ccccc} H^!(gf) & \to & \Sigma_{\Delta_{gf}} H^*(gf) & \to & \Sigma_{\Delta_{gf}} H^*(f) H^*(g) \\ \downarrow & & & & \downarrow \\ H^!(f)H^!(g) & \to & \Sigma_{\Delta_f} H^*(f) \Sigma_{\Delta_g} H^*(g) & \to & \Sigma_{\Delta_f} \Sigma_\Gamma H^*(f) H^*(g) \end{array}$$

where Σ_Γ is the suspension associated with the pair $(XY \to X, X \stackrel{id\times f}{\to} XY)$ and the right lower horizontal arrow is defined according to 1.2.4 by the pull-back square

$$\begin{array}{ccc} X^2_Y & \to & X^2_S \\ \downarrow & & \downarrow \\ X & \to & XY \end{array}$$

commutes.
Proof:



1.3 Transversal morphisms

1.3.1 A morphism $f : X \to S$ in C is called $\mathcal{H}$-transversal if the exchange morphism associated with the pull-back square

$$\begin{array}{ccc} X \times_S X & \to & X \\ \downarrow & & \downarrow \\ X & \to & S \end{array}$$

is an isomorphism. It is called universally $\mathcal{H}$-transversal if for any $g : S' \to S$ the projection $X \times_S S' \to S'$ is $\mathcal{H}$-transversal.

1.3.2 An isomorphism is a universally transversal morphism by 1.1.7.

1.3.3 Lemma *Let $X \stackrel{f}{\to} Y \stackrel{g}{\to} S$ be a composable pair of morphisms such that g is transversal and f is universally transversal. Then the composition gf is transversal. If g is also universally transversal then the composition is universally transversal.*
Proof: ????

1.3.4 Let $p : X \to S$ be a morphism and $x : S \to S$ a section of p. Consider a marked diagram

$$\begin{array}{ccccc} S & \stackrel{x!}{\to} & X & \stackrel{p*}{\to} & S \\ {\scriptstyle x} \downarrow {\scriptstyle *} & & {\scriptstyle x'} \downarrow {\scriptstyle *} & & {\scriptstyle x} \downarrow {\scriptstyle *} \\ X & \stackrel{\Delta!}{\to} & X \times_S X & \stackrel{pr_2*}{\to} & X \\ & & {\scriptstyle pr_1} \downarrow {\scriptstyle !} & & {\scriptstyle p} \downarrow {\scriptstyle !} \\ & & X & \stackrel{p*}{\to} & S \end{array} \tag{1}$$

where all three squares are pull-back squares. The 1-morphism $\mathsf{H}(S) \to \mathsf{H}(S)$ given by the right vertical and the upper horizontal sides of this diagram coincides with the composition $\Omega_{(X,x)}\Sigma_{(X,x)}$ and the 1-morphism given by the lower left "stairs" is canonically isomorphic to the identity 1-morphism of $\mathsf{H}(S)$. The exchange morphisms associated with the upper left square and the lower right square go in opposite directions such that in general this diagram does not give any 2-morphism between $\Omega_{(X,x)}\Sigma_{(X,x)}$ and the identity. If p is ($\mathcal{H}$-)transversal then the exchange morphism associated with the lower right square has an inverse wich can be composed with the exchange morphism given by the upper left square. Thus for any *transversal* morphism p and any section x of p there is a canonical 2-morphism $Id \to \Omega_{(X,x)}\Sigma_{(X,x)}$. Similarly from the marked diagram

$$\begin{array}{ccccc} S & \stackrel{x*}{\to} & X & \stackrel{p!}{\to} & S \\ {\scriptstyle x} \downarrow {\scriptstyle !} & & {\scriptstyle x'} \downarrow {\scriptstyle !} & & {\scriptstyle x} \downarrow {\scriptstyle !} \\ X & \stackrel{\Delta*}{\to} & X \times_S X & \stackrel{pr_2!}{\to} & X \\ & & {\scriptstyle pr_1} \downarrow {\scriptstyle *} & & {\scriptstyle p} \downarrow {\scriptstyle *} \\ & & X & \stackrel{p!}{\to} & S \end{array} \tag{2}$$

obtained from the first one by the exchange of $*$'s and !'s we get for any transversal p and any section x of p a canonical 2-morphism $\Sigma_{(X,x)}\Omega_{(X,x)} \to Id$.

1.3.5 Theorem *If $p : X \to S$ is a universally $\mathcal{H}$-transversal morphism and x is a section of p then the 2-morphisms*

$$Id \to \Omega_{(X,x)}\Sigma_{(X,x)}$$

$$\Sigma_{(X,x)}\Omega_{(X,x)} \to Id$$

satisfy the adjunction axiom.

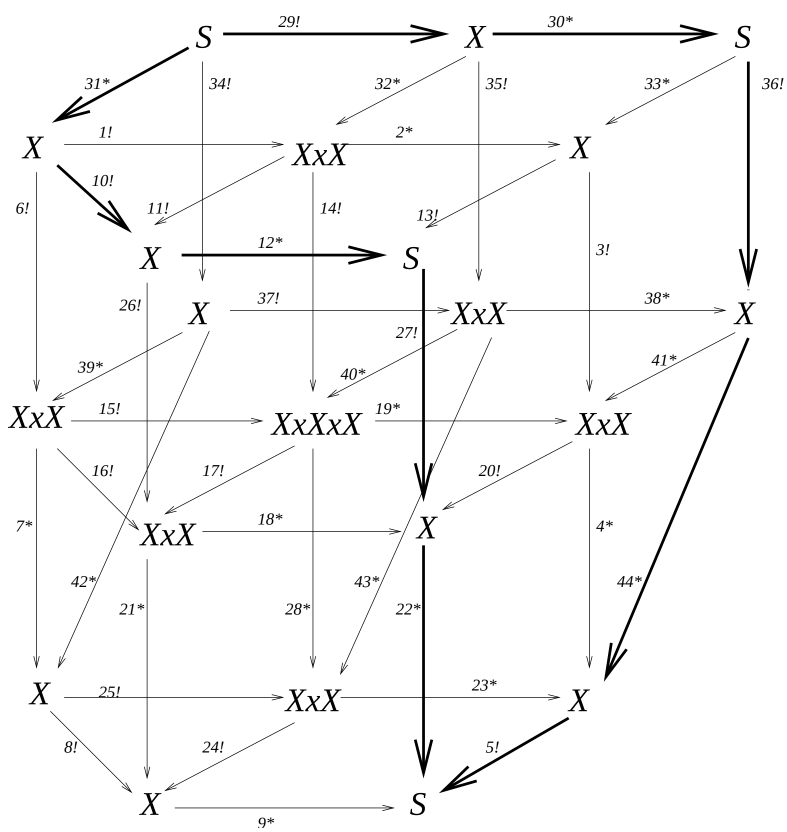
Proof: We have to verify that the compositions

$$\Omega_{(X,x)} \to \Omega_{(X,x)}\Sigma_{(X,x)}\Omega_{(X,x)} \to \Omega_{(X,x)}$$

and

$$\Sigma_{(X,x)} \to \Sigma_{(X,x)}\Omega_{(X,x)}\Sigma_{(X,x)} \to \Sigma_{(X,x)}$$

coincide with the corresponding identity 2-morphisms. One can easily see that these two compositions are dual in the sense of 1.2.3 and therefore it is sufficient to show that the first one equals identity. The main marked diagram for the proof looks as follows:



For the convenience of further reference we numbered all the arrows. The right vertical face of the diagram is the diagram (2) defining the 2-morphism $Id \to \Omega\Sigma$ and the upper horizontal face is the diagram (1) defining the 2-morphism $\Sigma\Omega \to Id$. The whole diagram is the union of the front part which is the product of

$$\begin{tikzcd}
X \arrow[r, "\Delta !"] \arrow[dr, "id!"'] & X \times_S X \arrow[r, "pr_2 *"] \arrow[d, "pr_1"', "!"] & X \arrow[d, "!"] \\
& X \arrow[r, "*"] & S
\end{tikzcd}$$

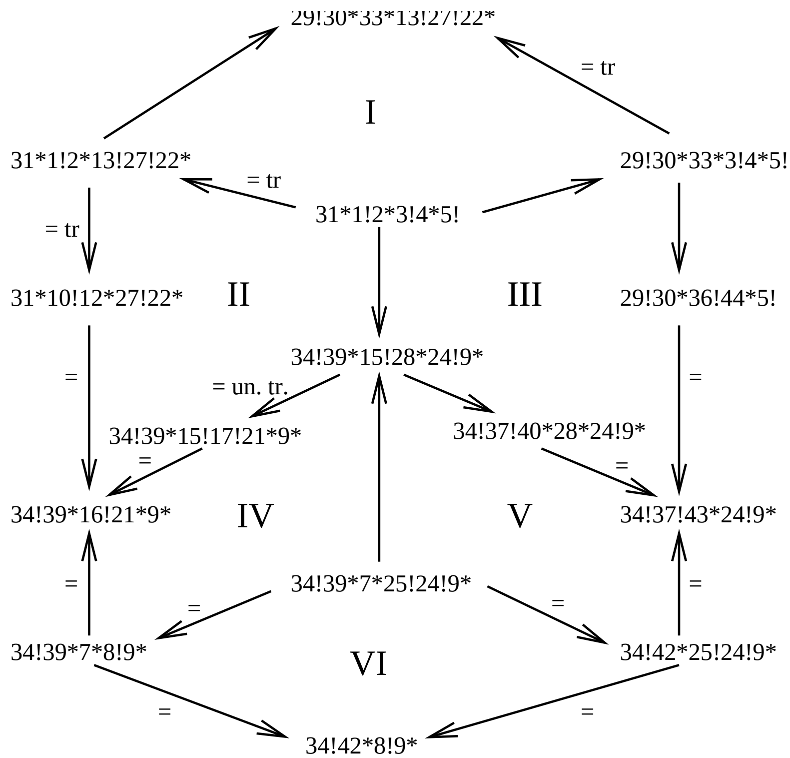
with $S \overset{x!}{\to} X \overset{p*}{\to} S$ and the back part which is the product of

$$\begin{tikzcd}
S \arrow[r, "x!"] \arrow[d, "x"', "*"] & X \arrow[d, "\Delta"', "*"] \arrow[dr, "id*"] & \\
X \arrow[r, "id\times x\,!"] & X \times_S X \arrow[r, "pr_1 *"] & X
\end{tikzcd}$$

with $X \overset{\Delta!}{\to} X \times_S X \overset{pr_2*}{\to} X$ over X along

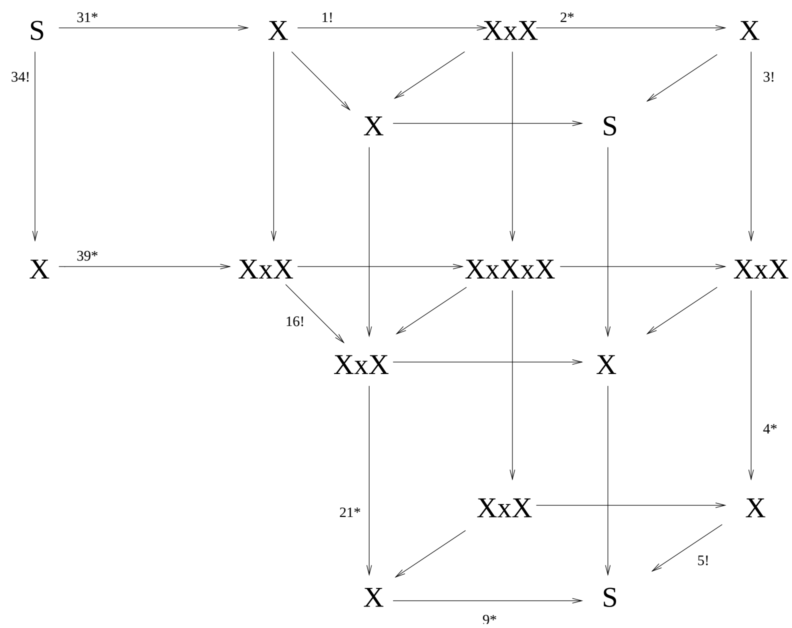
$$\begin{pmatrix} X \\ {\scriptstyle id\times x}\downarrow{\scriptstyle !} \\ X \times X \\ {\scriptstyle pr_1}\downarrow{\scriptstyle *} \\ X \end{pmatrix} \times_X \left(X \overset{\Delta!}{\to} X \times X \overset{pr_2*}{\to} X\right) = \left(X \overset{\Delta!}{\to} X \times X \overset{pr_2*}{\to} X\right) \times_S \begin{pmatrix} S \\ {\scriptstyle x}\downarrow{\scriptstyle !} \\ X \\ {\scriptstyle p}\downarrow{\scriptstyle *} \\ S \end{pmatrix}$$

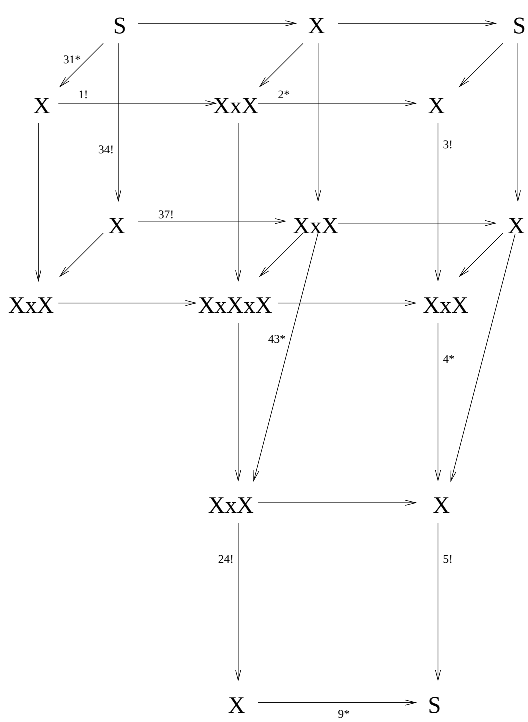
Consider the following diagram whose vertices are 1-morphisms given by paths in our main diagram:



The edges of this diagram are the 2-morphisms represented by appropriate combinations of the exchange 2-squares and composition isomorphisms which can be seen directly from the main diagram. The equality sign idicates that the corresponding 2-morphism is an isomorphism. The mark "tr" after the equality sing means that it is an isomorphism because p is a transversal morphism. The mark "un. tr." means that it is an isomorphism because p is universally transversal. The 2-morphisms with unmarked equalities are isomorphisms either because they are just the composition isomorphisms of the 2-functors H^* and $H^!$ or because they are exchange morphisms associated with the squares where one pair of morphisms consists of identities.

Let us show that each of the polygons (I)-(VI) is commutative. The square (I) is commutative by the 2-category axioms. The commutativity of (II) and (III) can be seen respectively using the e-functor axioms from the following two subdiagrams of our main diagram:





The commutativity of pentagons (IV) and (V) follows directly from the vertical and horizontal composition axioms. Finally the square (VI) is again commutative by the 2-category axioms.

Observe now that commutativity of (IV) together with the fact that the arrows marked by the equality sign are isomorphisms imply that the vertical arrow $34^!39^*7^*25^!24^!9^* \to 34^!39^*15^!28^*24^!9^*$ is an isomorphism. The commutativity of (V) now implies that the arrow

$$34^!39^*15^!28^*24^!9^* \to 34^!37^!40^*28^*24^!9^*$$

is also an isomorphism. From this it is easy to see that the composition

$$31^*10^!12^*27^!22^* \to 31^*1^!2^*13^!27^!22^* \to 29^!30^*33^*13^!27^!22^* \to$$

$$\to 29^!30^*33^*3^!4^*5^! \to 29^!30^*36^!44^*5^!$$

equals the composition

$$31^*10^!12^*27^!22^* \to 34^!39^*16^!21^*9^* \to 34^*39^*7^*8^!9^* \to 34^!42^*8^!9^* \to$$

$$\to 34^!42^*25^!24^!9^* \to 34^!37^!43^*24^!9^* \to 29^!30^*36^!44^*5^!$$

where we use the inverses to isomorphisms where necessary.

1.3.6 As one can see from the proof of 1.3.5 it remains valid under slightly weaker conditions. Namely instead of requiring p to be universally transversal it is sufficient to require both p and the projection $X \times_S X \to X$ to be transversal.

1.4 Cotransversal morphisms

1.4.1 A morphism $f : X \to S$ is called $\mathcal{H}$-cotransversal if the diagonal morphism $\Delta_f : X \to X \times_S X$ is $\mathcal{H}$-transversal. A morphism is called universally $\mathcal{H}$-cotransversal if the diagonal morphism $\Delta_f : X \to X \times_S X$ is universally $\mathcal{H}$-transversal.

A morphism $f : X \to S$ is called $\mathcal{H}$-bitransversal if it is $\mathcal{H}$-transversal and $\mathcal{H}$-cotransversal. A morphism is called universally $\mathcal{H}$-bitransversal if it is universally $\mathcal{H}$-transversal and universally $\mathcal{H}$-cotransversal.

1.4.2 A monomorphism is a universally cotransversal morphism by 1.3.2.

1.4.3 Lemma *Let* $X \xrightarrow{f} Y \xrightarrow{g} S$ *be a composable pair of universally cotransversal morphisms. Then the composition* gf *is universally cotransversal.*

Proof: The diagonal morphism $\Delta_{gf} : X \to X \times_S X$ for gf is the composition of the diagonal morphism $\Delta_f : X \to X \times_Y X$ for f and the morphism $X \times_Y X \to X \times_S X$ which fits into the pull-back square

$$\begin{array}{ccc} X \times_Y X & \to & Y \\ \downarrow & & \downarrow\Delta_g \\ X \times_S X & \to & Y \times_S Y \end{array}$$

The statement of the lemma follows now from 1.3.3.

1.4.4 For a cotransversal morphism f the 2-morphism $Id \to \Sigma_{\Delta_f}$ of 1.2.7 is an isomorphism. The composition of the morphism $H^!(f) \to \Sigma_{\Delta_f} H^*(f)$ of 1.2.12 with its inverse gives a morphism $H^!(f) \to H^*(f)$. If f is a monomorphism this morphism coincides with the morphism defined in 1.1.3.

1.4.5 Lemma *Let* $X \xrightarrow{f} Y \xrightarrow{g} S$ *be a composable pair of universally cotransversal morphisms. Then the square*

$$\begin{array}{ccc} H^!(gf) & \to & H^*(gf) \\ \downarrow & & \downarrow \\ H^!(f)H^!(g) & \to & H^*(f)H^*(g) \end{array}$$

commutes.

Proof: Consider the diagram

$$\begin{array}{ccccc} H^!(gf) & \to & \Sigma_{\Delta_{gf}} H^*(gf) & \leftarrow & H^*(gf) \\ & & \downarrow & & \downarrow \\ & & \Sigma_{\Delta_{gf}} H^*(f)H^*(g) & \leftarrow & H^*(f)H^*(g) \\ \downarrow & & \downarrow & & \downarrow\cong \\ & & \Sigma_{\Delta_f}\Sigma_{\Gamma} H^*(f)H^*(g) & \leftarrow & H^*(f)H^*(g) \\ & & \uparrow & & \uparrow\cong \\ H^!(f)H^!(g) & \to & \Sigma_{\Delta_f} H^*(f)\Sigma_{\Delta_g} H^*(g) & \leftarrow & H^*(f)H^*(g) \end{array}$$

The left hand side hexagon is the commutative diagram of 1.2.13. The upper square on the right commutes by axioms of a 2-category, the middle square commutes by 1.2.8 and the lower square by 1.2.11. Since f is cotransversal and g is universally cotransversal all the horizontal arrows on the right hand side are isomorphisms. Thus the up-going arrow is an isomorphism which together with the commutativity of the hexagon and the squares implies the statement of the lemma.