# C-systems - from monads and from universes in categories

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The first few steps in all approaches to the semantics of dependent type theories remain insufficiently understood. The constructions which have been worked out in detail in the case of a few particular type systems by dedicated authors are being extended to the wide variety of type systems under consideration today by analogy. This is not acceptable in mathematics. Instead we should be able to obtain the required results for new type systems by *specialization* of general theorems formulated and proved for abstract objects the instances of which combine together to produce a given type system.

One such class of objects is the class of C-systems introduced by John Cartmell in his 1978 Ph.D. thesis [1] under the name "contextual categories".

By a (pre-)category $C$ we mean a pair of sets $Mor(C)$ and $Ob(C)$ with four maps

$$\partial_0, \partial_1 : Mor(C) \to Ob(C)$$

$$Id : Ob(C) \to Mor(C)$$

and

$$\circ : Mor(C)_{\partial_1} \times_{\partial_0} Mor(C) \to Mor(C)$$

which satisfy the well known conditions of unity and associativity (note that we write composition of morphisms in the form $f \circ g$ or $fg$ where $f : X \to Y$ and $g : Y \to Z$). These objects would be usually called categories but we reserve the name "category" for those uses of these objects that are invariant under the equivalences.

**Definition 1** *A C-system is a pre-category $CC$ with additional structure of the form*

1. *a function $l : Ob(CC) \to \mathbf{N}$,*
2. *an object pt of $CC$ such that $\{pt\} = l^{-1}(0)$,*
3. *a map $ft : Ob(CC) \to Ob(CC)$ such that if $l(X) > 0$ then $l(ft(X)) = l(X) - 1$ and $ft(pt) = pt$,*
4. *for each $X \in Ob(CC)$ a morphism $p_X : X \to ft(X)$,*
5. *for each $X \in Ob(CC)$ such that $l(X) > 0$ and each morphism $f : Y \to ft(X)$ an object $f^*X$ and a morphism*

$$q(f, X) : f^*X \to X$$

*such that the following additional conditions are satisfied:*

1. *pt is a final object of* $CC$

2. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$ *and* $f : Y \to ft(X)$ *one has* $l(f^*(X)) > 0$, $ft(f^*X) = Y$ *and the square*

$$\begin{array}{ccc} f^*X & \xrightarrow{q(f,X)} & X \\ p_{f^*X}\downarrow & & \downarrow p_X \\ Y & \xrightarrow{f} & ft(X) \end{array} \tag{1}$$

   *is a pull-back square,*

3. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$ *one has* $id^*_{ft(X)}(X) = X$ *and* $q(id_{ft(X)}, X) = id_X$,

4. *for* $X \in Ob(CC)$ *such that* $l(X) > 0$, $g : Z \to Y$ *and* $f : Y \to ft(X)$ *one has* $(gf)^*(X) = g^*(f^*(X))$ *and* $q(gf, X) = q(g, f^*X)q(f, X)$.

For an alternative definition see [5].

## C-systems from monads on sets

For a mode general version of the following construction see [4]. Let $R$ be a monad on $Sets$. Let $CC(R,R)$ be the pre-category whose set of objects is $Ob(CC(R,R)) = \amalg_{n\geq 0} Ob_n$ where

$$Ob_n = R(\emptyset) \times \ldots \times R(\{1,\ldots,n-1\})$$

and the set of morphisms is

$$Mor(CC(R,R)) = \coprod_{m,n\geq 0} Ob_m \times Ob_n \times R(\{1,\ldots,m\})^n$$

with the obvious domain and codomain maps. The composition of morphisms is defined in the same way as in the Kleisli category $C(R)$ of $R$ such that the mapping $Ob(CC(R,R)) \to \mathbf{N}$ which sends all elements of $Ob_n$ to $n$, is a functor from $CC(R,R)$ to $C(R)$. The associativity of compositions follows immediately from the associativity of compositions in $C(R)$.

Note that if $R(\emptyset) = \emptyset$ then $CC(R,R) = \emptyset$ and otherwise the functor $CC(R,R) \to C(R)$ is an equivalence, so that in the second case $C(R)$ and $CC(R,R)$ are indistinguishable as categories. However, as pre-categories they are quite different.

The pre-category $CC(R,R)$ is given the structure of a C-system as follows. The final object is the only element of $Ob_0$, the map $ft$ is defined by the rule

$$ft(T_1, \dots, T_n) = (T_1, \dots, T_{n-1}).$$

The canonical pull-back square defined by an object $(T_1, \ldots, T_{n+1})$ and a morphism

$$(f_1, \ldots, f_n) : (R_1, \ldots, R_m) \to (T_1, \ldots, T_n)$$

is of the form:

$$\begin{array}{ccc} (R_1, \ldots, R_m, T_{n+1}(f_1/1, \ldots, f_n/n)) & \xrightarrow{(f_1,\ldots,f_n,m+1)} & (T_1, \ldots, T_{n+1}) \\ {\scriptstyle (1,\ldots,m)}\downarrow & & \downarrow{\scriptstyle (1,\ldots,n)} \\ (R_1, \ldots, R_m) & \xrightarrow{(f_1,\ldots,f_n)} & (T_1, \ldots, T_n) \end{array} \qquad (2)$$

**Proposition 2** *With the structure defined above $CC(R,R)$ is a C-system.*

## A special case - syntactic monads

Consider a signature $\Sigma$ given by a set $Op$ of operations and the arity function $Ar : Ob \to list\ \mathbf{N}$. An operation $\Theta$ of arity $(0,1)$ should be thought of as being of the form $\Theta(t_1, x_1.t_2)$. For example, usual algebraic operations will have arities of the form $(0, \ldots, 0)$ where the number of 0's is the number of arguments and "quantifiers" such as $\forall$ and $\exists$ will have arity 1. The $\lambda$-abstraction also has arity 1.

To any such signature $\Sigma$ one associates a class of expressions with bindings. Then the sets $R(X)$ of such expressions with free variables from the set $X$ modulo the $\alpha$-equivalence form a monad. For a universal characterization of such monads see [2]. For a formal construction of such monads using nominal sets see [4]. Applying the previous construction we get a C-system defined by $\Sigma$.

## Type theories of Martin-Lof genus

For a signature $\Sigma$ as above consider four classes of "sentences". Sentences in each class are of the form $\Gamma \vdash \mathcal{J}$. In each case $\Gamma$, the context, is a sequence of the form $x_1 : T_1, \ldots, x_n : T_n$ for some $n \geq 0$ where $T_i \in R(\{x_1, \ldots, x_{i-1}\})$ and the $\mathcal{J}$ parts are:

$$
\begin{array}{ll}
\Gamma \vdash T \;\; type & T \in R(\{x_1, \ldots, x_n\}) \\
\Gamma \vdash t : T & T, t \in R(\{x_1, \ldots, x_n\}) \\
\Gamma \vdash T = T' & T, T' \in R(\{x_1, \ldots, x_n\}) \\
\Gamma \vdash t = t' : T & T, t, t' \in R(\{x_1, \ldots, x_n\})
\end{array}
$$

A type system of the Martin-Lof genus is specified by a choice of $\Sigma$ and by a choice of the subsets of *derivable* sentences of each class such that a certain set of conditions holds.

This approach to formulating type theory through these four classes of sentences was introduced by Per Martin-Lof in a remarkable paper called "Constructive mathematics and computer programming".

Since we are only interested in the $\alpha$-equivalence classes of judgements we may assume that the variables declared in the context are taken from the set of natural numbers such that the first declared variable is 1, the second is 2 etc. Then, the set of judgements of the form

$$(1 : A_1, \ldots, n : A_n \vdash A\,type)$$

(in the notation of Martin-Lof "$A\,type\,(1 \in A_1, \ldots, n \in A_n)$") can be identified with the set of judgements of the form

$$(1 : A_1, \ldots, n : A_n, n+1 : A \rhd)$$

stating that the context $(1 : A_1, \ldots, n : A_n, n+1 : A)$ is well-formed.

With this identification the type theory is specified by four sets $C, \widetilde{C}, Ceq$ and $\widetilde{Ceq}$ where

$$C \subset \coprod_{n \geq 0} R(\emptyset) \times \ldots \times R(\{1, \ldots, n-1\})$$

$$\widetilde{C} \subset \coprod_{n \geq 0} R(\emptyset) \times \ldots \times R(\{1, \ldots, n-1\}) \times R(\{1, \ldots, n\}) \times R(\{1, \ldots, n\})$$

$$Ceq \subset \coprod_{n \geq 0} R(\emptyset) \times \ldots \times R(\{1, \ldots, n-1\}) \times R(\{1, \ldots, n\})^2$$

$$\widetilde{Ceq} \subset \coprod_{n \geq 0} R(\emptyset) \times \ldots \times R(\{1, \ldots, n-1\}) \times R(\{1, \ldots, n\})^2 \times R(\{1, \ldots, n\})$$

**Theorem 3** *There is a bijection between the quadruples of subsets* $C, \widetilde{C}, Ceq, \widetilde{Ceq}$ *as above satisfying the conditions required from a type theory of Martin-Lof genus and pair of the form* $(CC', \sim)$ *where* $CC'$ *is a C-subsystem of* $CC(R, R)$ *and* $\sim$ *is a regular congruance relation on* $CC'$.

**Proof**: See [5] and [4].

**Corollary 4** *Any type theory of Martin-Lof genus defines a C-system, namely* $CC'/sim$.

## C-system from a universe in a category

**Definition 5** *Let $\mathcal{C}$ be a category. A universe on $\mathcal{C}$ is a morphism $p : \widetilde{U} \to U$ together with a mapping which assigns to any morphism $f : X \to U$ in $\mathcal{C}$ a pull-back square*

$$\begin{array}{ccc} (X; f) & \xrightarrow{Q(f)} & \widetilde{U} \\ {\scriptstyle p_{(X,f)}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

In what follows we will write $(X; f_1, \ldots, f_n)$ for $(\ldots((X; f_1); f_2)\ldots; f_n)$. For details of the following construction see [3].

For $\mathcal{C}$, a universe $p$ in $\mathcal{C}$ and a final object $pt$ of $\mathcal{C}$ we define a C-system $CC = CC(\mathcal{C}, p)$ as follows:

Objects of $CC$ are sequences of the form $(F_1, \ldots, F_n)$ where $F_1 \in Hom_{\mathcal{C}}(pt, U)$ and $F_{i+1} \in Hom_{\mathcal{C}}((pt; F_1, \ldots, F_i), U)$. Morphisms from $(G_1, \ldots, G_n)$ to $(F_1, \ldots, F_m)$ are given by

$$Hom_{CC}((G_1, \ldots, G_n), (F_1, \ldots, F_m)) =$$

$$Hom_{\mathcal{C}}((pt; G_1, \ldots, G_n), (pt; F_1, \ldots, F_m))$$

and units and compositions are defined as units and compositions in $\mathcal{C}$ such that the mapping $(F_1, \ldots, F_n) \to (pt; F_1, \ldots, F_n)$ is a full embedding of the underlying category of $CC$ to $\mathcal{C}$. The image of this embedding consists of objects $X$ for which the canonical morphism $X \to pt$ is a composition of morphisms which are (canonical) pull-backs of $p$. We will denote this embedding by $int$.

The final object of $CC$ is the empty sequence (). The map $ft$ sends $(F_1, \ldots, F_n)$ to $(F_1, \ldots, F_{n-1})$. The canonical morphism $p_{(F_1,\ldots,F_n)}$ is the projection

$$p_{((pt;F_1,\ldots,F_{n-1});F_n)} : ((pt; F_1, \ldots, F_{n-1}); F_n) \to (pt; F_1, \ldots, F_{n-1})$$

The canonical pull-back square are of the form

$$\begin{array}{ccc} (G_1, \ldots, G_n, f \circ F_{m+1}) & \xrightarrow{q(f)} & (F_1, \ldots, F_{m+1}) \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_F} \\ (G_1, \ldots, G_n) & \xrightarrow{f} & (F_1, \ldots, F_m) \end{array} \tag{3}$$

where $p_F = p_{(F_1,\ldots,F_{m+1})}$, $p_G = p_{(G_1,\ldots,G_n,f\circ F_{m+1})}$ and $q(f)$ is the unique morphism such that $q(f) \circ p_F = p_G \circ f$ and

$$int(q(f)) \circ Q(F_{m+1}) = f \circ Q(F_{m+1}).$$

The unity and composition axioms for the canonical squares follow immediately from the unity and associativity axioms for compositions of morphisms in $C$.

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$$p_{((pt;F_1,\ldots,F_{n-1}),F_n)} : ((pt; F_1, \ldots, F_{n-1}), F_n) \to (pt; F_1, \ldots, F_{n-1})$$

The canonical pull-back squares are of the form

$$\begin{array}{ccc} (G_1, \ldots, G_n, F_{m+1}f) & \xrightarrow{q(f)} & (F_1, \ldots, F_{m+1}) \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_F} \\ (G_1, \ldots, G_n) & \xrightarrow{f} & (F_1, \ldots, F_m) \end{array} \tag{4}$$

where $int(p_F) = p((pt; F_1, \ldots, F_{n-1}), F_n)$,

$$int(p_G) = p((pt; G_1, \ldots, G_{n-1}), F_{m+1} \circ f)$$

and $q(f)$ is the morphism such that $p_F q(f) = f p_G$ and

$$Q(F_{m+1}) int(q(f)) = Q(F_{m+1} f).$$

## Syntax and semantics of dependent type theories

1. Signature defines the monad of raw expressions.

2. Derivation rules define the subsets of the derivable sentences of four kinds and, by the first of our constructions, a C-system $CC(\mathbf{T})$.

3. Derivation rules also define additional operations on this C-system.

4. The *Initiality Theorem* asserts that C-system with these operations is the initial object among all C-systems with such operations.

5. A mathematical category with a special morphism in it (such as, for example, the universal Kan fibration in well-ordered simplicial sets) defines a C-system $CC(\mathcal{C}, p)$ by the second of our constructions.

6. Structures on this morphism arising from geometric or other mathematical considerations define on this C-system operations of the form associated with the derivation rules of the type system.

7. By initiality theorem we obtain a homomorphism of C-systems

$$CC(\mathbf{T}) \to CC(\mathcal{C}, p)$$

i.e. a model for $\mathbf{T}$.

## The $(\Pi, \lambda)$-structures on C-systems

**Definition 6** *Let $CC$ be a C-system. A pre-$(\Pi, \lambda)$-structure on $CC$ is a pair of functions*

$$\Pi : Ob_{\geq 2} \to Ob$$
$$\lambda : \widetilde{Ob}_{\geq 2} \to \widetilde{Ob}$$

*such that:*

1. $ft(\Pi(\Gamma)) = ft^2(\Gamma)$,
2. $\partial(\lambda(s)) = \Pi(\partial(s))$.

For a pre-$(\Pi, \lambda)$-structure $(\Pi, \lambda)$ and $\Gamma \in Ob$ the function $\Pi$ defines, in view of the first condition of Definition 6, a function

$$\Pi^{\Gamma} : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

and the function $\lambda$ defines, in view of the first and the second conditions of Definition 6, a function

$$\lambda^{\Gamma} : \widetilde{Ob}_2(\Gamma) \to \widetilde{Ob}_1(\Gamma)$$

The second condition also implies that the square:

$$\begin{array}{ccc} \widetilde{Ob}_2(\Gamma) & \xrightarrow{\lambda^\Gamma} & \widetilde{Ob}_1(\Gamma) \\ \partial\downarrow & & \downarrow\partial \\ Ob_2(\Gamma) & \xrightarrow{\Pi^\Gamma} & Ob_1(\Gamma) \end{array} \tag{5}$$

commutes. One can easily see that the notion of a pre-$(\Pi, \lambda)$-structure could be equally formulated as two families of functions $\Pi^\Gamma$ and $\lambda^\Gamma$ such that the squares (5) commute.

**Definition 7** *A pre-*$(\Pi, \lambda)$*-structure is called a* $(\Pi, \lambda)$*-structure if the following conditions hold:*

1. *for any* $\Gamma \in Ob_{\geq 2}$ *the square (5) is a pull-back square,*
2. *for any* $f : \Gamma' \to \Gamma$ *the square*

$$\begin{array}{ccc} Ob_2(\Gamma) & \xrightarrow{\Pi^\Gamma} & Ob_1(\Gamma) \\ f^*\downarrow & & \downarrow f^* \\ Ob_2(\Gamma') & \xrightarrow{\Pi^{\Gamma'}} & Ob_1(\Gamma') \end{array} \tag{6}$$

   *commutes,*

3. *for any* $f : \Gamma' \to \Gamma$ *the square*

$$\begin{array}{ccc} \widetilde{Ob}_2(\Gamma) & \xrightarrow{\lambda^\Gamma} & \widetilde{Ob}_1(\Gamma) \\ f^*\downarrow & & \downarrow f^* \\ \widetilde{Ob}_2(\Gamma') & \xrightarrow{\lambda^{\Gamma'}} & \widetilde{Ob}_1(\Gamma') \end{array} \tag{7}$$

   *commutes.*

**Theorem 8** *Let $\mathcal{C}$ be a locally cartesian closed category with a final object. Let $p : \widetilde{U} \to U$ be a morphism with a universe structure on it. Let $P$, $\widetilde{P}$ be a pair of morphisms that make the square:*

$$\begin{array}{ccc}
\underline{Hom}_U(\widetilde{U}, \underline{U} \times \widetilde{U}) & \stackrel{\widetilde{P}}{\longrightarrow} & \widetilde{U} \\
\downarrow p_2 & & \downarrow p \\
\underline{Hom}_U(\widetilde{U}, \underline{U} \times U) & \stackrel{P}{\longrightarrow} & U
\end{array}$$

*a pull-back square.*

*Then one can construct, in a functorial way, a $(\Pi, \lambda)$-structure on $CC(\mathcal{C}, p)$.*

**Proof**: See [].

## References


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