Paul Bernays Lectures, 2014

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Foundations of Mathematics: their past, present and future.

Part II. The story of set theory (so far).

In the first lecture we discussed the questions of what is a foundation of mathematics and how foundations of mathematics relate to mathematics.

While trying to answer the first question we observed that there are two streams in the story of foundations. I called these streams "Foundations 1" and 'Foundations 2".

Foundations 1 is defined as the study of "the basic mathematical concepts and how they form hierarchies of more complex structures and concepts".

Foundations 2 is defined as the study of "the fundamentally important structures that form the language of mathematics also called metamathematical concepts".

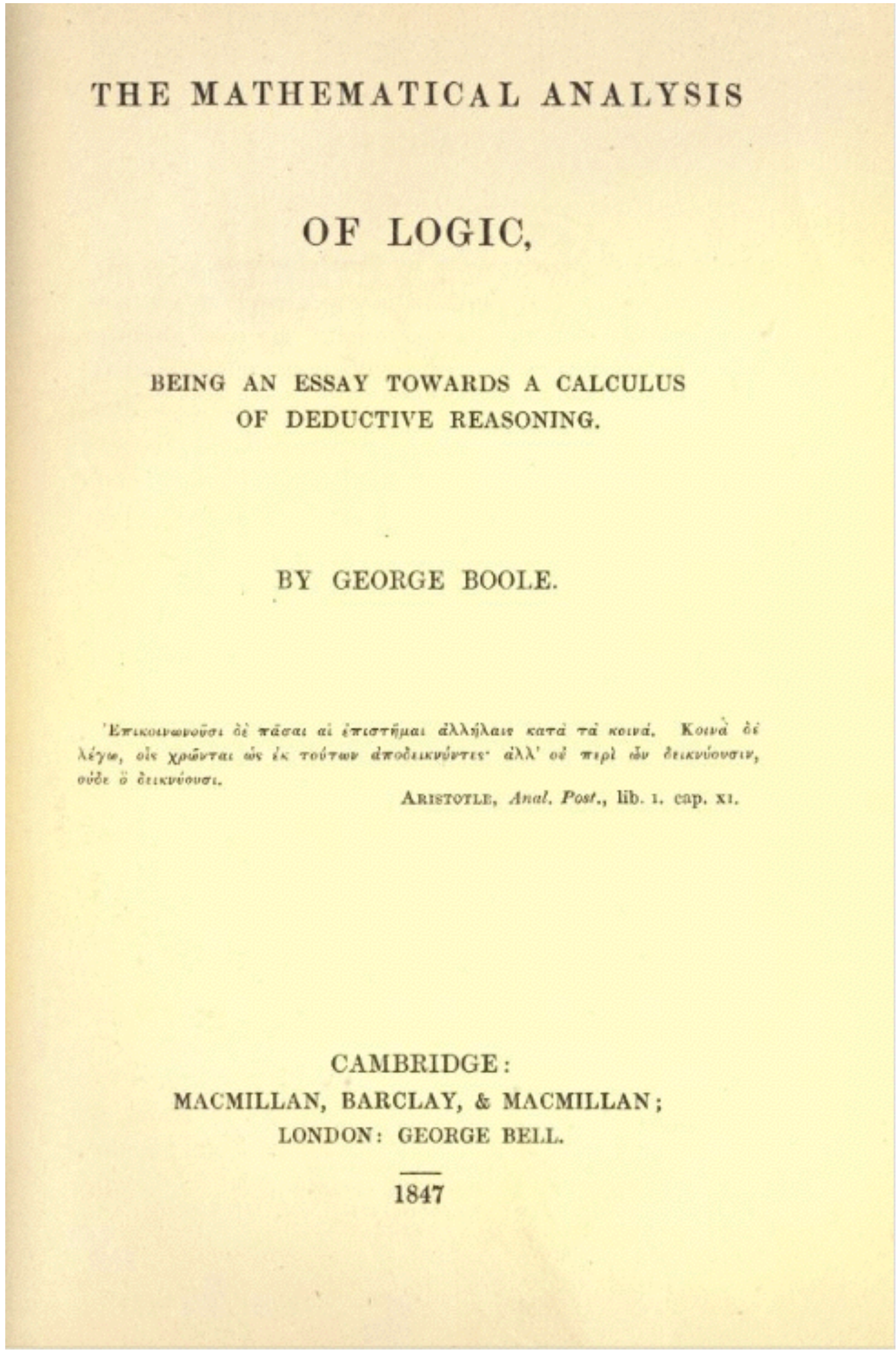
I showed multiple ideas related to Foundations I, starting from comments in the work of the 5th century neoplatonic philosopher Proclus. Many of these ideas could be traced back to an approach attributed to Pythagoreans.

Much less was written by mathematicians on Foundations 2. This remained a part of Logic with the main reference being Aristotle's Organon. Logic was considered to be higher in the hierarchy of sciences than Mathematics and the flow of ideas from mathematics to logic was inhibited.

The new chapter in Foundations 2 and with it in Foundations in general starts, as far as I could find, with a paper "A mathematical analysis of logic" by George Boole from 1847.

In it Boole develops a symbolic calculus for what today we would call the logic of sub-classes of one ambient class. He calls this class "Universe" and denotes by 1.

Boole however never says "a sub-class of 1" but simply "a class".



THE MATHEMATICAL ANALYSIS

OF LOGIC,

BEING AN ESSAY TOWARDS A CALCULUS
OF DEDUCTIVE REASONING.

BY GEORGE BOOLE.

Ἐπικοινωνοῦσι δὲ πᾶσαι αἱ ἐπιστῆμαι ἀλλήλαις κατὰ τὰ κοινά. Κοινὰ δὲ λέγω, οἷς χρῶνται ὡς ἐκ τούτων ἀποδεικνύντες· ἀλλ' οὐ περὶ ὧν δεικνύουσιν, οὐδὲ ὃ δεικνύουσι.

ARISTOTLE, *Anal. Post.*, lib. I. cap. XI.

CAMBRIDGE:
MACMILLAN, BARCLAY, & MACMILLAN;
LONDON: GEORGE BELL.

1847

For Boole a class is a non-empty class. This allows him to consider the transformation from “All Xs are Ys” to ‘Some Xs are Ys” to be lawful. It also allows him to translate “Some Xs are Ys” into “xy=v” without having to say explicitly that “v” here refers to a non-empty class.

This viewpoint implies for example that his notation 0 does not refer to a class at all and the expression xy=0 for the proposition “No Xs are Ys” does not translate in his mind, as it does in ours, to “the class of both Xs and Ys equals to the empty class” but remains a purely formal notation.

He writes

“It may happen that the simultaneous satisfaction of equations thus deduced, may require that one or more of the elective symbols should vanish. This would only imply the nonexistence of a class:“

Meaning here by “vanish” that some symbols could not be used anymore where we would just say that this symbol refers to the empty class. And he continues with what is probably the first formulation of the concept of inconsistency in the formal logic:

“… it may even happen that it may lead to a final result of the form $1 = 0$, which would indicate the nonexistence of the logical Universe.

Such cases will only arise when we attempt to unite contradictory Propositions in a single equation.“ (Boole, 1847, p.68)

A few years later Boole wrote a book "The Laws of Thought". In it he introduces a very important distinction between propositions primary and secondary. He writes:

"Every assertion that we make may be referred to one or the other of the two following kinds. Either it expresses a relation among things, or it expresses, or is equivalent to the expression of, a relation among propositions. […] The former class of propositions, relating to things, I call "Primary;" the latter class, relating to propositions, I call "Secondary."

(Boole, 1853, p.37 of Project Gutenberg's digitization)

This distinction later evolved into the concepts of the "first order logic" and the "second order logic".

He then says that any primary proposition can be transformed into a secondary proposition:

"If instead of the proposition, "The sun shines," we say, "It is true that the sun shines," we then speak not directly of things, but of a proposition concerning things […] Every primary proposition may thus give rise to a secondary proposition, viz., to that secondary proposition which asserts its truth, or declares its falsehood." (loc.cit., p. 38)

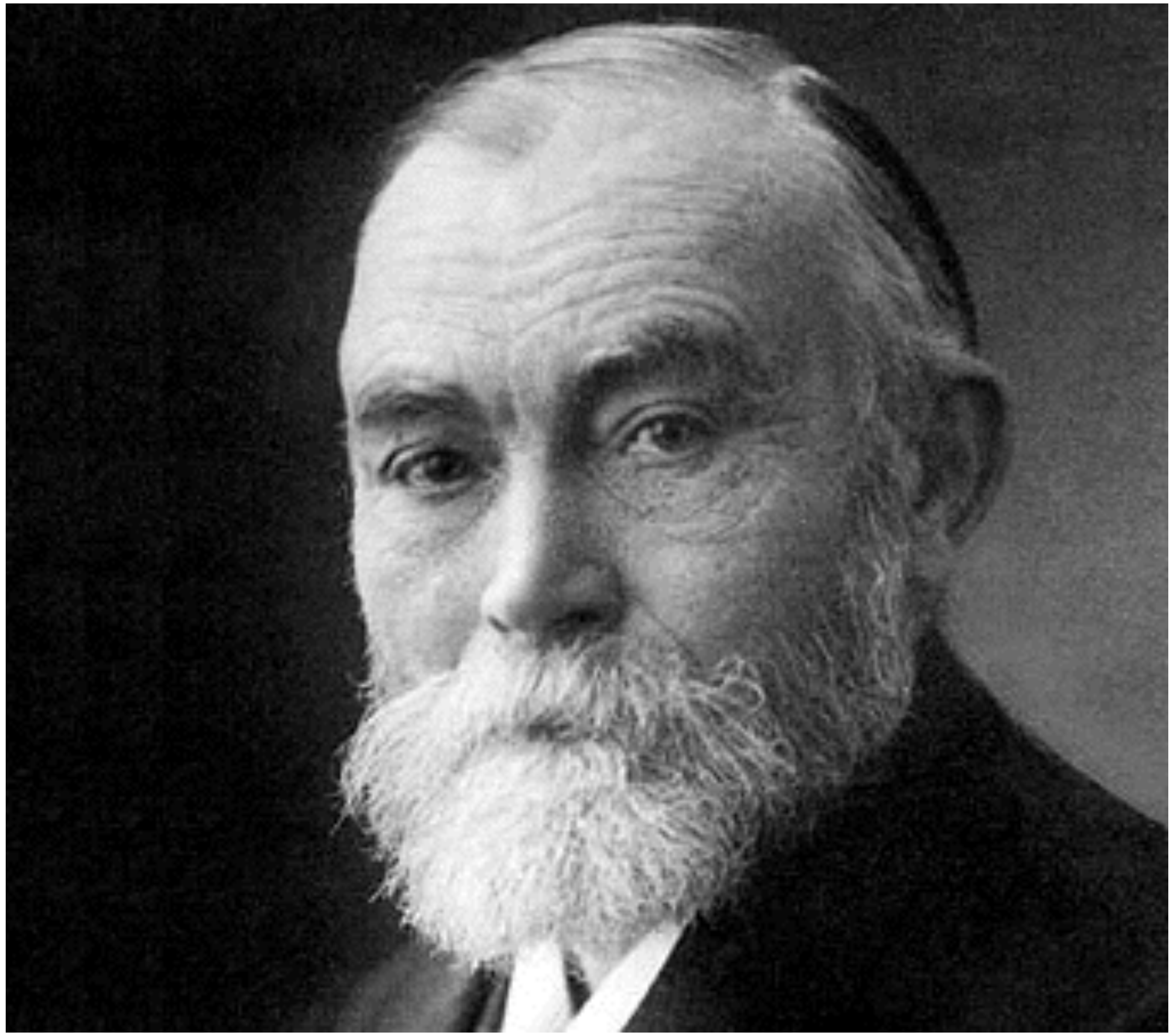
The equivalence of these two forms is known in type theory as propositional extensionality: P <=> (P = True).

It is the simplest particular case of the Univalence Axiom.

The next reference point in the development of Foundations 2 is an 1879 paper by Gottlob Frege "A formula language, modeled on that of arithmetic, for pure thought".

It is a paper that introduced a lot of ideas and it would deserve a careful discussion which I do not have time for now.

Let us just make a note that this is the second reference point in the modern history of Foundations 2.



Gottlob Frege (1848-1925)

While these momentous events were happening in Foundations 2, development also continued in the parts of mathematics that will in the later years form the most fundamental layers of new Foundations 1.

The "theory of forms" which Grassmann wrote about was beginning to develop into concrete proposals.

Bernhard Riemann in a visionary 1854 lecture "On the Hypotheses which lie at the Bases of Geometry" first discusses "continuous or discrete manifoldness" and then proceeds to write about continuous manifoldness after a remark that the notions of discrete manifoldness are being very common.

George Cantor, in 1883, wrote a paper called "'"'Foundations of the general theory of manifolds (Mannigfaltigkeitslehre)". He explains the word "manifold" as follows:

"I use this word to designate a very broad theoretical concept which I hitherto used only in the special form of a theory of geometric or arithmetical sets. In general, by a "manifold" or "set" I understand every multiplicity which can be thought as one […] ."

(Georg Cantor, Grundlagen, 1883, p. 916 of "From Kant to Hilbert", v.2)

Cantor around 1870 (from Wikipedia)

This paper also contains the following remarkable words:

”Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established." (Georg Cantor, Grundlagen, 1883, p. 896 of “From Kant to Hilbert”, v.2)

Contrast these words with the following words of Boole:

“An almost boundless diversity of theorems, which are known, and an infinite possibility of others, as yet unknown, rest together upon the foundation of a few simple axioms;” (Boole, 1853, p.3 of Project Gutenberg’s digitization)

Realization of the vision so well expressed by Boole for the theory of sets invented by Cantor was attempted by Ernst Zermelo.

In an interesting and elegant 1908 paper "Investigations into the foundations of set theory" he writes:

"Now in the present paper I intend to show how the entire theory created by G. Cantor and R. Dedekind can be reduced to a few definitions and seven "principles", or "axioms", which appear to be mutually independent."

The theory that Zermelo invented was later extended and made more precise and became known as ZFC - the Zermelo-Fraenkel set theory with the axiom of Choice.

At about the same time, in 1903, Bertrand Russell starts his treatise "Principles of Mathematics" with a definition:

"Pure Mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants."

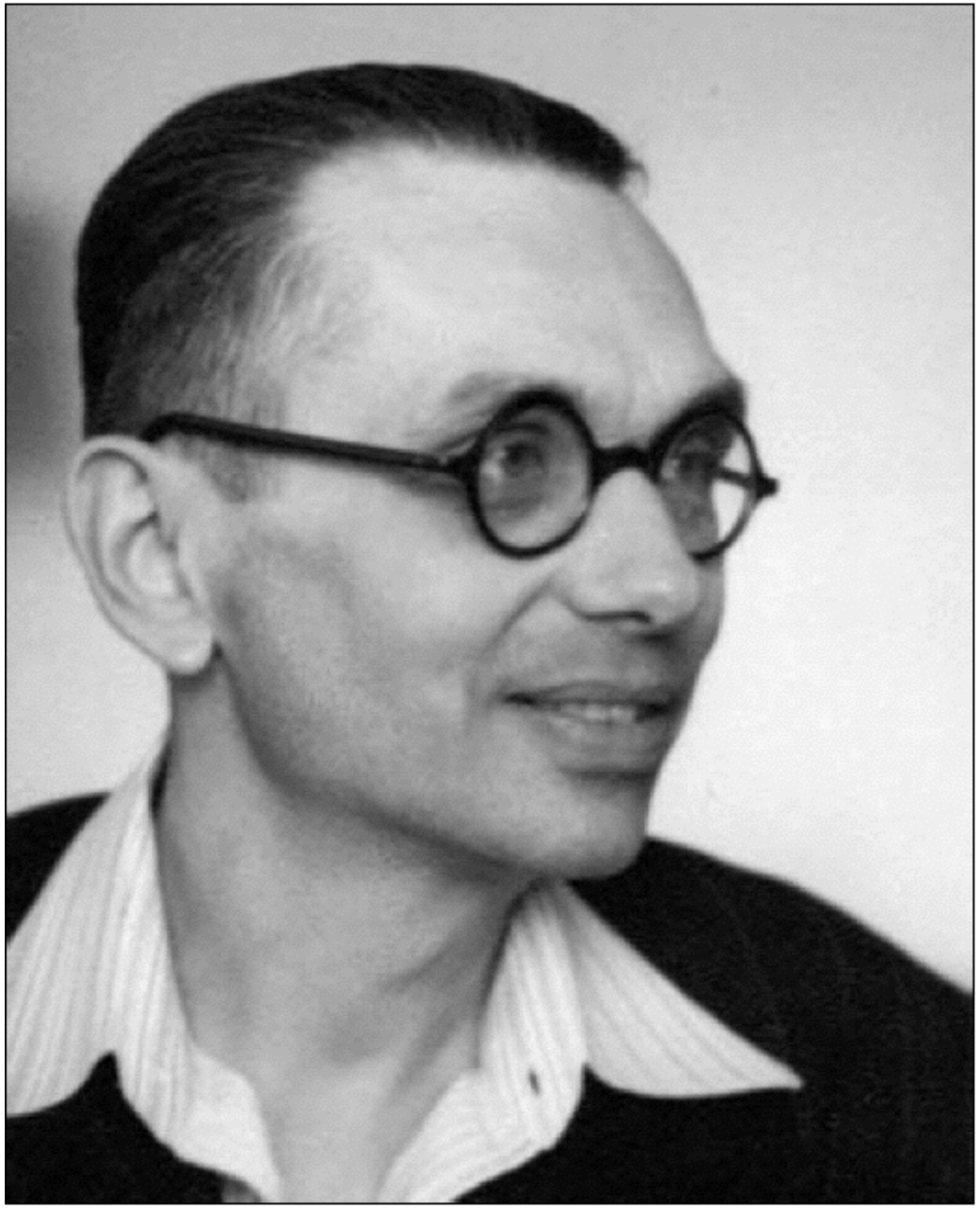
The efforts to reduce mathematics to logic, which were motivated by an ancient and powerful idea that placed Logic higher in the hierarchy of sciences than Mathematics, would continue for many decades.

But a development in Foundations 2 put a limit to these attempts.

This development is the discovery, in the Fall of 1930, by a young Austrian mathematician Kurt Godel, of two theorems. These theorems are known today as the first and the second incompleteness theorem.

The first theorem implied that none of the known approaches to formalization of reasoning can be used to create a consistent formal deduction system where all questions about finite integers could be answered.

The second theorem implied that none of the systems where even the basic questions could be answered can be proved to be consistent.



Kurt Godel 1950

These results can probably be compared in their effect to the discovery that no multiple of the diagonal of a square can be measured by the side of this square that is attributed to Pythagoreans.

Pythagoreans are said to have had a model of the world based on ratios of integers. The result about the diagonal pushed this model from the all encompassing status to a lower position.

Something similar has probably happened to the relative positions of Logic and Mathematics after Godel’s discoveries.

Let me fast forward now to the 1980ies when I was beginning to learn pure mathematics.

What were the states of Foundations 1 and 2 at that time?

The new vision of Foundations I was that mathematics was based on the theory of sets.

Finite integers and operations on them were defined in terms of finite sets so that arithmetic could be founded on the study of finite sets.

Axiomatic geometry was not a part of mainstream mathematics anymore. Instead there appeared the fields of topology and of differential and algebraic geometry.

The division of mathematics into the study of discrete versus continuous also disappeared since continuous was now defined in terms of the discrete using the ideas of actual infinity.

This approach to Foundations I was very successful and productive.

What about Foundations 2? Here the story is more complex.

From some point on it was officially accepted as a postulate that the set theory, which mathematics was based on, was successfully formalized in ZFC.

ZFC is a particular theory in the so called predicate logic. Along with the acceptance of ZFC it was accepted that predicate logic is an adequate, and in fact the only acceptable one, formal language of pure mathematics.

This led to the state of affairs when the development of Foundations 2 in as much as they were related to pure mathematics and mathematical education stopped.

The real state of affairs in Foundations 2 was extremely different.

The second half of the 20th century was a period of unprecedented innovation and development in Foundations 2. Almost all these developments occurred outside of the area of predicate logic and so remained virtually invisible to pure mathematics.

These developments were due to the appearance of computers. Recall that both Boole and Frege tried to create formula languages for human thought.

The needs of computer science required the ability to communicate human thoughts to computers so that computers could perform tasks based on these thoughts. Computers only understand formal languages.

The theory and practice of programming languages became the main source of new ideas in Foundations 2.

It is commonly said, and I have repeated it myself, that the great advances of pure mathematics in the second half of the 20th century are due to the establishment of a strong foundation in the form of set theory.

It seems to me now, after I learned more about the history of mathematics, that this saying needs a careful qualification.

The notion of a set, being the simplest case of a general form, is fundamental for mathematics.

The main concepts of set theory discovered by Georg Cantor, such as the concept of a bijection, truly belong to the deepest levels of the modern hierarchy of mathematical concepts.

However, the theory of sets that is at the core of modern pure mathematics have never been successfully formalized.

The formal theory ZFC, a development of Zermelo's attempt, is not an adequate formalization of the set theory which is used in mathematics.

And I believe that it was impossible at that time to formalize set theory properly because such formalization required much more advanced Foundations 2 than were available at that time.

Only now adequate formalizations of the intuitive set theory used by mathematics becomes feasible.

And most likely such a formalization will be first achieved in the framework of a more general theory of forms that Univalent Foundations are based on.