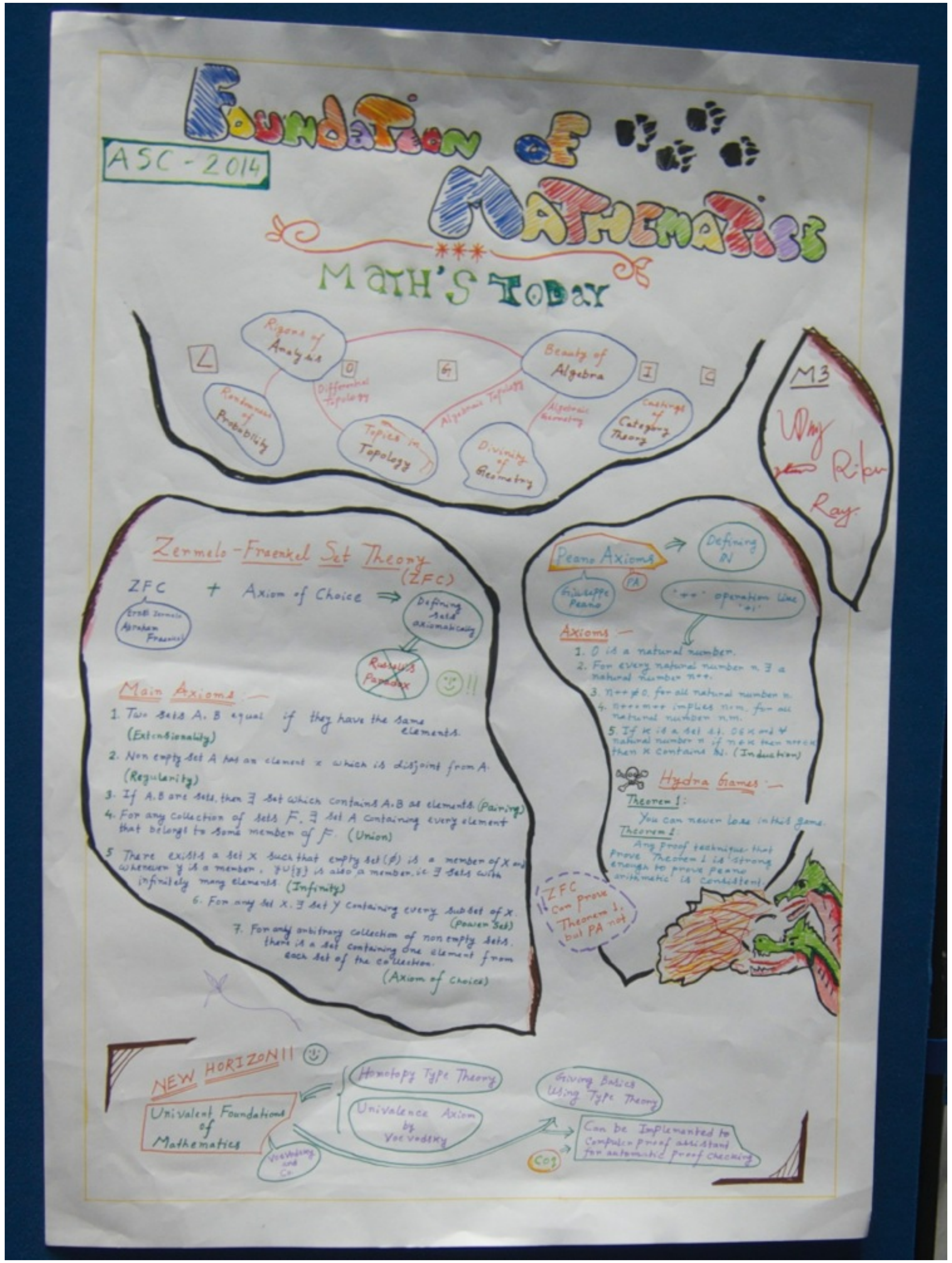
Paul Bernays Lectures, 2014

Prof. Vladimir Voevodsky

from the Institute for Advanced Study, Princeton, NJ, USA

Foundations of Mathematics: their past, present and future.

Part I. To the history of the conception.



A poster made by the students of the 8th Asian Science Camp in Singapore.

What are “foundations of mathematics”?

Let’s see what Wikipedia has to say about it:

Foundations of mathematics is the study of the basic mathematical concepts (number, geometrical figure, set, function...) and how they form hierarchies of more complex structures and concepts, especially the fundamentally important structures that form the language of mathematics (formulas, theories and their models giving a meaning to formulas, definitions, proofs, algorithms...) also called metamathematical concepts, with an eye to the philosophical aspects and the unity of mathematics.

From https://en.wikipedia.org/wiki/Foundations_of_mathematics (Aug. 30, 2014)

We can see here the mixture of two meanings that makes talking and thinking about foundations of mathematics very difficult.

On the one had there is the study of "the basic mathematical concepts and how they form hierarchies of more complex structures and concepts"

On the other hand: "the fundamentally important structures that form the language of mathematics also called metamathematical concepts".

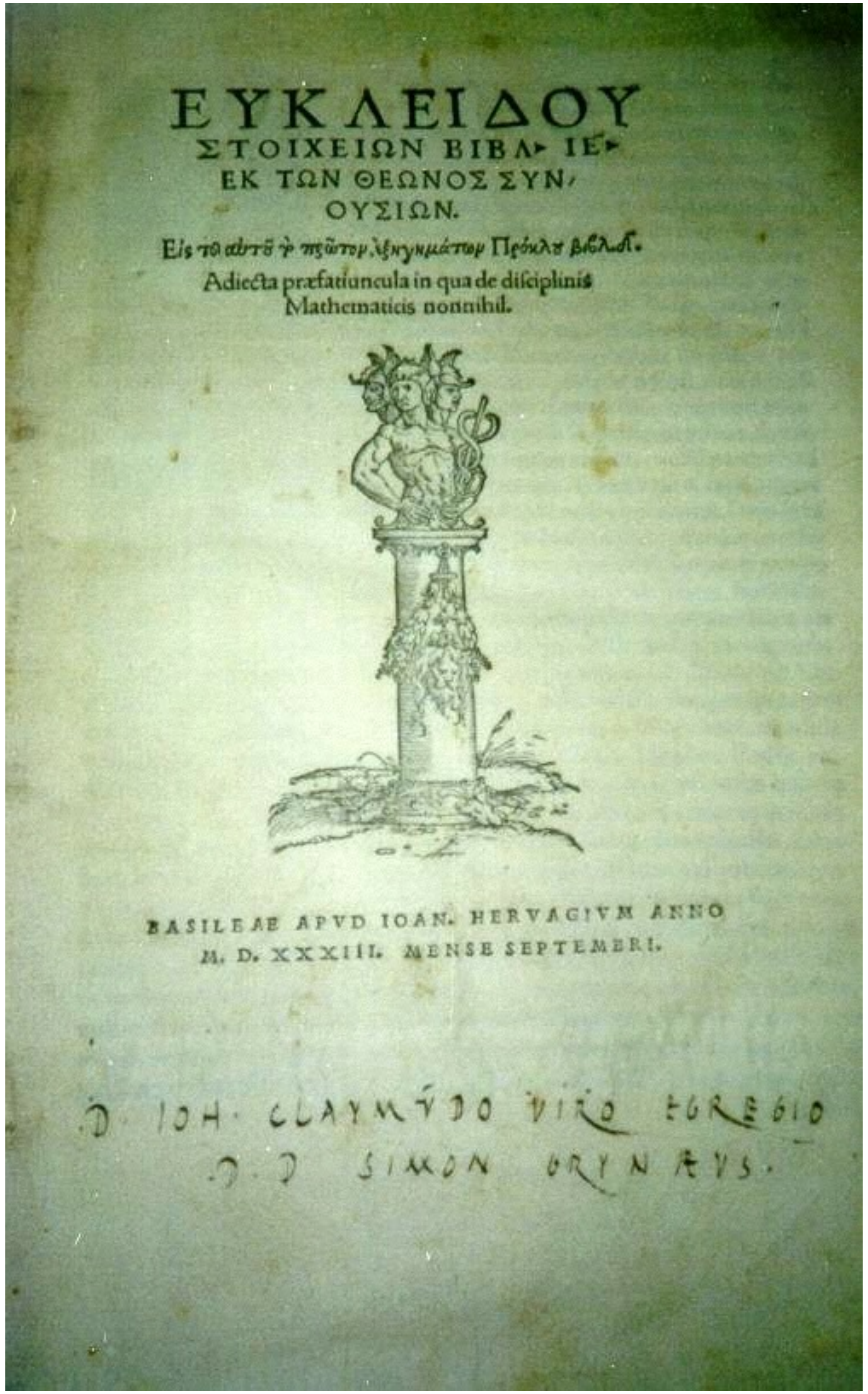
Let us find some examples of discourse related to the first and the second meanings of "foundations of mathematics" in a book that I have spent many hours with while learning for these lectures.

This book is the commentary to the first book of Euclid's elements by Proclus in translation by Glenn R. Morrow.

Proclus was a pagan who died in Athens at the end of the fifth century AD.

He was the head of the main non-christian philosophical and religious tradition of his time.

This tradition is known today as neoplatonism.



This commentary comes with a prologue that contains an extensive discussion of how ancients thought about mathematics.

The most important element of the neoplatonic view is summarized by the quote:

"… mathematics occupies a middle ground between the intelligible and the sense worlds and exhibits within itself many likenesses of divine things and also many paradigms of physical relations, …" (Proclus, 35).

Proclus discusses also the neoplatonic values related to mathematics.

It is amazing how close these values agree with the values that I learned in the mathematical community in Moscow in nineteen-eighties. Here is a sentence that is eerily similar, if much more eloquent in form, to what I remember telling many times to people who asked me about my studies:

"We must therefore posit mathematical knowledge and the vision that results from it as being worthy of choice for their own sakes, and not because they satisfy human needs." (Proclus 27-28)

Proclus tells of two approaches to the structuring of mathematical science. This is about "Foundations I":

"The Pythagoreans considered all mathematical science to be divided into four parts: one part they marked of as concerned with quantity, the other half with magnitude; and each of these they posited as twofold. A quantity can be considered in regard to its character by itself or in its relation to other quantity, magnitudes as either stationary or in motion." (Proclus, 35)

The primary divisions is known today as the division between the discrete ("quantities") and continuous ("magnitudes").

BTW - the separation of variables in programming languages into types first appeared in Fortran (1957) and there were exactly two types - integers (discrete) and floating point (continuous).

The secondary divisions in the Pythagorean classifications are less clear to us. Proclus illustrates them as follows:

“Arithmetics then studies quantity as such, music the relations between quantities, geometry magnitudes at rest, spherics magnitudes inherently moving.” (Proclus, 35)

The connection of music to ratios of integers is more or less clear. “Spherics” refers to the mathematical basis of astronomy.

Another approach attributed by Proclus to a more recent source is as follows:

“But others, like Geminus, think that mathematics should be divided differently; they think of one part as concerned with intelligibles only and of another as working with perceptibles and in contact with them.

Of the mathematics that deals with intelligibles they posit arithmetic and geometry as the two primary and the most authentic parts, while the mathematics that attends to sensibles contains six sciences: mechanics, astronomy, optics, geodesy, canonics (music theory), and calculation.”   (Proclus, 38)

The primary division here is the one very familiar from today - the division of mathematics into pure and applied. And the secondary divisions are also quite easy to comprehend in modern terms.

The division of mathematics into "classical" and "constructive", which is extremely important in the dynamics of development of the Univalent Foundations today, is not directly reflected in Proclus.

What is addressed by him however are arguments about a related distinction- the one between "theorems" and "problems" where a "problem" is understood as an inquiry the answer to which is a construction of an object.

This discussion about theorems versus problems is about metamathematical issues, it is "Foundations 2".

To understand the following quotes one needs to know that "proposition" is used in the sense of any inquiry about mathematical objects:

"Some of the ancients, however, … , insisted on calling all propositions "theorems", considering "theorems" to be a more appropriate designation than "problems" for the objects of the theoretical sciences, especially since these sciences deal with eternal things. There is no coming to be among eternals, and hence a problem has no place here, proposing as it does to bring into being or to make something not previously existing, such as to construct an equilateral triangle … .

Others, on the contrary, … , thought it correct to say that all inquiries are problems … . " (Proclus, 77-78)

These arguments are remarkably alive today.

Just a few months ago we had a heated discussion about it on the homotopy type theory mailing list under the guise of the discussion whether one can consider all types as propositions and all objects as proofs.

And here is a related quote from an important paper by Per Martin-Lof.

Correlatively, the third form of judgment may be read not only

a is an object of type (element of the set) *A* ,

a is a proof of the proposition *A* ,

but also

a is a program for the problem (task) *A* .

The equivalence of the first two readings is the by now well-known correspondence between propositions and types discovered by CURRY (1958, pp. 312–315) and HOWARD (1969), whereas the transition from the second to the third is the KOLMOGOROV (1932) interpretation of propositions as problems or tasks (Ger. *Aufgabe*).

Per Martin-Lof, “Constructive mathematics and computer programming”, 1979, p. 162

As we move from Proclus forward in time we find more about Foundations 1. Here is Gauss (around 1801):

"The subject of mathematics is all extensive magnitudes (those in which parts can be conceived); intensive magnitudes (all non-extensive magnitudes) insofar as they depend on the extensives. To the former class of magnitudes belong space (or geometrical magnitudes which include lines, surfaces, solids, and angles), time, number; to the latter: velocity, density, rigidity, pitch of tone, intensity of tones and of light, probability, etc."

(cited by Lewis in "H. Grassmann's 1844 Ausdehnungslehre and Schleiermacher's Dialektik" p. 106)

And here is Hermann Grassmann. The introduction to his main work Ausdehnungslehre of 1844 contains several important ideas which are revealing of the state of Foundations I at his time. The pages refer to the 1995 edition of the english translation.

First he distinguishes mathematics from philosophy by their methods:

"… in [philosophy] the overview of the whole predominates, and its development consists precisely in the gradual ramification and articulation of the whole, in the [mathematics] the interconnection of particulars is emphasized, and separate, independent developments combine together, each becoming only a factor in the following concatenation." (Grassmann, p.30)

We can see that Grassmann sees mathematics not as coming from one foundation that is the root of the rest of mathematics. Instead, mathematics is seen as rising from many independent roots and acquiring wholeness only through the intertwining of the lines arising from these roots.

Later in the text Grassmann proposes that there should be a single general theory underlying mathematics and he describes his vision as follows:

"By the general theory of forms we mean that series of truths that relate to all branches of mathematics in the same way, and which thus assume only the general concepts of equality and difference, conjunction and separation." (Grassmann, p.33)

He further partitions mathematics first into the theory of "continuous form, or magnitude in the narrow sense" and "discrete or conjunctive form" and then into "that arising from the equal [which] we may call the algebraic form" and "that from the different [which we may call] the combinatorial form". (Grassmann, p. 25-26)

As we can see the idea of partitioning mathematics into two halves and then into two halves again which Proclus attributed to Pythagoreans is very persistent.

All these quotes are about Foundations 1.

Traditionally, one would probably say that Foundations 2 arose in the western world in the form of a more general field of the science of logical discourse. The main ancient source for this science is the collection of works of Aristotle known as Organon.

In the neoplatonic Proclus there is little discussion about Foundations 2 other than the part about "theorems" versus "problems".

There is also an attempt to explain the distinction between hypotheses, postulates and axioms using ideas from "inspired Aristotle" but Proclus's exposition of these ideas in the Commentary diverges from what is found in Aristotle's Posterior Analytics (a part of the Organon).

We will consider the development of Foundations 1 and Foundations 2 in the 19th and 20th century in the next lecture.

At the end of this lecture let me take a look not forward, but backward in time from the time of Plato and Aristotle.

Any approach to foundations of mathematics that aims at being complete must be able to provide an analysis of the foundation comprised from a number system and methods of reasoning used to perform inferences about numbers in this system.

A lot is known about the history of number systems. The slowness with which they evolved gives us some idea of the complexity of the concept and action systems that underlines them and that are now almost invisible to us being embedded at the deeper layers of our psyche.

We can try to use the distinction between Foundations 1 and Foundations 2 to gain some clarity in the structure of these ancient foundational systems.

To represent any number as it is done, for example, in the decimal number system in terms of units, tens, hundreds etc. is an example of a hierarchy of concepts used to order the infinite realm of individual numbers.

The concepts of "greater" and "less" and "equal" are originally metamathematical. This can be seen from the fact that in the ancient number systems such as the Babylonian one they were expressed by the words of the common language just as today we use the words of common languages to express logical connectives.

More importantly, metamathematical are the algorithms used for calculation.

To understand number systems as examples of a general concept of a foundational system we need a concept that would include methods of computation as a part of a foundations.

Today, in particular in the Univalent Foundations, one of the most interesting and complex problems is the integration of abstract reasoning with computations in one system.

It turns out that consistency is a property of a given system of computation combined with principles of abstract reasoning.

Methods of computation, just like axioms, combine together in a dependent manner. Of three methods A, B and C combining A and B can be consistent as well combing B and C but combining A, B and C together may lead to inconsistencies.

Maybe thinking about algorithms used by the ancients in their mathematics can help us to be more creative in the design of our new integrated formal reasoning systems for the future.