Univalent Foundations and Set Theory

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Univalent foundations

- are based on a class of formal deduction systems which we currently refer to as *homotopy type theories*. These theories are extensions of Martin-Lof type theories which are compatible with the univalent model.

The univalent model is that which connects univalent foundations with set theoretic foundations and serves as means to ensure that the deduction systems which we construct are at least as consistent as ZFC.

The main difficulty in explaining univalent foundations lies in the fact that type theories encompass a wide class of deduction systems which currently does not have a generally accepted definition.

Sentences of type systems

In a typical type system one works with four kinds of sentences:

$$x_1 : T_1, \ldots, x_n : T_n \vdash T \quad type$$

$$x_1 : T_1, \ldots, x_n : T_n \vdash t : T$$

$$x_1 : T_1, \ldots, x_n : T_n \vdash T = T'$$

$$x_1 : T_1, \ldots, x_n : T_n \vdash t = t' : T$$

where $x_1, \ldots, x_n$ are names of variables and $T_1, \ldots, T_n, T, t, t'$ are expressions of the system of expressions underlying the type system.

Derivation rules

Type system is specified by the underlying system of expressions and a collection of derivation rules which show which sentences are derivable in the theory. For example most type systems would have the following derivation rule for sentences of the second kind where the object expression t starts with λ:

$$\frac{\Gamma, x : T \vdash o : T'}{\Gamma \vdash \lambda x : T, o : \prod x : T, T'}$$

the following derivation rule for sentences of the first kind where the type expression starts with Id:

$$\frac{\Gamma \vdash o_1 : T \quad \Gamma \vdash o_2 : T}{\Gamma \vdash Id\, T\, o_1\, o_2 \quad type}$$

h-levels

May be the most important feature of homotopy type theories is the notion of h-levels. Given a type expression T in context Γ and an object n of type nat of natural numbers we can write a new type expression $isofhlevel\, n\, T$ in Γ. This expressions are defined inductively from the standard type constructors as follows:

$$isofhlevel\, 0\, T = \sum x : T, \prod x' : T, Id\, T, x, x'$$

$$isofhlevel\, (1+n)\, T = \prod x : T, \prod x' : T, isofhlevel\, n\, (Id\, T, x, x')$$

Proving that a type expression is of h-level n means constructing (in a given context) of an object of type $isofhlevel\, n\, T$ i.e. deriving a sentence of the form

$$\Gamma \vdash o : isofhlevel\, n\, T$$

h-levels (cont.)

For example the type *nat* is provably of h-level 2 which intuitively corresponds to sets and the unit type and empty type are of h-level 1.

The univalent model suggests the following interpretation of types of small h-levels:

Types of h-level 1 are propositions. Inhabited types of h-level 1 are provable propositions. Thus the proposition as types correspondence is modified in homotopy type theory by interpreting only types of h-level 1 as propositions.

Types of h-level 2 are sets. Thus most of mathematics deals with types of h-level 2 and their objects.

Types of h-level 3 correspond to groupoids. For example the collection of all finite groups in homotopy type theory is a type of h-level 3.

Models of type theories

Equivalence classes of sentences of the first and second kind of a type system T modulo the equivalence relations defined by sentences of the third and the fourth kind are denoted $B_{n+1}(T)$ and $\widetilde{B}_n(T)$ where n is the length of the context. One also formally adds $B_0(T)$ which contains one element pt.

The properties of sentences such as their ability under substitution e.g.

$$\frac{\Gamma \vdash o : S \quad \Gamma, x : S, \Gamma' \vdash T \quad type}{\Gamma, \Gamma'[o/x] \vdash T[o/x] \quad type}$$

define a number partial defined operations on B_n's and $\widetilde{B}_n$'s. With this partially defined operations B-sets of a type theory form a model of a quasi-algebraic theory.

Models of type theories (cont.)

Derivation rules of a type system add operations and equations to the theory such that ultimately the system of B-sets or B-system of a type theory becomes *initial* model of a complex quasi-algebraic theory. A model of the type theory is a homomorphism from its B-system to another B-system. Since under the right choice of operations B-sets of a type system form an initial model of a theory constructing a model of a type system is equivalent to constructing a B-system with a list of additional operations and axioms.

Models of type theories (cont.)

The key machinery for constructing such B-systems is the machinery of categories with universes. Let $\mathcal{C}$ be a category and $p : \widetilde{U} \to U$ a morphism. Let us fix a final object pt of $\mathcal{C}$. Let us further fix for any morphism $f : X \to U$ a pull-back square

$$\begin{array}{ccc} (X,f) & \xrightarrow{(q,f)} & \widetilde{U} \\ {\scriptstyle p_{X,f}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

and let us write $(X, f_1, \ldots, f_n)$ etc. for the obvious iterated construction.

Models of type theories (cont.)

Now let B_n be the set of sequences $(f_1, \ldots, f_n)$ where $f_1 : pt \to U$ and $f_{i+1} : (pt, f_1, \ldots, f_i) \to U$ and let $\widetilde{B}_n$ be the set of sequences $(f_1, \ldots, f_n, g)$ where $(f_1, \ldots, f_n)$ is as before and $g : (pt, f_1, \ldots, f_n) \to \widetilde{W}$.

One easily constructs a B-system structure on these sets which is determined up to a canonical isomorphism by the equivalence class of $(\mathcal{C}, p)$. Let us denote this B-structure as $BB(\mathcal{C}, p)$.

Models of type theories (cont.)

Finally, when $\mathcal{C}$ is a locally cartesian closed category, various additional type-theoretic structures of the B-sets corresponding to $(\mathcal{C}, p)$ can in all the interesting cases we know of be obtained from natural structures on the morphism $p : \widetilde{W} \to W$. For example the standard group of structures which correspond to the introduction-elimination rules of the dependent products are generated on $BB(\mathcal{C}, p)$ from a pair of morphisms $\widetilde{P}$ and P such that the following square is pull-back:

$$\begin{array}{ccc} \underline{Hom}_U(\widetilde{U}, \underline{U} \times \widetilde{U} & \stackrel{\widetilde{P}}{\longrightarrow} & \widetilde{U} \\ {\scriptstyle \underline{Hom}_U(\widetilde{U}, Id_U \times p)}\downarrow & & \downarrow{\scriptstyle p} \\ \underline{Hom}_U(\widetilde{U}, \underline{U} \times U) & \stackrel{P}{\longrightarrow} & U \end{array}$$

Univalent model

The current construction of the univalent model for type theory such as that of Coq which assumes an infinite sequence of universes $U_i : U_{i+1}$ is based on $\mathcal{C}$ being the category of simplicial sets relative to ZFC with $\omega + 2$ universes.

The largest universe is used to define the set of objects of the category of simplicial sets. The next smaller one is used to define

$$p : \widetilde{U} \to U$$

Univalent model (cont.)

The formal definition of $p : \widetilde{U} \to U$ is as follows. The set U_n of n-simplexes of U is the set of isomorphism classes of Kan fibrations $X \to \Delta^n$ given together with well-orderings on the sets X_n and such that all X_n belongs to one of the universes $U_0, \ldots, U_i, \ldots$.

It is easy to see that p itself is a Kan fibration. A much less trivial fact is that U is a Kan simplicial set and that p is a *univalent fibration*.

Univalent model (cont.)

In the univalent model a sentence $x_1 : T_1, \ldots, x_n : T_n \vdash T \quad type$ is interpreted as a tower of fibrations of (Kan) simplicial sets

$$[T_1, \ldots, T_n, T] \to [T_1, \ldots, T_n] \to \ldots \to [T_1] \to [pt]$$

and a sentence $x_1 : T_1, \ldots, x_n : T_n \vdash t : T$ as a section

$$[t] : [T_1, \ldots, T_n] \to [T_1, \ldots, T_n, T]$$

of the first fibration in this tower.

Univalent model (cont.)

In particular a type expression in the empty context $\vdash T \quad type$ is interpreted as a Kan simplicial set and an object of a type in the empty context $\vdash t : T$ as a point of the Kan simplicial set corresponding to T.

The standard constructors are interpreted as follows:

1. $[\prod x : T, T'] =$ the simplicial set of sections of the fibration $[T, T'] \to [T]$
2. $[\sum x : T, T'] = [T, T']$
3. $[Id\ T\, o_1\, o_2] =$ simplicial paths from $[o_1]$ to $[o_2]$ in $[T]$
4. $[nat]$ - the usual set of natural numbers considered as a simplicial set.

Univalent model (cont.)

Most importantly for a universe U_i of the type theory the interpretation $[U_i]$ is the simplicial subset of U such that the set of n-simplxes $(U_i)_n$ is the set of fibrations $X \to \Delta_n$ such that for each m the set of m-simplexes X_m of X is in the set-theoretic universe number i. If we define $\widetilde{U}_i$ as the sentence $X : U_i \vdash X \quad type$ then the obvious projection $[\widetilde{U}_i] \to [U_i]$ is the universal Kan fibration with fibers in the i-th universe.

Univalent model (cont.)

One verifies relatively easily that the univalent model maps h-levels of types to levels of homotopy types as follows:

1. types of h-level 0 map to contractible simplicial sets
2. types of h-level 1 map to simplicial sets which are homotopy equivalent to either a point or to the empty set
3. types of h-level 2 map to simplicial sets which are homotopy equivalent to sets
4. types of h-level n+2 map to simplicial sets representing homotopy n-types.

Current developments

Currently we are developing new type theories more complicated than the standard Martin-Lof type theory and at the same time more convenient for practical formalization of complex mathematics.

Such type theories may easily have over a hundred derivation rules. Thus a careful and formalizable approach is needed to show that the newly constructed type theory is at least as consistent as ZFC with a given structure of universes.

More precisely we need to be able to formally show that the univalent model extends to these new theories.

Current developments (cont.)

The first step in the development of the machinery which one needs to be able to verify such statements is Initiality Theorem. The theorem should state that the B-sets of a type system with given set of derivation rules form the initial model of an extension of the quasi-algebraic theory of B-systems extended by operations and equations which are formally read from the derivation rules.

So far we do not have even a proper formulation of such a result but all our experience shows that it is possible and that we are moving in the direction where we will be able to state and prove it.

Current developments (cont.)

The second step in the development of such a machinery should be Operations Theorem which shows how to obtain operations of the kind considered in the Initiality Theorem on B-systems $BB(\mathcal{C}, p)$ for locally Cartesian closed categories from structures on p formulated in the internal language of locally Cartesian closed categories. This step is also becoming intuitively more clear. In this step in particular the ideas and language of LF (Logical Framework) appear in a very suggestive way.

Conclusion

In my vision the more and more complex homotopy type theories will become the languages in which practical unified formalization of all mathematics is done. The set theory will remain the most important benchmark of consistency. The theory of interpretations of type theories in set-theoretic objects will be formalized in relatively simple type theories and each new addition to the practical language will require formal "certification" by showing, through formally constructed interpretation, that it is at least as consistent as ZFC.