Talk at WoLLIC (Upenn, May 2011) by Vladimir Voevodsky

Univalent Foundations are:

1. Foundations of mathematics which can be used both for constructive and for non-constructive mathematics.
2. Foundations which naturally include "axiomatization" of the categorical and higher categorical thinking.
3. Foundations which can be conveniently formalized using the well known in computer science class of languages called Martin-Lof type systems.
4. Foundations which are based on direct axiomatization of the "world" of homotopy types instead of the "world" of sets.

A bit of history

1. Multiple attempts to use existing foundations (ZFC - Zermelo-Freankel theory with the Axiom of Choice) as the basis of formalization of mathematics in the language of proof assistants such as Coq all led to very unnatural constructions.
2. In 1996, Martin Hofmann and Thomas Streicher constructed a new semantics for type theory which interpreted types not as sets but as groupoids.
3. In 2005/2006 Steve Awodey and his students discovered the connection between the Martin-Lof identity types and factorization axioms of the abstract homotopy theory. This led to their interpretation as path spaces.

A bit of history (cont.)

4. At about the same time I introduced the idea of univalent fibrations and conjectured that there exists a semantics for Martin-Lof type systems which interprets universes as bases of the univalent fibrations.

5. In the fall of 2009 I understood how to combine the ideas of Steve Awodey with my ideas to obtain a far reaching generalization of the groupoid interpretation. It was eventually called the univalent model of Martin-Lof type theories.

6. In February 2010 I started to write a Coq library of mathematical constructions based on the univalent model. See http://github.com/vladimirias/Foundations

Some of the key ideas

1. Univalent model suggest a definition for the type of "weak equivalences" between two types. These equivalences satisfy all the standard properties one would expect from equivalences between higher groupoids or homotopy types. Most of such properties have now been fromaly proved. All constructions written in the context of the univalent semantics are invariant under these equivalences.

2. Not all types are to be interpreted as sets. There is a filtration on type expressions by their "h-levels".

   (a) Types of h-level 0 are "contractible" i.e. equivalent to the one point type.

   (b) Types of h-level 1 correspond to "propositions".

(c) Types of h-level 2 correspond to sets.

(d) Types of h-level 3 correspond to groupoids.

(e) Types of higher levels correspond to higher groupoids or, equivalently, to more general homotopy types.

3. Types with decidable equality have level ≤ 2 e.g. the usual inductive types such as natural numbers, trees etc. are sets.

4. Typical examples of types of level > 2 are universes.

5. The univalent model satisfies a new axiom which is called the univalence axiom. It imposes the condition that the identity type between two types is naturally weakly equivalent to the type of weak equivalences between these types.

Some of the key ideas (cont.)

6. The univalence axiom implies the functional extensionality both for "straight" functions and for dependent functions. It also implies that two logically equivalent "propositions" (types of h-level 1) are equal.

7. The univalence axiom implies that the universe of types of h-level n has h-level $n+1$. In particular, the type of "propositions" is a "set" and the type of "sets" is a "groupoid".

8. The univalence axiom implies similar statements for types with structures e.g. one can prove using the univalence axiom that the identity type between two groups is equivalent to the type of isomorphisms between these groups.

Some of the key ideas (cont.)

9. Unlike many other axioms (e.g. the axiom of excluded middle), the univalence axiom is expected "to have computational content". In other words computation/normalization should be extendable in a certain sense to terms which involve the univalence axiom. For example there is the following precise:

   Conjecture 1. There exists a terminating algorithm which for any term expression t of type nat constructed using the univalence axiom (in any way) returns a term expression $t1$ of type nat which does not use univalence axiom (in any way) and a proof that $t = t1$ (which may use the univalence axiom).

Current state of development.

1. The basic properties of weak equivalences, h-levels etc. have been formalized in Coq. These results do not depend on the univalence axiom. Some of the proofs require functional extensionality.

2. The current approach to the universe management in Coq is not flexible enough for many of the more advanced and interesting applications of the univalent approach. I am talking to the Coq development people about this issues. At the moment, in order not to stall the development of interesting mathematics I am using a patch provided by Hugo Herbelin which switches off the universe consistency verification in Coq.

3. Modulo the disclaimer of the previous paragraph we have been able to formalize a very important construction - the construction of true set-quotients of types. This opens the way for the formalization of many areas of mathematics which were unaccessible to direct type-theoretic formalization due to the usual problems with quotients in type theory.

4. There is a growing community of people working on the issues connected with the univalent foundations. There will be a full year program on this topic at the Institute for Advanced Study in 2012-2013 co-organized by Steve Awodey, Thierry Coquand and myself. For information on the program See http://www.math.ias.edu/sp/univalent .