Univalent Foundations of Mathematics

Talk at IAT by Vladimir Voevodsky

July 11 , 2011

Which problems of set theoretic foundations the univalent foundations solve and which they do not attempt to solve ?

1. Univalent foundations do not attempt to solve any possible consistency issues - the consistency problems in the univalent foundations are essentially the same as in the set-theoretic ones.

2. Univalent foundations unlike the set-theoretic ones admit convenient formalization in the class of languages which the most advanced proof assistants of today use.

3. Univalent foundations naturally include "axiomatization" of categorical and higher categorical thinking.

What is different and what is the same in set-theoretic and univalent foundations.

1. As set theoretic foundations the univalent foundations are ”universal” i.e. can be used for systematization and formalization of all areas of mathematics.
2. Unlike set-theoretic foundations which are formulated in the language of first order logic the univalent foundations are formulated in languages of a completely different class called Martin-Lof type theories.
3. Univalent foundations are based on direct axiomatization of the ”world” of homotopy types instead of the ”world” of sets.
4. Univalent foundations are formulated in constructive terms.

A bit of history

1. Multiple attempts to use existing foundations (ZFC - Zermelo-Fraenkel theory with the Axiom of Choice) as the basis of formalization of mathematics in the language of proof assistants such as Coq all led to very unnatural constructions.

2. In 1996, Martin Hofmann and Thomas Streicher constructed a new semantics for type theory which interpreted types not as sets but as groupoids.

3. In 2005/2006 Steve Awodey and his students discovered the connection between the identity types and factorization axioms of the abstract homotopy theory. This led to the interpretation of identity types as path spaces.

A bit of history (cont.)

4. At about the same time I introduced the idea of univalent fibrations and conjectured that there exists a semantics for Martin-Lof type systems which interprets universes as bases of the univalent fibrations.

5. In the fall of 2009 I understood how to combine the ideas of Steve Awodey with my ideas to obtain a far reaching generalization of the groupoid interpretation. It was eventually called the univalent model of Martin-Lof type theories.

6. In February 2010 I started to write a Coq library of mathematical constructions based on the univalent model. See http://github.com/vladimirias/Foundations

What homotopy theory has to do with foundations of mathematics? I.

Consider ZFC with two universes (inaccessible cardinals) $U1$ and $U2$ such that $U1 \in U2$ and $U1 \subset U2$. Let $FSets(U1)$ be the set of finite sets in $U1$ and $FSets(U2)$ be the set of finite sets in $U2$. Then $FSets(U1) \neq FSets(U2)$ and, moreover, $FSets(U1)$ is not even isomorphic to $FSets(U2)$.

This is a major reason why practicing mathematicians do not think about $FSets$ as about a set. Instead they say, let's consider $FSets(U1)$ and $FSets(U2)$ as categories. Two categories are "the same" if they are equivalent and $FSets(U1)$ and $FSets(U2)$ are equivalent as categories for any $U1$ and $U2$.

Let's go one step further. What is then a finite category? This notion should be invariant under equivalences so the only reasonable definition is to say that a category is finite if all its sets of morphisms are finite and the set of isomorphism classes of objects is finite.

Consider all finite categories in $U1$ and $U2$. These collections of objects have category structures where morphisms are functors. But now we'll discover that the categories of all finite categories in $U1$ and $U2$ are *not* equivalent since isomorphisms between categories define isomorphisms between their sets of objects and a finite category in our sense may have an arbitrary large set of objects.

The solution is to consider the collection of all finite categories as a 2-category. Then, as 2-categories finite categories in $U1$ and $U2$ will be "the same" i.e. will be 2-equivalent.

This construction can be iterated and we come to the conclusion that if we want the world of finite sets to be independent on the universe we need to introduce categories, if we want the world of finite categories to be independent of the universe we need to introduce 2-categories etc.

One of the issues here is that giving a good definition of an n-category is very non-trivial. Fortunately we do not really need the whole concept of an n-category for our purpose since the concept of an equivalence requires us only to know what are isomorphisms between objects.

What homotopy theory has to do with foundations of mathematics? II.

A category where all morphisms are isomorphisms is called a groupoid. As we saw we may consider the collections of finite sets in $U1$ and $U2$ as groupoids where morphisms are isomorphisms of sets. Then on the next step we will only need the collection of all finite groupoids which will form a 2-groupoid etc. For the whole hierarchy we will only need to know a good definition of an n-groupoid.

Remarkably, it is much easier to define n-groupoids than n-categories due to something called the Grothendieck correspondence.

The Grothendieck correspondence says that up to an equivalence n-groupoids are the same as topological spaces with no non-trivial homotopy groups π_i for $i > n$ up to homotopy equivalence.

Let us say that a space or a homotopy type has h-level 0 if it is contractible , h-level 1 if the space of paths between any two points is contractible and h-level $n + 1$ if the space of paths between any two points is of h-level n. For $n \geq 2$ being of h-level n is equivalent to the condition that $\pi_i = 0$ for $i > n - 2$.

The Grothendieck correspondence allows us to think of homotopy types of h-level $n + 2$ instead of thinking about n-groupoids.

For example we should have a well defined homotopy type of all finite sets, a well defined homotopy type of all homotopy types with finite π_0 and π_1's and with $\pi_i = 0$ for $i > 1$ etc.

Such homotopy types do indeed exist and can be defined. Up to a canonical homotopy equivalence they are independent from the universe structure of the set theory which we start with.

For example the homotopy type of the groupoid of finite sets and their isomorphisms is of the form $\amalg_{n\geq 0} B\Sigma_n$ where $B\Sigma_n$ is the classifying space of the group of permutation of n elements.

Basics of the univalent foundations

The basic elementary entities in the univalent foundations are types and terms of a given type. There is definition of an equivalence between two types (an equivalence is a term of a type constructed in a certain way from the two types in question). All construction one can syntactically describe are invariant under the equivalences.

There are enough constructions to define the type of natural numbers, the type of functions between any two types, the types of pairs or more generally n-tuples, the types of equalities between two terms of a given type etc.

There is also, in the standard version an infinite hierarchy of universes.

For any n there is a definition of what it means for a given type to be of h-level n. There is only one (up to an equivalence) type of h-level 0 namely the "one point" type. The types of h-level 1 correspond to "propositions", terms of such types are proofs of the corresponding propositions . Types of h-level 2 correspond to sets. For example the type of natural numbers and the type of finite binary trees have h-level 2. The type of finite sets in a given universe is not a set but a type of h-level 3. Up to an equivalence it does not depend on the ambient universe.

Similarly there is a definition of an function of h-level n. Functions of h-level 0 are equivalences, functions of h-level 1 are "inclusions".

The logic (intuitionistic one) is formalized in terms of operations on types of h-level 1 . The set theoretic mathematic is formalized in terms of structures and operations on types of level 2. The category theory and more general mathematics of the "categorical" level is formalized in terms of structures and operations on types of h-level 3 etc.

To connect the univalent foundations with the set-theoretic ones one constructs a model of the underlying type theory which interprets types as homotopy types. This model maps type expressions of h-level 1 to homotopy types of h-level 1 i.e. to the homotopy types of the one point set and the empty set. The types of h-level 2 are mapped to sets and the types of higher h-levels to the homotopy types of the corresponding h-levels.

In particular, as one would expect, the type of finite sets is mapped to the homotopy type $\amalg_{n \geq 0} B\Sigma_n$.

Current state of development.

1. The basic properties of weak equivalences, h-levels etc. have been formalized in Coq.

2. The current approach to the universe management in Coq is not flexible enough for many of the more advanced and interesting applications of the univalent approach. I am talking to the Coq development people about this issues.

3. We have been able to formalize a very important construction - the construction of true set-quotients of types. This opens the way for the formalization of many areas of mathematics which were unaccessible to direct type-theoretic formalization due to the problems with quotients in type theory.

4. There is a growing community of people working on the issues connected with the univalent foundations. There will be a full year program on this topic at the Institute for Advanced Study in 2012-2013 co-organized by Steve Awodey, Thierry Coquand and myself. For information on the program see http://www.math.ias.edu.