What if current foundations of mathematics are inconsistent?

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Goedel's second incompleteness theorem

Theorem (Goedel) It is impossible to prove the consistency of any formal reasoning system which is at least as strong as the standard axiomatization of elementary number theory ("first order arithmetic").

It was first publicly mentioned by Goedel in an abstract submitted on Oct. 23, 1930.

Von Neumann, using Goedel's proof of the first incompleteness theorem independently recognized that it implies the second incompleteness theorem and wrote to Goedel in November 1930:

"Thus, I think that your result has solved negatively the foundational question: there is no rigorous justification for classical mathematics."

(Von Neumann to Goedel, November 29, 1930)

Goedel's paradox:

- We know that the first order arithmetic is consistent.
- It can be proved that it is impossible to prove that the first order arithmetic is consistent.

What are the choices?

- If we somehow "know" that the first order arithmetic is consistent than we should be able to transform this knowledge into a proof and then the second incompleteness theorem is false as stated.
- Admit a possibility of “transcendental”, provably unprovable knowledge.
- Admit that the sensation of knowing in this case is an illusion and that the first order arithmetic is inconsistent.

Two arguments in the support of the "know":

- "Formulas as subsets" interpretation of the first order arithmetic.
- Gentzen's argument using induction on the structure of the proofs.

What the “first order arithmetic” is and what does one mean by its (in-) consistency?

The first order arithmetic is a mathematical object which belongs to the class of objects which are called formal theories and to its sub-class of formal theories in first order logic.

A formal theory consists of the following four components:

1. Two alphabets, one for "special symbols" and one for the “names of variables”.

2. Syntactic rules which, for any sequence of names of variables x,y,z,... specify which sequences of letters from both alphabets are called "formulas with free variables x,y,z,...”.

3. Deduction rules which are algorithms for constructing closed formulas (i.e. formulas without free variables), from (collections of) closed formulas.

4. A collection closed formulas which are called axioms.

A closed formula is called a theorem if it can be obtained from axioms by means of the deduction rules.

The theories of first order logic have among their special symbols the following symbols:

- $\forall$ - "for all"
- $\exists$ - "there exists"
- $\vee$ - "or"
- $\wedge$ - "and"
- $\Rightarrow$ - "implies"
- $\neg$ - "not"

There are also parentheses and . which are used just as in the natural languages for "punctuation".

A first order theory is called inconsistent if there is a closed formula A such that both A and $\neg$A are theorems.

The particular formal system of the first order arithmetic contains in addition to the basic ones the special symbols =, +, *, >, 0 and 1.

A typical closed formula in first order arithmetic may look like that:

$$\forall n.\exists m.3*n*n*+5*m*m+7=17*m*n$$

which translates into:

"for all n there exists m such that

$$3n^3 + 5m^2 + 7 = 17mn$$ "

This translation into english can be extended to formulas with free variables x,y,z,... which are interpreted as descriptions of subsets in the set of sequences of natural numbers of the form (x,y,z,...).

For example,

$$\exists m.3*n*n*n+5*m*m+7=17*m*n$$

is interpreted as the description of the subset which consists of natural numbers n such that there exists m such that

$$3n^3 +5m^2 +7=17mn$$

It is precisely the existence of this intuitive interpretation which was the original justification for the consistency of the first order arithmetic.

The main problem with this argument is that a general formula even with one free variable describes a subset of natural numbers for which one can prove, using an argument similar to the one which is used in Goedel's proof, that there is not a single number n which can be shown to belong to this subset or not to belong to it.

There is another argument which is often cited as a proof of consistency of first order arithmetic which has been invented by Gerhard Gentzen (1909-1945) .

While Gentzen's reduction argument leads to many very interesting developments it can not be used as a proof of consistency. In relation to the consistency issue the only thing which it shows is that any inconsistency will define a non-terminating decreasing sequence of "ordinals less that ε_0".

What would inconsistency of the first order arithmetic mean for mathematics?

- Inconsistency of the first order arithmetic implies inconsistency of most other foundational systems. For example it implies inconsistency of set theory.

- Inconsistency of the first order arithmetic implies inconsistency of the constructive (or "intuitionistic") arithmetic. It was shown by Goedel in a 1933 paper.

The nature of Goedel's argument shows that it is impossible to construct foundations for mathematics which will be provably consistent.

What we need are foundations which can be used to construct reliable proofs despite being inconsistent.

One possible candidate for such foundations is being developed now using a class of formal systems which are called constructive type theories.

The key property of constructive type theories which is important to us is that a proof of a formula in such theories is not a sequence of deduction rules which connect this formula to the axioms but is, itself, a formula in the same language.

In constructive type theory even if there are inconsistencies one can still construct reliable proofs using the following "workflow":

- A problem is formalized.
- A solution is constructed using all kinds of abstract concepts. This is the creative part.
- An algorithm which verifies "reliability" is applied to the constructed solution (e.g a proof). If this algorithm terminates then we know that we have a good solution of the original problem. If it does not then we may have to start looking for another solution.

There are probably many different ways to ensure "reliability" of a solution or a proof. For example, a solution is reliable if the corresponding expression has a normal form which belongs to a subset of the general formal system for which consistency can be proved.

Summary:

- I suggest that the correct interpretation of Goedel's second incompleteness theorem is that it provides a step towards the proof of inconsistency of many formal theories and in particular of the "first order arithmetic".
- Such an interpretation has important constructive consequences for epistemology and other areas of philosophy.
- In mathematics we will have to learn how to use inconsistent theories to construct reliable proofs - examples from constructive type theories show that it is possible.