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# PART I

# About Mathematics

# Four Levels of Mathematics:

1. Elementary

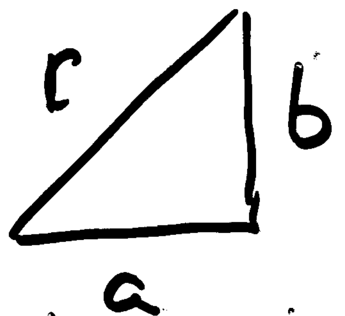
2. "Higher"

3. Modern

4. Synthetic

# Elementary:

- Elem. Geometry





$$c^2 = a^2 + b^2$$

- Elem. Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

School. Very Old
( > 1000 years)

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Higher:

- Analysis

$$\int x\,dx = \frac{x^2}{2}$$

- Differential Equations:

$$\dot{f} = af + b$$

- Differential Geometry
- Probability

# Modern:

- Modern Algebra

  e.g. Galois theory,

  Group theory

- Basic Topology

- Logic and Set Theory

  e.g. Goedel Theorem

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<u>Bourbaki</u> –

a group of French mathematicians Tried to create full description of modern mathematics.

Synthetic:

- Representation Theory
- Algebraic Geometry
- Homotopy Theory
- Differential Topology

⑥

Usually people who do not specialize in Math. know only levels 1 and 2. Sometimes heard about 3.

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# Main reason:

- Elementary - integrated into everyday life
- Higer - integrated into most sciences
- Modern - integrated into a few sciences
- Synthetic - very poorly integrated

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One of the most important challenges today - learn how to use mathematics of higher levels in other branches of knowledge.

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Another very impo-
rtant task -
- use opportunities
of modern computers
to realize the dream
of Bourbaki.

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# PART II

# Motivic Homotopy Theory

The part of the <u>FRONTIER</u> where the motivic homotopy theory lies and where most (but <u>NOT ALL</u>) activity is, is above LEVEL 4.

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# Motivic Homotopy Theory

How to apply the ideas of homotopy theory to algebraic geometry

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To connect alg-
ebraic geometry
and homotopy
one uses –

Category Theory

Category theory –

– an example of "new" mathematics.

Not a development of existing fields.

Appeared in late 1950-s

# Other "new" fields:

- Complexity theory
- Analysis (long ago)
- Set theory (19-th century)

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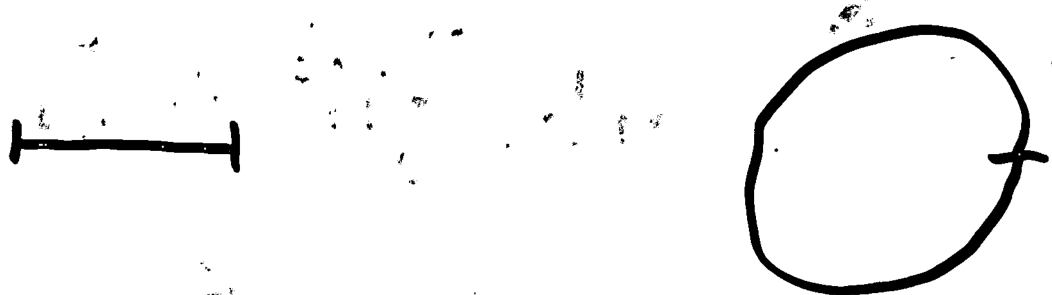
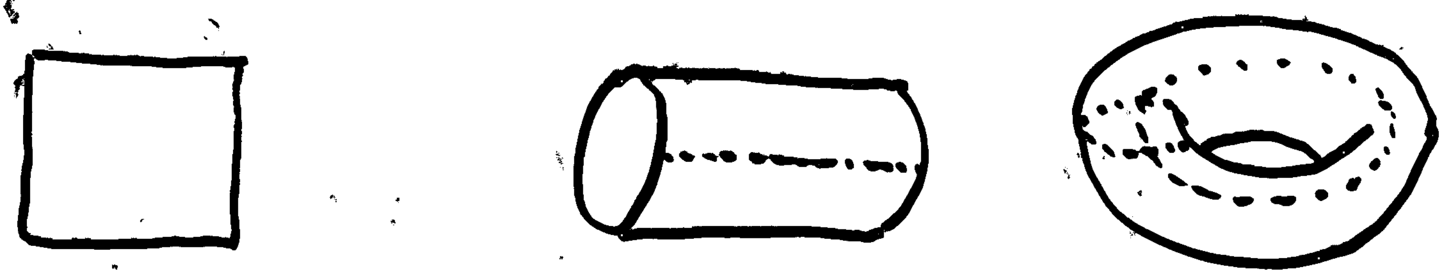
# Topology –

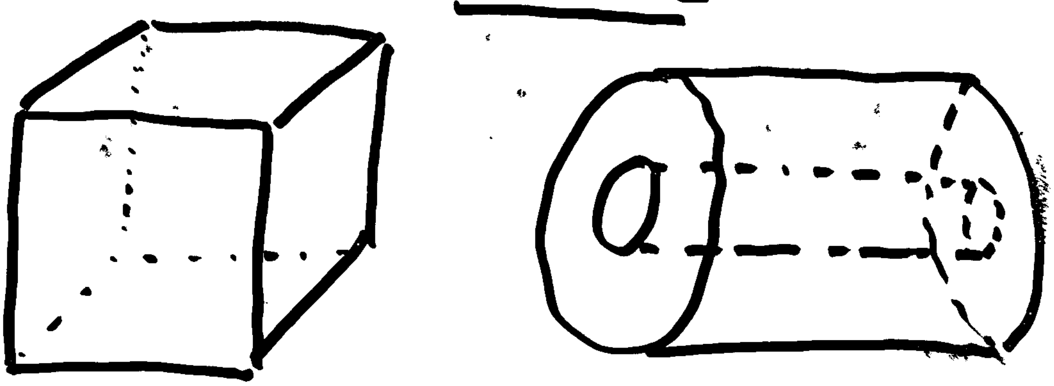
– studies spaces:

• 

dim = 1

• 

dim = 2

• 

?

dim = 3

# Two main invariants:

- Dimension

$$\dim(\vdash\!\dashv) \neq \dim(\square)$$

$$\dim(\bullet) = \dim(\bullet\ \bullet)$$

- Number of connected components

$$\pi_0(\vdash\!\dashv) = \pi_0(\square)$$

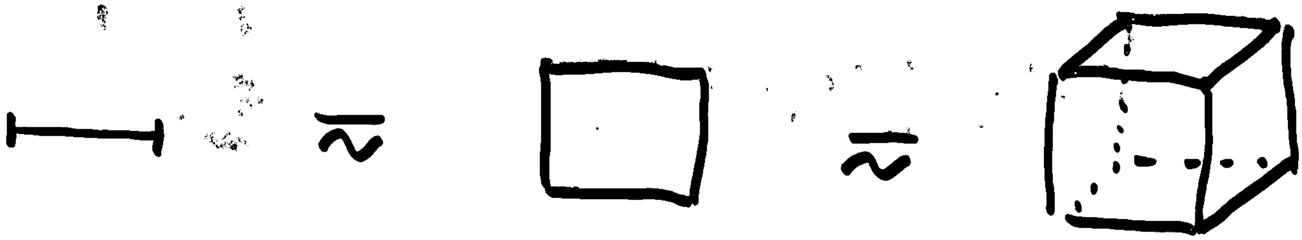
$$\pi_0(\bullet) \neq \pi_0(\bullet\ \bullet)$$

Homotopy theory -

- studies higher

generalizations of

$\pi_0, \pi_1, \pi_2, \ldots$



$\bullet \not\simeq \bullet \; \bullet \not\simeq \bullet \; \bullet \; \bullet$

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# A Category:

- A set of objects
- For any two objects $X, Y$ a set of morphisms $\mathrm{Mor}(X, Y)$
- For $X, Y, Z$ and $f : X \to Y$ $g : Y \to Z$ composition $g \circ f : X \to Z$

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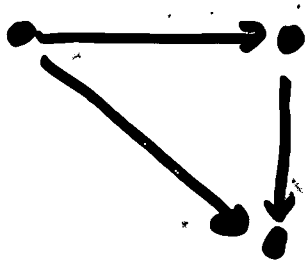
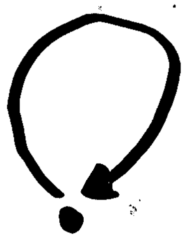
# Examples:





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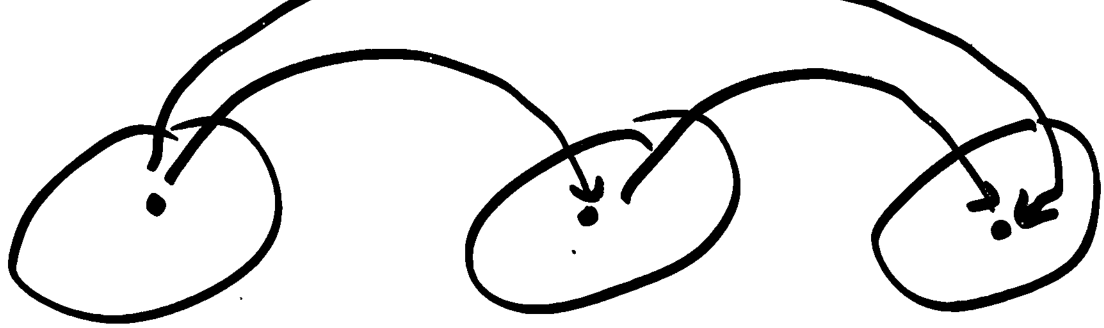
Top – the category of topological spaces.

Objects: spaces

Morphisms: continuous maps

Compositions:



In $\underline{\text{Top}}$ one has:

- $\mathrm{Hom}(\bullet, X)$ – points of $X$
  $\parallel$
  Mor

- $\mathrm{Hom}([0,1], X)$ – paths in $X$

- $$\begin{tikzcd}
\mathrm{Hom}([0,1], X) \arrow[d, shift right, "pt \xrightarrow{0} {[0,1]}"'] \arrow[d, shift left, "{[0,1]} \xleftarrow{1} pt"] \\
\mathrm{Hom}(pt, X)
\end{tikzcd}$$

To define $\pi_0(X)$

need to know only

$\mathrm{Mor}([0,1], X)$

$\mathrm{Mor}(pt, X)$

and two maps

$$\mathrm{Mor}([0,1], X) \downarrow\downarrow \mathrm{Mor}(pt, X)$$

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# Category of algebraic equations:

- Objects

$(x_1, \dots x_n ; f_1, \dots f_m)$

$f_i = f_i(x_1, \dots x_n) - a$

polynomial.

<u>Morphisms</u>:

$$(x_1, \dots x_n;\ f_1, \dots f_m)$$

$$\downarrow$$

$$(y_1, \dots y_{m'};\ g_1, \dots g_{m'})$$

$$\begin{cases} \varphi_1(x_1, \dots x_n) \\ \vdots \\ \varphi_{n'}(x_1, \dots x_n) \end{cases}$$ such that

$g_1(\varphi_1, \dots \varphi_{n'})$, ...

$g_{m'}(\varphi_1, \dots \varphi_{n'})$ can be expressed through $f_1, \dots f_m$.

# Examples:

- **Objects**:

$$\mathbb{A}^n = (x_1, \dots x_n; \emptyset)$$

$$\mathbb{A}^0 = pt$$

$$GL_n = (x_{ij}, t; \det(x_{ij}) \cdot t = 1)$$

- **Morphisms**:

$$\mathrm{Hom}(pt, (x_1, \dots x_n; f_1, \dots f_m))$$

$$= \left\{ \begin{array}{l} \text{the set of solutions} \\ \text{of the system } \begin{array}{l} f_1 = 0 \\ \vdots \\ f_m = 0 \end{array} \end{array} \right.$$

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Hom (pt, $Gl_n$) –

– the set of inver-
tible matrices $n \times n$.

Hom (pt, $\mathbb{A}^1$) = numbers

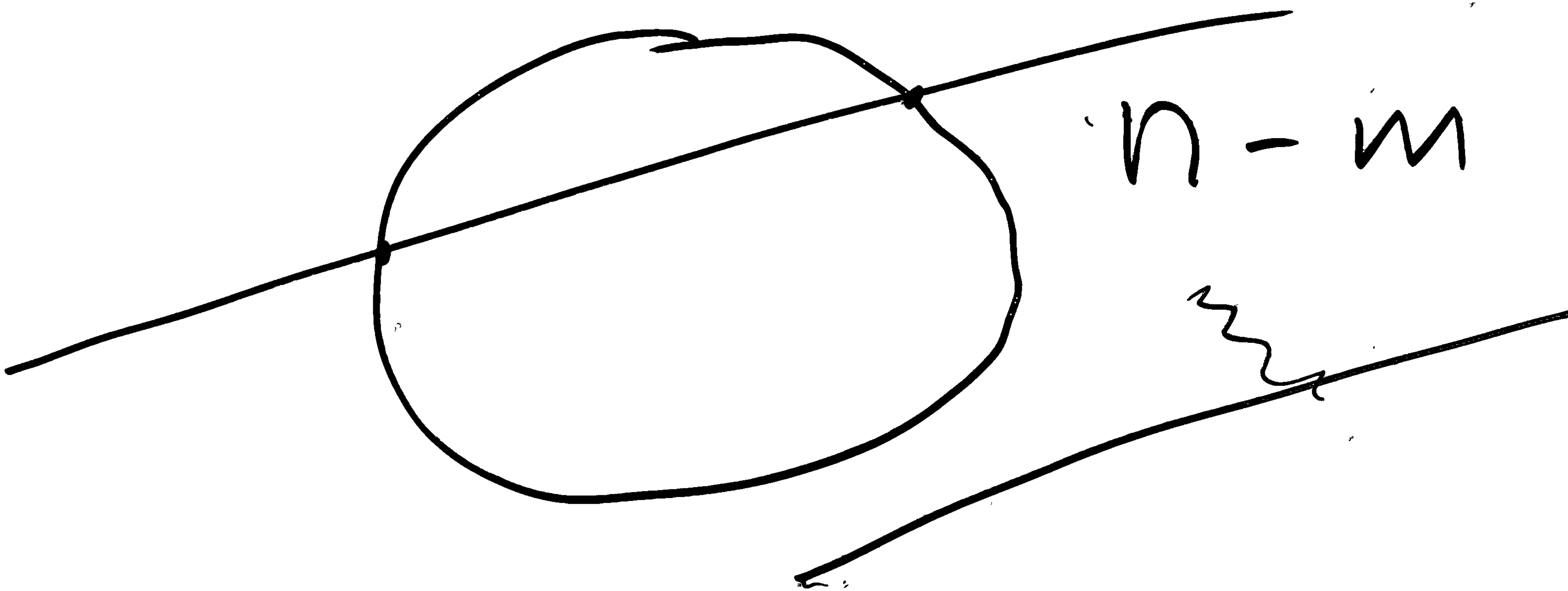
$$pt \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{A}^1$$

same as

$$pt \underset{1}{\overset{0}{\rightrightarrows}} [0,1]$$

$x_1, \ldots x_n \qquad f_1, \ldots f_m$

$$x_1^2 + x_2^2 = 1$$



$$(x_1, \ldots x_n ; \emptyset)$$

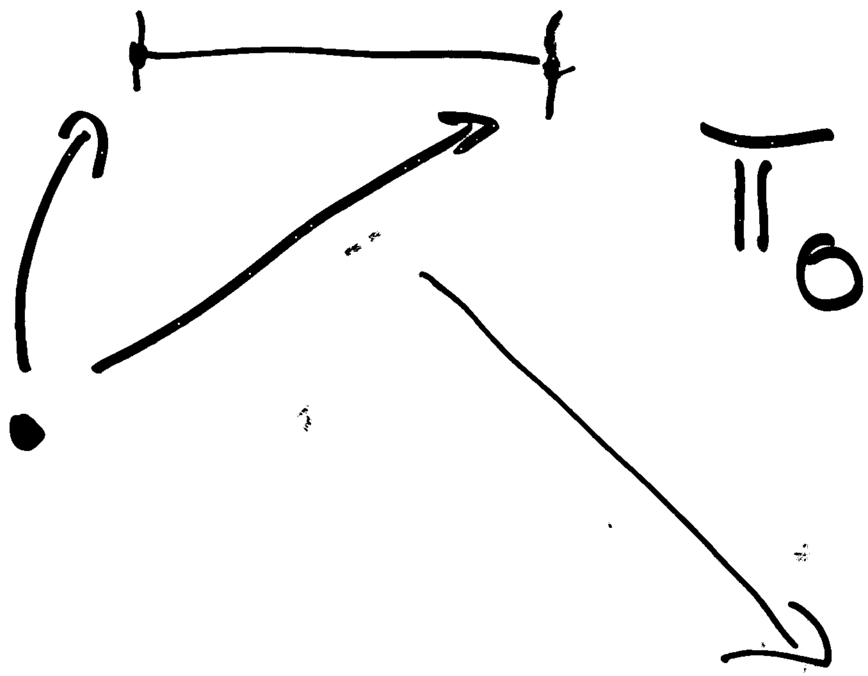
$$\shortparallel$$

$$\mathbb{A}^n$$

$$\mathbb{A}^0 = (\emptyset \_ \emptyset)$$

$$\shortparallel$$

$$pt$$




- A category
- an interval $I$
- $pt \underset{1}{\overset{0}{\rightrightarrows}} I$