PART I
About Mathematics
Four Levels of Mathematics:

1. Elementary
2. "Higher"
3. Modern
4. Synthetic
Elementary:

- Elem. Geometry
  \[ c^2 = a^2 + b^2 \]
  \[ \frac{c}{b} \]

- Elem. Algebra
  \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

School. Very Old
(> 1000 years)
Higher:

- Analysis
  \[ \int x \, dx = \frac{x^2}{2} \]

- Differential Equations:
  \[ \dot{f} = af + b \]

- Differential Geometry

- Probability
Modern:

- Modern Algebra
  e.g. Galois theory, Group theory

- Basic Topology

- Logic and Set Theory
  e.g. Gödel Theorem
Bourbaki - a group of French mathematicians

Tried to create full description of modern mathematics.
Synthetic:

- Representation Theory
- Algebraic Geometry
- Homotopy Theory
- Differential Topology
Usually people who do not specialize in Math know only levels 1 and 2. Sometimes heard about 3.
Main reason:

- Elementary - integrated into everyday life
- Higher - integrated into most sciences
- Modern - integrated into a few sciences
- Synthetic - very poorly integrated
One of the most important challenges today is to learn how to use mathematics in higher levels of knowledge.
Another very important task — use opportunities of modern computers to realize the dream of Bourbaki.
PART II

Motivic Homotopy Theory
The part of the frontier, where the motivic homotopy theory lies and where most (but not all) activity is, is above level 4.
Motivic Homotopy Theory

How to apply the ideas of homotopy theory to algebraic geometry
To connect algebraic geometry and homotopy, one uses - Category Theory
Category theory - an example of "new" mathematics. Not a development of existing fields. Appeared in late 1950s.
Other "new" fields:

- Complexity theory
- Analysis (long ago)
- Set theory (19th century)
Topology - studies spaces:

- $\dim = 1$
- $\dim = 2$
- $\dim = 3$
Two main invariants:

- Dimension
  \[ \dim(-) \neq \dim(\square) \]
  \[ \dim(\cdot) = \dim(\cdot\cdot) \]
- Number of connected components

\[ \Pi_0(-) = \Pi_0(\square) \]
\[ \Pi_0(\cdot) \neq \Pi_0(\cdot\cdot\cdot) \]
Homotopy theory - studies higher generalizations of \( \Pi_0, \Pi_1, \Pi_2, \ldots \)

\[
I \cong \square \cong \Box
\]
A Category:

- A set of objects
- For any two objects $X, Y$ a set of morphisms $\text{Mor}(X, Y)$
- For $X, Y, Z$ and $f: X \rightarrow Y$, $g: Y \rightarrow Z$, composition $g \circ f: X \rightarrow Z$
Examples:

\[\text{Diagram 1} \quad \text{Diagram 2} \quad \text{Diagram 3}\]
Top - the category of topological spaces.

Objects: spaces

Morphisms: continuous maps

Compositions:
In Top one has:

- \( \text{Hom}(\bullet, X) \) - points of \( X \)
- \( \text{Mor} \)
- \( \text{Hom}([0,1], X) \) - paths in \( X \)
- \( \text{Hom}([0,1], X) \)
- \( \text{Hom}(\text{pt}, X) \)
To define \( \pi_0(x) \) need to know only
\[ \text{Mor} \left( [0,1], x \right) \]
\[ \text{Mor} \left( \text{pt}, x \right) \]
and two maps
\[ \text{Mor} \left( [0,1], x \right) \]
\[ \downarrow \downarrow \]
\[ \text{Mor} \left( \text{pt}, x \right) \]
Category of algebraic equations:

- Objects

\((x_1, ..., x_n; f_1, ..., f_m)\)

\(f_i = f_i(x_1, ..., x_n) - \text{a polynomial.}\)
Morphisms:

\[(x_1, \ldots, x_n) \xrightarrow{f_1, \ldots, f_m} \]
\[\downarrow\]
\[(y_1, \ldots, y_m; g_1, \ldots, g_m)\]

\[\Psi_1 (x_1, \ldots, x_n) \]
\[\vdots\]
\[\Psi_n (x_1, \ldots, x_n) \quad \text{such that}\]
$g_1(\psi_1, \ldots, \psi_n), \ldots$

$g_m(\psi_1, \ldots, \psi_n)$ can be expressed through $f_1, \ldots, f_m$. 
Examples:

- **Objects:**
  
  \[ \mathbb{A}^n = (x_1, \ldots, x_n; \emptyset) \]
  
  \[ \mathbb{A}^0 = \text{pt} \]

  \[ \text{GL}_n = (x_{ij}, t_j; \det(x_{ij}) \cdot t = 1) \]

- **Morphisms:**

  \[ \text{Hom}(\text{pt}, (x_1, \ldots, x_n; f_1, \ldots, f_m)) = \{ \text{the set of solutions} \}
  
  \begin{cases} 
  \text{of the system } \ f_i = 0 \\
  \ f_m = 0
  \end{cases} \]
$\text{Hom (pt, } GL(n) \text{)} -$

- the set of invertible matrices non.

$\text{Hom (pt, } A' \text{)} = \text{ numbers}$

$\begin{array}{ccc}
\text{pt} & \overset{0}{\longrightarrow} & A' \\
\downarrow 1 & & \downarrow \text{same as} \\
\text{pt} & \overset{0}{\longrightarrow} & (0,1)
\end{array}$
\[ x_1, \ldots, x_n \]

\[ f_1, \ldots, f_m \]

\[ x_1^2 + x_2^2 = 1 \]

\[ n - m \]
\[ \text{Pt} = \mathcal{O}_{X_1} \quad \text{and} \quad \text{Pt} = \mathcal{O}_{X_2} \]
A category

\[ \text{pt} \xrightarrow{1} \text{an interval} \]

\[ H \]