Theorem Let $(H, f^*): Sch/S \longrightarrow Triang$ be a 2-functor with the following properties:

0. $H(\emptyset) = 0$
1. For any f, f^* has a right adjoint f_*
2. For a smooth f, f^* has a left adjoint $f_\#$ and the base change $(f^*, f_\#)$ is an isomorphism.
3. For a complimentary pair ~~of~~ (j, i) of open and closed embeddings the sequence $j_\# j^* \longrightarrow Id \longrightarrow i_* i^*$ extends to a distinguished triangle.
4. for any X the adjunction $\cancel{\varphi}\ Id \rightarrow p_{X,*} p_X^*$ where $p_X : \mathbb{A}^1_X \rightarrow X$ is an iso.
5. For any X the functor $\Sigma_T : H(X) \rightarrow H(X)$

   $\Sigma_T = \cancel{k_*}\ p_{X,\#}\, i_* \qquad i : X \xrightarrow{0} \mathbb{A}^1_X$

   is an equivalence.

Then (H, f^*) extends to a unique up to a strict iso. cross functor $(H, f^*, f_*, f^!, f_!)$ and one has:

- smooth morphisms are transversal for $(f^*, f^!)$
- proper morphisms are transversal for $(f_*, f_!)$

Intermediate Th. 1 Let $(M, f^*, f_*, f^!, f_!)$: $Sch/S \to Triang$ be ~~an~~ a cross functor such that the following conditions hold:

1. open emb. are upper transversal and closed emb. are lower transvercal
2. $\mathbb{A}^1_X \to X$ are ~~tran~~ upper transversal for all X.
3. if $j: U \to X$, $i: Z \to X$ are an open and a closed emb. such that $X = j(U) \amalg i(Z)$ then (i, j) are M-complimentary (Def. 1)
4. T-stability holds
5. $\mathbb{A}^1$-homotopy invariance holds.

Then proper morphisms are lower M-transversal.

Plan of the proof of <u>ITh. 1</u>:

- gen. lemma 1
- gen lemma 2
- $\Sigma_{p:\mathbb{A}^n_X\to X}\, p^* \cong p^*\,\Sigma^n_T$
- for an open $j: U \to X$, $j^*\Sigma_p \cong \Sigma_{j\circ p}\, j^*$

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & X \arrow[d, "p"] \\ & S \end{tikzcd}$$

prop: for $U \hookrightarrow \mathbb{A}^n_X$, $p_U: U \to X$, one has $p_U^! \approx p_U^*\,\Sigma^n$.

- let $a \to b$ be a 2-morphism. Assume that a has a right adjoint a_r and b has a left adjoint b_e and that

$$\begin{array}{l} a a_r \to b a_r \\ a b_e \to b b_e \end{array} \quad \text{are isomorphisms. Then}$$

$a \to b$ is an isomorphism.

- <u>Def</u>: $\mathbb{A}^1$-h. eq.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p_X"'] & Y \arrow[d, "p_Y"] \\ & S \end{tikzcd}$$

<u>L.</u> if f is an $\mathbb{A}^1$-h. eq. then

$$\begin{array}{l} p_{X,!}\, p_X^! \to p_{Y,!}\, p_Y^! \\ p_{Y,*}\, p_Y^* \to p_{X,*}\, p_X^* \end{array} \quad \text{are isos.}$$

L. same for a <u>local</u> $\mathbb{A}^1$-h. eq.

- Exact triangles: upper and lower.

- <u>Th.</u> under the assumptions of <u>I Th. 1</u> the morphisms $\mathbb{P}^n_X \to X$ are lower transversal

- <u>Example</u>: projective cones.

- End of th proof of I Th. 1 using Chow lemma and blow-up sequence.

$$\begin{tikzcd}
UV \arrow[d, "p'"'] \arrow[dr, "r"] & V \arrow[l, "j'"'] \arrow[d, "p"] & Z \arrow[l, "i'"'] \\
U \arrow[r, "j"] & X & Z \arrow[l, "i"']
\end{tikzcd}$$

$$r_! r^! \longrightarrow j_! j^! \oplus p_! p^! \longrightarrow Id \longrightarrow r_! r^! [1]$$

$$\begin{tikzcd}
r_! r^! \arrow[r] \arrow[d] & p_! p^! \arrow[r] \arrow[d] & i_* i^* \arrow[r] \arrow[d, "\wr", "\exists !"'] & r_! r^! [1] \arrow[d] \\
j_! j^! \arrow[r] & Id \arrow[r] & i_* i^* \arrow[r] & j_! j^! [1]
\end{tikzcd}$$

$$Id \longrightarrow i_* i^* = (pi')_* (pi')^* \xrightarrow{\sim}$$

$$\xrightarrow{\sim} p_* i'_* i'^* p^* \longrightarrow p_* j'_! j'^! p^*$$

$$\longrightarrow$$