Nov. 21, 98

$j : \mathbb{P}^n - pt \to \mathbb{P}^n$

$i \quad pt \to \mathbb{P}^n$

$$H_\ell(\bar p_n)\,\Omega_\Delta\,H_\ell(j)H(p_n) \longrightarrow H_\ell(\bar p_n)\,\Omega_\Delta\,H(\bar p_n) \longrightarrow H_\ell(\bar p_n)\,\Omega_\Delta\,H_r(i^{\#}) \quad (*)$$

$i_n : \mathbb{P}^{n-1} \to \mathbb{P}^n$

$$H_\ell(\bar p_{n-1})\,\Omega_{r_i}\,H(\bar p_{n-1}) \xrightarrow{f_1} H_\ell(\bar p_n)\,\Omega_\Delta\,H_\ell(j)H(p_n)$$

<u>Lemma 1</u> f_1 is an iso.

$$\begin{tikzcd}
\mathbb{P}^{n-1}\times\mathbb{P}^n \arrow[r, "pr_2"] \arrow[d] & \mathbb{P}^n \arrow[d, "\bar p_n"] \\
\mathbb{P}^{n-1} \arrow[u, bend left, "r_i"] \arrow[r, "\bar p_{n-1}"'] & S
\end{tikzcd}$$

$$(H_\ell(\bar p_{n-1})\,\Omega_{r_i}\,H(\bar p_{n-1}))_r = H_r(\bar p_{n-1})\,\Sigma_{r_i}\,H(\bar p_{n-1})$$

$$H_r(\bar p_{n-1})\,\Sigma_{r_i}\,H(\bar p_{n-1}) \longleftarrow H_\ell(\bar p_n)H_r(pr_2)H_r(r_{i_n})H(\bar p_{n-1})$$

$$\| $$

$$\xleftarrow{f_2} H_\ell(\bar p_n)H_r(i_n)H(\bar p_{n-1})$$

<u>Lemma 2</u> f_2 is an isomorphism (by induction)

Lemma 3 $\quad He(\bar{p}_n)\, \Omega_\Delta\, H_r(i_n) \xrightarrow{\sim} Id.$

Thus we get a distinguished triangle:

$$Id \longrightarrow (He(\bar{p}_n)\,\Omega_\Delta\, H(\bar{p}_n))_r \longrightarrow He(\bar{p}_n) H_r(i_n) H(p_{n-1})$$

But we have

$$He(\bar{p}_n) He(j') H(p_n') \longrightarrow He(\bar{p}_n) H(\bar{p}_n) \longrightarrow He(\bar{p}_n) H_r(i_n) H(p_{n-1})$$

where $\quad j' : \mathbb{P}^n - \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$

$\qquad\quad p_n' : \mathbb{P}^n - \mathbb{P}^{n-1} \longrightarrow S.$

Thus there is an isomorphism

$$(He(\bar{p}_n)\,\Omega_\Delta\, H(\bar{p}_n))_r \cong He(\bar{p}_n) H(\bar{p}_n)$$

$$\|$$

$$H_r(\bar{p}_n)\, \Sigma_\Delta\, H(\bar{p}_n)$$

It remains to show that the diagram

$$\begin{tikzcd}
H_\ell(\bar{p}_n)H(\bar{p}_n) \arrow[r, "\oplus"] \arrow[dr, "\beta * \mathrm{Id}_{H(\bar{p}_n)}"'] & (H_\ell(\bar{p}_n)\Omega_\Delta H(\bar{p}_n))_r \arrow[d, "\varphi"] \\
& H_r(\bar{p}_n)\Sigma_\Delta H(\bar{p}_n)
\end{tikzcd}$$

commutes.

~~Consider the morphism~~ left half of the distinguished
The right adjoint to the ~~distinguished~~ triangle
(*) is of the form:

$$\cancel{to}\ H_r(\bar{p}_n)\Sigma_\Delta H(\bar{p}_n) \to H_r(p_n)H(j)\Sigma_\Delta H(\bar{p}_n)$$

Thus we have a commutative diagram:

$$\begin{tikzcd}
H_\ell(\bar{p}_n)H(\bar{p}_n) \arrow[r] \arrow[d, "\varphi \circ \oplus"'] & H_\ell(\bar{p}_n)H_r(i_n)H(\bar{p}_{n-1}) \arrow[d, "f_2"] \\
H_r(\bar{p}_n)\Sigma_\Delta H(\bar{p}_n) \arrow[dr] & H_r(\bar{p}_{n-1})\Sigma_{r_i} H(\bar{p}_{n-1}) \\
& H_r(p_n)H(j)\Sigma_\Delta H(\bar{p}_n) \arrow[u, "(f_1)_r"']
\end{tikzcd}$$

We have to check that the same diagram commutes if $\varphi \circ \oplus$ is replaced by $\beta * Id_{H(\bar{p}_n)}$.

The composition of $\beta * Id_{H(p_n)}$ with the arrow

$$\begin{tikzcd} H_r(\bar{p}_n)\Sigma_\Delta H(\bar{p}_n) \arrow[r] & H_r(\bar{p}_n)H(j)\Sigma_\Delta H(\bar{p}_n) \arrow[d] \\ & H_r(\bar{p}_{n-1})\Sigma_{m_i} H(\bar{p}_{n-1}) \end{tikzcd}$$

is given by the diagram:

$$\begin{tikzcd} S \arrow[r, "r"] & S \arrow[r, "r"] & S \arrow[r, "r"] & S \\ \mathbb{P}^{n-1} \arrow[u, "m"] \arrow[r, dash, "r"] \arrow[d, "r"] & \mathbb{P}^n - pt \arrow[u, "m"] \arrow[r, "r"] \arrow[d, "r"] & \mathbb{P}^n \arrow[u, "m"] \arrow[r, "r"] \arrow[d, "r"] & \mathbb{P}^n \arrow[u, "m"] \arrow[d, "r"] \\ \mathbb{P}^{n-1} \times \mathbb{P}^n \arrow[r, "r"] \arrow[d, "e"] & \mathbb{P}^n_{pt} \times \mathbb{P}^n \arrow[r, "r"] \arrow[d, "e"] & \mathbb{P}^n \arrow[r, "r"] \arrow[d, "e"] & \mathbb{P}^n \arrow[d, dash, "e"] \\ \mathbb{P}^{n-1} \arrow[r, "r"] & \mathbb{P}^n - pt \arrow[r, "r"] & \mathbb{P}^n \arrow[r, "r"] & S \end{tikzcd}$$

The composition of the upper horizontal arrow and f_2 is given by

$$\begin{tikzcd}
S' \arrow[r, "r"] & S \\
\mathbb{P}^{n-1} \arrow[r, "r"] \arrow[u, "m"] \arrow[d, "r"'] & \mathbb{P}^n \arrow[u, "m"'] \arrow[d, "r"] \\
\mathbb{P}^{n-1} \times \mathbb{P}^n \arrow[r, "r"] \arrow[d, "e"'] & \mathbb{P}^n \arrow[d, "e"] \\
\mathbb{P}^{n-1} \arrow[r, "r"'] & S
\end{tikzcd}$$

Commutativity is now obvious.

-plan 2-

The composition is a split-mono thus $H_e(\bar{p}_n) \to H_r(\bar{p}_n)\Sigma_\Delta$ is split-mono.

$$\begin{tikzcd}
H_e(p_n)\Omega_\Delta H(p_n)H_r(\bar{p}_n) \arrow[r] \arrow[d] & H_r(\bar{p}_n)H(\bar{p}_n)H_r(\bar{p}_n) \arrow[d] \\
H_e(p_n)\Omega_\Delta \arrow[r] & H_r(\bar{p}_n)
\end{tikzcd}$$

The composition is split-epi thus $H_e(\bar{p}_n)\Omega_\Delta \to H_r(\bar{p}_n)$ is split-epi. Thus $H_e(\bar{p}_n) \to H_r(\bar{p}_n)\Sigma_\Delta$ is an iso.

Theorem $H_e(\bar{p}_n) \longrightarrow H_r(\bar{p}_n)\Sigma_\Delta$ is an isomorphism.

Plan of the proof: Induction by n. The ~~case~~ case $n=0$ is obvious.

Lemma Assuming that the statement holds for $\bar{p}_{n-1}$ the P_r-exchange morphism for a pull-back square:

$$\begin{tikzcd}
\mathbb{P}^{n-1}_X \arrow[r, "\bar{p}_{n-1,X}"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
\mathbb{P}^{n-1}_S \arrow[r, "\bar{p}_{n-1,S}"] & S
\end{tikzcd}$$

such that f is smooth is an isomorphism ($H_e(f)H_r(\bar{p}_{n-1,X}) \xrightarrow{\sim} H_r(\bar{p}_{n-1,S})H_e(f')$)

Lemma Σ_Δ is an equivalence

Lemma (pp. 1-) Under the inductive assumption

1. $H_e(\bar{p}_n)H(\bar{p}_n) \longrightarrow H_r(\bar{p}_n)\Sigma_\Delta H(\bar{p}_n)$ ~~are~~
2. $H_e(\bar{p}_n)\Omega_\Delta H(\bar{p}_n) \longrightarrow H_r(\bar{p}_n)H(\bar{p}_n)$

~~are~~ isomorphisms

Lemma 1. and 2. are adjoint to each other.

Inductive step:

$$\begin{tikzcd}
H_e(\bar{p}_n) \arrow[r] \arrow[d] & H_r(\bar{p}_n)\Sigma_\Delta \arrow[d] \\
H_e(\bar{p}_n)H(\bar{p}_n)H_e(\bar{p}_n) \arrow[r] & H_r(\bar{p}_n)\Sigma_\Delta H(\bar{p}_n)H_e(\bar{p}_n)
\end{tikzcd}$$