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Defn: A 2-pretheory is a 2-functor

$H: (\text{Sch}/S)^{\text{op}} \rightarrow \mathcal{C}$ at such that :

a. 2-additivity For X, Y over S

the functor $H(i_X) \times H(i_Y): H(X \amalg Y) \rightarrow H(X) \times H(Y)$ is an equivalence.

b. For a smooth morphism $p: X \rightarrow S$

the functor $H(p)$ has a left adjoint $H_e(p)$ and for a pull-back square

$$\begin{array}{ccc} X' & \xrightarrow{f_X} & X \\ p'^* & & \downarrow p \\ S' & \xrightarrow{f_S} & S \end{array}$$

the canonical morphism

$$H(f_S) H_e(p) \longrightarrow H_e(p') H(f_X)$$

is an isomorphism.

c. For a closed embedding $i: Z \rightarrow S$

the functor $H(i)$ has a right adjoint, $H_r(i)$.

Def2: A \mathcal{D} -pretheory of type I is a \mathcal{D} -pretheory such that:

- a. the cat. $H(X)$ are additive ~~and~~
- b. i. for any point ∞ of P^1 over ~~X~~ the sequence

$$H_e(p_\infty) H(p) \rightarrowtail \text{red} \rightarrowtail H_p$$

$$H_e(p_\infty) H(p_\infty) \xrightarrow{c_\infty} H_e(p) H(p) \xrightarrow{e_\infty} H_e(p) H_r(\infty)$$

where: $U = P^1 - \infty(X)$

$$p: P^1 \rightarrow X$$

$$p_\infty: P^1 U \rightarrow X$$

is split-exact.

ii. the functors ~~to~~ $H_e(p) H_r(\infty)$ are equivalences of $H(X)$.

3. the adjunctions ~~to~~ $H_e(p_\infty) H(p_\infty) \rightarrow \text{Id}$ are isomorphisms.

Lemma 3 If all the categories $H(X)$ are additive then the functors $H(f)$ (and thus $H_e(p)$ and $H_r(\alpha)$) are additive.

Proof: Follows from axioms a & b Def. 1 applied to $S' \amalg S' \rightarrow S \amalg S$
 $\downarrow \quad \downarrow$
 $S' \xrightarrow{f} S \quad \blacksquare$

Normal Assumption

Denote the functor $H_e(p)H_r(\alpha)$ for $p: \mathbb{P}_X^I \rightarrow X$ and α a section of p by Σ_α . Assume that Def. 2(a, b₁, b₂) hold.

Define $\varphi_{xy}: \Sigma_\alpha \rightarrow \Sigma_y$ as the composition:

Denote by $\pi: H_e(p)H(p) \rightarrow H_e(p_\mu)H(p_\mu)$ the composition of the adjunction

$H_e(p)H(p) \rightarrow \text{Id}$ with the inverse to

the adjunction $H_e(p_\mu)H(p_\mu) \rightarrow \text{Id}$ which

exists by 2.b2. The diagram

$$\begin{array}{ccc} \text{He}(pu)H(pu) & \xrightarrow{\iota_x} & \text{He}(p)H(p) \\ \downarrow & & \downarrow \\ \text{Id} & \longrightarrow & \text{Id} \end{array}$$

commutes and thus π is a projection for the i_x . Since By 2b1 there is a unique section $\lambda_x : \text{He}(p)H_r(x) \rightarrow \text{He}(p)H(p)$ such that the ~~diagram~~ sequence

$$\text{He}(p)H_r(x) \xrightarrow{\lambda_x} \text{He}(p)H(p) \longrightarrow \text{He}(pu)H(pu)$$

is exact.

Lemma 4: For any x, y the composition

~~is~~ $\varphi_{xy} : \Sigma_x \xrightarrow{\lambda_x} \text{He}(p)H(p) \xrightarrow{G_y} \Sigma_y$ is an iso, $\varphi_{xx} = \text{Id}$, $\varphi_{xy} = \varphi_{yx}^{-1}$ and $\varphi_{xz} = \varphi_{x\bar{x}} \varphi_{yz} \varphi_{xy}$.

By Lemma 4 all the functors Σ_x are canonically isomorphic and from this point we denote Σ_x by Σ and \mathcal{L}_x by \mathcal{L} . Note that the morphisms $\sigma_x : \text{He}(p)\text{H}(p) \rightarrow \Sigma$ do depend on x .

Lemma 5 There are canonical isomorphisms

$$\text{H}(f)\Sigma \rightarrow \Sigma \text{H}(f)$$

(RHS)

$$\Sigma \text{He}(p) \rightarrow \text{He}(p)\Sigma$$

Proof: Use 1b. for the square

Lemma Assume that Σ is an equivalence and let \mathcal{R} be the right adjoint inverse (see ?). Then there are ~~and~~ unique isomorphisms

$$\text{H}(f)\Sigma \rightarrow \mathcal{R}\text{H}(f) \quad \text{and}$$

$$H_e(p) \mathcal{S} \rightarrow \mathcal{S} H_e(p)$$

such that

Define a morphism

$$\boxed{\beta : H_e(p) H_e(p) \mathcal{S}_\Delta \rightarrow \text{Id}}$$

as the composition :

$$H(p) H_e(p) \mathcal{S}_\Delta \longrightarrow H_e(\text{pr}_2) H(\text{pr}_1) \mathcal{S}_\Delta$$

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\text{pr}_2} & \mathbb{P}^1 \\ \text{pr}_1 \downarrow & & \downarrow P \\ \mathbb{P}^1 & \xrightarrow{P} & X \end{array} \quad \begin{array}{c} H_e(\text{pr}_2) H_r(\Delta) \mathcal{S}_\Delta \\ \Sigma_\Delta \mathcal{S}_\Delta \\ \text{Id.} \end{array}$$

$$\boxed{\beta : \text{Id} \rightarrow H_e(p) \mathcal{S}_\Delta H(p)}$$

as the composition

$$\text{Id} \rightarrow \mathcal{S}_0 \Sigma_0 \xrightarrow{\delta \lambda} \mathcal{S} H_e(p) H(p) \longrightarrow H_e(p) \mathcal{S} H(p)$$

Theorem 1

~~\mathcal{E}_P and $H_e(p) \mathcal{S}_0$ define an adjunction making $H(p)$ into a right adjoint to~~

~~Proof: We have to verify with set $F = H(p)$, $G = H_e(p) \mathcal{S}_0$. We have to verify (by \hookrightarrow) that~~

Lemma 6 The composition

$$H(p) \xrightarrow{\text{Id}_{\mathcal{S}_0}} H(p) H_e(p) \mathcal{S}_0 H(p) \xrightarrow{\beta \circ \text{Id}} H(p)$$

is an isomorphism.

Proof:

Consider the diagram:

$$\begin{array}{c}
 H(p) \xrightarrow{\cong} H(p) \mathcal{S}\Sigma \rightarrow H(p) \mathcal{S}H(p)H(p)H(p) \\
 \downarrow \\
 H(p)He(p)\mathcal{S}H(p) \longrightarrow H(p)He(p)H(p)\mathcal{S} \\
 \downarrow \qquad \qquad \qquad | \\
 He(pr_1)H(pr_1)\mathcal{S}H(p) = He(pr_2)H(pr_2)H(p)\mathcal{S} \\
 \downarrow \qquad \qquad \qquad | \circ_\Delta \\
 He(pr_2)H_r(\Delta)\mathcal{S}H(p) \xrightarrow{\cong} H\Sigma H(p)\mathcal{S} \\
 \Downarrow \qquad \qquad \qquad \Downarrow \\
 H(p)
 \end{array}$$

There is (by 2b1) an isomorphism

$H(p)\mathcal{S}\Sigma \rightarrow H(p)\Sigma\mathcal{S}$ such that

$$\begin{array}{ccc}
 H(p)\mathcal{S}\Sigma & \longrightarrow & H(p)\Sigma\mathcal{S} \\
 \text{Id} \circ \alpha \downarrow & & \downarrow \text{Id} \circ \beta \circ \text{Id} \\
 H(p)\mathcal{S}He(p)H(p) & = & H(p)He(p)H(p)\mathcal{S}
 \end{array}$$

commutes. Thus it is sufficient to show that the composition

$$H(p)\Sigma \rightarrow H(p)He(p)H(p) \rightarrow He(pr_2)H(pr_2)H(p) \xrightarrow{\circ_\Delta} \Sigma H(p)$$

is an isomorphism. One verifies that it is in fact the canonical isomorphism of Lemma 5. Lemma 6 is proven.

Lemma 7

$$\begin{array}{ccc} \text{Me}(p) \otimes_{\mathbb{D}} \text{M}(p) & \xrightarrow{3 \circ \text{Id}} & \text{Me}(p) \otimes_{\mathbb{A}} \text{M}(p) \text{Me}(p) \otimes_{\mathbb{A}} \text{M}(p) \\ & & \downarrow \text{Id} \circ \beta \circ \text{Id} \\ & & \text{Me}(p) \otimes \text{M}(p) \end{array}$$

is an isomorphism.

Proof: Consider:

$$\begin{array}{c} \text{M}(p) \otimes \text{M}(p) \cong \otimes \sum \text{M}_i(p) \otimes \text{M}_{\bar{i}}(p) \rightarrow \otimes \text{Me}(p) \text{M}(p) \text{Me}(p) \otimes \text{M}(p) \\ \swarrow \qquad \downarrow \qquad \searrow \text{Id} \\ \text{Me}(p) \otimes \text{M}(p) \text{Me}(p) \otimes \text{M}(p) = \otimes \square \\ \downarrow \\ \otimes \cancel{\text{Me}(p) \text{M}(p) \text{M}(p) \text{M}(p)} - \text{Me}(p) \otimes \text{Me}(p) \text{M}(p) \text{M}(p) = \otimes \square \\ \downarrow \\ \text{Me}(p) \otimes \sum \otimes \text{M}(p) = \otimes \square \\ \downarrow \\ \text{Me}(p) \otimes \text{M}(p) \end{array}$$