Def: A noncommutative CY var. of dim = d is a compact, homologically smooth, triangulated Aoo-category with Serre functor \( S \simeq [d] \).

\( A \) Aoo-algebra, \( \mathcal{C} = \text{A-mod} \), \( m_k : A^{\otimes k} \to A \) \( k \geq 1 \)

- compact: \( \text{rank } \text{Hom}(A, m_k) < \infty \)
- hom. smooth: \( A \) is a perfect \( A-A \)-bimodule.

Lemma: if \( A \) is hom. smooth & compact then \( \text{rank } \text{Hom}(A, A) < \infty \).

Conj.: a) If X NCCY then the moduli space \( M_X \) is smooth (focus on this)
  b) \( \exists \) embedding of \( M_X \) into an affine space
  c) \( \exists \) global moduli stack \( M_X^{\text{glob}} \), which is an orbifold.

\( \triangleright \) Toen's work

- \( X \) CY, \( \text{dim } X = d \), \( X/\mathcal{C} \), \( P \) a generator of \( \text{Perf}(X) \)
  \( A = \Omega^{\otimes \infty}(X, \text{End } P) \) dg-algebra, \( \text{vol} = \text{holm vol} \)
  \( \alpha \mapsto \int_X \text{tr}(\alpha)^{\langle \text{vol} \rangle} \) gives a nondegenerate pairing on \( \text{Hom}^0(A) \)

- Let \( A \) = compact (weakly unital) Aoo-algebra

  \[ \text{CC}_*(A) := (\mathbb{C}, (A, A)[[u^{-1}]], b + uB)^{\text{deg } u = 2} \]
  Hochschild chains
  Hochschild diff.

  \[ \text{Let } [\gamma] \in H^d(\text{CC}_*(A)) \]
  \( (A, m) \) is a subcomplex of the Hochschild complex;
  \[ (A, m) \longrightarrow k[-d] \]

  Def: \( \gamma \) is homologically nondegenerate if \( \text{Tr}(\gamma)(ab) \) is a nondegenerate pairing.
Prop: \[ \text{such } \varphi \text{ give rise to an isom class of scalar products on a minimal model of } A. \]

Def: \[ \text{ } \varphi \text{ gives rise to a } CY \text{ structure of degree } d \text{ on } A. \]

Conj 2: \[ \text{the 2 definition of NC } CY \text{ structures } \begin{cases} [-S \cong [d] \\ -Tr_{(n)} \text{ nondeg pairing} \end{cases} \text{ are equivalent for } \text{smooth } \text{ and } A. \]

Also:
\[ A^! = \text{Hom}(A, A \otimes A) \text{ inverse duality bimodule } \]
\[ (F \mapsto F \otimes k_x^{-1}[-\dim X]) \]

Prop: \[ \text{if } A \text{ is smooth, then } A^! \in \text{Perf}(A-\text{mod}) \]

Conj 2': \[ \text{The existence of a } CY \text{ structure of degree } d \text{ is equivalent to } A^! \cong A[d]. \]

(proved by V. Ginzburg for case } A = \text{assoc. algebra}. \]

Smoothness of moduli space: involves noncomm. Hodge-to-De Rham degeneration

(char k = 0)

Def: \[ CC_{*}(A) = (CC_{*}(A)[[w]], b + wB) \text{ negative cycle complex } \]

Def: \[ \text{Say } A \text{ satisfies degeneration property if } \]
\[ H^*(CC_{*}(A)/[w]) \text{ is flat } k[[w]]/[[w]] \forall w \geq 1 \]
\[ \Rightarrow H^*(CC_{*}(A)) \text{ is a flat } k[[w]] \text{-module } \]
\[ \text{i.e. a vector bundle over } A^d_{\text{trivial } [-2]} \]
\[ \text{Generic fiber } = HP (A) \text{ periodic cyclic homology } \]
\[ 2/2 \text{-graded } \]

Fiber at } w = 0 \cong HH_0(A) \]

Conj 3 (degeneration of Hodge-to-De Rham) \[ \text{IF } A \text{ finite dim. smooth, then degeneration property holds.} \]

- Recently announced by Kaledin in case } A \text{ is entirely in degrees } \geq 0.
Conway of Conj.-3:

Let $A$ be CY algebra. Then assuming Conj.-3, $M_A$ is smooth.

NB: $M_A = \text{we can define}$

1) $A_{oo}$-structure (on a fixed underlying vector space, define $m_k$)

2) Bilinear pairing $(\cdot, \cdot): A_{oo} \otimes \mathbb{k}[-d] \rightarrow \mathbb{k}$ st. $(m_k(a_0, \ldots, a_n), a_n)$ is cyclically invariant

In case 1), $TM = \text{Hochchild cohomology}$ is related in CY case.

In case 2), $TM = \text{cyclic cohomology.}$

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Costello, Kontsevich-Soibelman, Katzarkov-pankov

\[ M_{\overline{g},n} \rightarrow M_{\bar{g},n} \]

\[ \text{Cohomological FT} \]

\[ \text{Top FT} \]