1. Configuration Spaces, Simultaneous Resolutions, and Braid Group actions on categories.

Joint work with P. Seidel, M. Khovanov

Dehn Twists:

$$(X_\omega) = \left\{ \sum_{i=1}^{n} x_i^2 = t^2 \leq (C^{n+2}, \omega_{\text{standard}}) \right\}$$

Lagrangian $S^n$ $x: \mathbb{R} \to \mathbb{R}$

$X_e$ $X_0$
2. Symplectically this is a fibre bundle

Symplectic connection: horizontal subspace $T_{X_t}^\perp \omega$.

All fibres $(X_t, \omega) \cong (T^*S^n, \omega_{\text{standard}})$

Monodromy $\in \text{Aut}(X_t, \omega)$

Dehn twist (Arnold, Seidel)

$\text{Picard-Lefschetz}$
In \( \text{Fuk}(X_t) \),

\[
\begin{align*}
\text{HF}^*(L,A) @ L & \xrightarrow{ev} A \\
& \xrightarrow{T_* A} [W]
\end{align*}
\]

\[ev = \text{id} \in \text{HF}^*(\text{HF}^*(L,A) @ L, A) \]

\[= \text{HF}^*(L,A)^* \otimes \text{HF}^*(L,A)\]

Tighten loop about \( t=0 \)

\( \Rightarrow \) different \( T_* \) due to curvature of symplectic connection \( \in \text{Ham}(X_t, \omega) \)

\[\lim_{t \to 0}
\]
Simultaneous Resolution

Family $X_t$ of $X_t$: simultaneous resolution is a family $\hat{X}_{\nu} \to X$ s.t. $\hat{X}_{\nu}$ is a resol of rings $V_{\nu}$, and $X_t$ is smooth.
5. $\Phi$ for our family and $n \neq 2$.

$\Phi$ for $n = 1$

$\Phi$ for $n = 2$ since $(T_L)_x \neq id$

But does exist $(n = 2)$ after base change $t \mapsto t^2$
6. \[ \{ x_1^2 + x_2^2 + x_3^2 = t^2 \} \subseteq \mathbb{C}^4 \]

Total space 3-fold \( \text{ODP} \).

\[ x = x_1 + i x_2 \quad y = x_1 - i x_2 \]

\[ \{xy = zw\} \subseteq \mathbb{C}^4 \]

Use ratios \( \frac{x}{z} = \frac{w}{y} \in \mathbb{P}^1 \) to resolve.

\( \hat{X} = \{ xy = zw \} \subseteq \mathbb{C}^4 \times \mathbb{P}^1 \)

Atiyah

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Not full blow up of singular point 0!

This would give \( \mathbb{P}^1 \times \mathbb{P}^1 \) except divisor.

Central fibre has 2 ruled. cpts.

Blow down one ruling \( \Rightarrow \hat{X} \)

other ruling \( \Rightarrow \) flop of \( \hat{X} \) using \( \frac{w}{z} = \frac{\bar{z}}{\bar{w}} \).
So $T^2_L \equiv \text{id}$ as diffeo.

$(T^2_L)_x = \text{id}$

But $T^2_L \in \text{Aut}(X_t,\omega)$ and $t \not\equiv \text{id}$. Seidel

$\hat{\chi}$ has degenerate symplectic form $\omega|_{\mathbb{P}^1} = 0$. 
8. Pullback of \( \omega \) to genuine symplectic form

(\( \exists L, L \neq S^2 \) no longer Lagrangian)

Get hyperkähler rotation of \( X_t \) as \( \hat{X}_0 \).

\[ X_t \text{ holomorphic symplectic } \omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i \]

\[ L = \sum dz_i \wedge d\bar{z}_{i+1} \]

\[ \omega|_L = 0, \quad \text{Im} \omega|_L \neq 0, \quad \text{Re} \omega|_L = \text{vol}_L \quad \text{whg} \]

Hyperkähler (HK)

Sim. on \( T^*P^1 \), \( \text{Re} \Omega|_C = 0, \) \( \omega|_{P^1} = \text{vol}_{P^1} \).

\[ \text{Im} \Omega|_C = 0, \]

HK maybe self mirror (at different level)

In some limit/time?? MS is HK rot??

\[ D^b(Fuk(x)) \leftrightarrow \text{Slags} \leftrightarrow \text{cx Lag} \in D^b(c_{1k}(x)) \]
In our case we pass to limit and get in central fibre $\hat{\mathcal{X}}_0$ the graph $\Delta_{\hat{\mathcal{X}}_0} \psi(C \times C)$.

Use $\mathcal{U} \in D^b(\hat{\mathcal{X}}_0 \times \hat{\mathcal{X}}_0)$ as Fourier-Mukai kernel.

\[ \Rightarrow T_C : D^b(\hat{\mathcal{X}}_0) \to D^b(\hat{\mathcal{X}}_0) \]

\[ T_C (\mathcal{E}) = \pi_2^\times \left( (\pi_1^\times \mathcal{E}) \otimes \mathcal{U} \right) \Delta_{\hat{\mathcal{X}}_0 \times \hat{\mathcal{X}}_0} \]

Acts as id on $\mathcal{E}$ s.t. $\text{Supp}(\mathcal{E}) \cap C = \emptyset$.
10. Exact sequence on $X_0 \times X_0$:

$0 \to \mathcal{O}_A \to \mathcal{O}_{\Delta \cup (c \times c)}(c,0) \to \mathcal{O}_{\Delta c}(c,0) \to 0$

$\mathcal{O}_c \otimes \mathcal{O}_c [1]$

gives exact triangle

$R\text{Hom}(\mathcal{O}_c, \mathcal{E}) \otimes \mathcal{O}_c \xrightarrow{\rho} \mathcal{E} \to \mathcal{T}_c \mathcal{E}$

Mirror to $T_0$. \( (T_0)^* [3] = [3] + (c^2) [C] [C] \)

Kontsevich: mirror of $T_0$ in auto of $D^b(c\mathfrak{h} - X_0)$, not auto of $X_0$. 

Configuration Space

\[ C_m = \{ \text{m distinct unordered pts in } \mathbb{C} \} \]

\[ = \{ \text{monic deg m poly w/ distinct roots} \} \]

\[ = \mathbb{C}^m \setminus \text{diagonals} \]

\[ \pi_1 (C_m) = \mathbb{B}_{m-1} \]

\( \mathbb{F} \) spaces \( X_2 \) parametrized by pts \( \Delta \in C_m \).
Eg (Donaldson, Khovanski-Seidel, Shapere-Verlin)

\[ X_2 = \left\{ \sum_{i=1}^{\infty} x_i^2 = p(t) = \prod_{i=1}^{n} (t - x_i)^2 \right\} \leq C^m \times C \]

Together form a family \( X = \bigcup_{x \in C_m} X_{x \in C_m} \) over \( C_m \). Symplectic.

Locally trivial symplectic fibre bundle \( \Rightarrow \pi_1(C_m) \to \text{Aut}(X_2, \omega) \)

\[ \text{grading...} \]

\[ \text{Aut}(\text{Fuk}(X_2, \omega)) \]

\[ \text{Aut}(\text{D}^b(\text{coh-mirror})) \]
Each $X_\delta$ fibred over $\mathbb{C}_x$ with fibres affine quadrics $\{\sum x_i^2 = p_i(t)\} \subseteq \mathbb{C}^n \times \mathbb{C}$.

The $i$th generator $T_i$ of $\Sigma_{n-1}$ acts by Dehn twist $T_{i,j}$ about Lagrangian $S^n$ fibred over paths between $x_i$ and $x_{i+1}$.
1a. Dim $n^2$. Hyperkähler!

Bring all $x_i$ together

$\Rightarrow$ Most singular fib. $X_0 = \{ \sum x_i^n = t^m \}$

An $m_1$ surface sing.

Restatement

Am $m_1$-chain of $(-2)$ R's.

Fits into simultaneous resolution with $X_2$'s after $\mathbb{P}_{m-1}$ base change

(i.e. base = \{ distinct ordered $\lambda_i \in \mathbb{C}^g$ \})

$\sum$ HK vector of $X_2$.
15. Prove that $T^{(\sigma)} : \text{D}^b(\mathbb{R}_0)$ give rep.

$$B_{n-1} \rightarrow \text{Aut}_{\text{eq}}(\text{D}^b(\mathbb{R}_0)).$$ (faithful)

$$\begin{cases} T_i \text{ and } T_i \in \mathfrak{t} \text{ and } T_i \text{ and } T_i \text{ and } T_i \\
T_i T_j = T_j T_i \text{ if } |i-j| \leq 1
\end{cases}
$$

Example: Matrices with distinct eigenvalues.

$\mathfrak{sl}(n, \mathbb{C}) \supset U = \{\text{distinct eigenvalues}\}
\xrightarrow{\text{adjoint action orbits}} C^n / S_n \supset C^n
$

Features of $U$ are regular orbits.

Symplectic: $B_{n-2} \rightarrow \text{Aut}(U_2, \omega)$, cf. Seidel-Smith.
16. Elvols collide $\iff$ singular fibres/obits.

Worst/central fibre = "nilpotent cone"

matrices of elvols 0.

Again, HK, F simultaneous resolution, Grothendieck after $\delta$ basechange.

\[
\begin{array}{c}
\{ A \in \text{SL}(n, \mathbb{C}) \} \\
\text{Flag } \langle V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n \rangle \\
\text{dim } V_i = i
\end{array}
\xrightarrow{A} \text{SL}(n, \mathbb{C})
\]

Smooth!

\[
\begin{array}{c}
elvols \\
C^n
\end{array}
\xrightarrow{\text{elvols}} \frac{C^n}{\Delta_0}
\]

Over $\text{UCSL}(n, \mathbb{C})$, horiz. map is $\phi_0$:
elvols distinct (+ ordered) so $V_i$ uniquely determined as $\langle V_i, \text{ith elvec} \rangle$. 
17. Fiber over \((\text{nilpotent cone}) \to \mathfrak{so}_3\)
in \(T^*\text{Flag}\), the rest of nil. cone
and the roots of general fibre \(U_2\).

Mirror!

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Monodromy action \(\mathbb{Z}_{n-2} \to \text{Aut}(U_2, \omega)\)
generated by fibred Dehn twists

\[\text{Locally } C^k \times (T^*S^2)\]

As \(\lambda_i \to \lambda_{i+1}\), collapse to fibred odp

\[\text{Locally } C^k \times \text{(surface odp)}\]

Mirror: take limits of graphs as \(\lambda_i \to \lambda_{i+1}\)
(and then all \(\lambda_i \to 0\))
Get, \( m \in T^*\text{Flag} \times T^*\text{Flag} \),

\[ \Delta_{T^*\text{Flag}} u(N_i \times_k N_i) \]

Here:

\[ F_i : = \{ \text{Flags } V_i \leq \ldots \leq V_{i-1} \leq V_{i+1} \leq \ldots \leq V_n = e^n \} \]

\[ \dim V_j = j \]

\[ F_{ij} : \text{ give } \pi_i^* T^* F_i \subset T^* \text{Flag} \]

\[ F_i \]

\[ \text{Call this } N_i \overset{C_i}{\hookrightarrow} T^* \text{Flag} \]

\[ \downarrow_{\pi_i} \]

\[ k_i \]

\[ P_i \text{ is a } T^1 \text{ bundle, } V_{N_i} \mid_{P_i} = O(-2) \]

\( k_i \) - locus of codps when contract \( T^1 \)’s

In locally about \( N_i \), \( T^* \text{Flag} \) looks like \( T^*P^1 \times k_i \).
19. Do signed version of $D^b(\text{surface})$ Dehn twist $T_{0_c}$ about $(-2) - \Pi'$. $C \subseteq \text{Surface}$.

In that case $C \xrightarrow{L} S$

$\uparrow$

point

$R\text{Hom}(U_{0_c}, E) \otimes U_{0_c} \xrightarrow{ev} E$

$\circ L^* p_c^{*} E \rightarrow E$

Find that FMT with $U_{\Delta U(N_i \times_{k_i} N_i)}(N_i, 0)$ is the functor

$E \rightarrow \text{Cone}(L_i^* \pi_i^* \pi_i^{*l^!} E \rightarrow E)$

Call this $T_i : D^b(\text{flag}) \rightarrow$
Then

- $T_i$ invertible
- $T_i T_j \equiv T_j T_i \quad |i-j| > 1$
- $T_i T_i T_i \equiv T_i T_i T_i T_i$

- Essentially 2d result
- Also easy
- Hard

- Can extend to action of affine braid group.
- Also to other groups - Bezrukavnikov, Mirković, Rumynin.

**Mirror Symmetry**

- Smoothing of singularity $\Leftrightarrow$ Resolution of (dual) singularity
- Vanishing cycle $L$ vanishing cycle $L$
- Exceptional set $E$

Hyperkähler case - self dual

Hyperkähler rotation: $\text{Stays} \rightarrow \text{Complex Stays}$

$D^b(Fuk) \rightarrow \text{Part of } D^b(\text{coh})$
Braid cobordisms

There is more structure on \( \{ \text{Functors } D^b(x) \} \):
- natural transformations between them.
- Hom's between Fun kernels
  \( \varphi^m \in D^b(x \times \times x) \)

Can define a categorification of \( B_n \) action
(e.g. \( D^b(x) \) or \( K(x) \)) to be an action of
the braid cobordism category

Objects: braids

Morphisms: cobordisms

\( x \to x \to x \to \cdots \)

Such simple structure elements

\( \text{id} \)
$T_i$ defined as cone

$$(l_i^* \circ \Pi_i \circ l_i^* \to \text{id})$$

so has natural morphism from id.

$\Delta U(N_i \times_k N_i)(N_i, 0)$ has natural morphism from $U_\Delta : V \to \Delta U(N_i \times_k N_i)(N_i, 0)$

To positive braid cobordism $\| \to \Sigma$

associate this nat. transf. $\text{id} \to T_i$.

Negative nat. transf. trickier.

Describe in surface case $\Sigma_i : T_i \to \text{id}$

(Flag is family version)
The above defines an action of $G$ on $D_0(Tr_{\Lambda})$. The corresp. map - moves correspond to not $\mathbb{C}$

Show this defines a map - moves correspond to $H^2(\mathbb{C}, G)$. $

\mathbb{C}$ is spanned by $\mathbb{C}$, with $v \in H^2(\mathbb{C}, G)$.

\( T_i = (R_{\mathbb{C}}(e_i, e_i) \circ \theta^i) \quad \text{id} \)

\( \left[ x_{-i} - x_i \right] \quad \text{id}_{[2]} \)

$\mathbb{H}^2(\mathbb{C}, e)$ is dual to $[C] \in H^2(\mathbb{C}, e)$. (C's span vector space $V \leq H^2(\mathbb{C}, e)$ with $v \in V$, $v \neq 0$.)