Symplectic homology: \( X \) (smooth complex) affine alg. var., \( K_X \cong \Theta_X \)
Make \( X \) into a sympl. ndl: \( c_0 = -dd^c h \) (w/ conical boundary)
\( \rightsquigarrow SH_* (X) \) (co-dim? usually) 2-graded vector space

\[ 
\begin{align*}
\text{Ex:} & \quad SH_* (\mathbb{C}^n) = 0 \\
& \quad SH_* (X \times \mathbb{C}) = 0 \quad \text{(subcritical case - Elashberg)} \\
& \quad SH_* ((\mathbb{C}^n)^*) = H^{-k}(LT^n) \quad \text{(cotangent bundle case - Viterbo etc...) Schwarz} \\
\end{align*} 
\]

\( \text{NB:} \) not computable in general.

Lefschetz fibrations: \( \pi: X \to \mathbb{C} \) affine Lefschetz fibration
(nondeg. critical pts, + condition on behavior at \( \infty \))
Fix a smooth fiber \( Y \). For every critical point \( \tau \), choose a path
\( \rightsquigarrow \) get a vanishing cycle = Lagrangian sphere \( \subset Y \).
Fix a basis of such vanishing cycles, denote it by \( (L_1, \ldots, L_n) \subset Y \).

\[ 
\text{Ex:} \text{ mirror of } \mathbb{CP}^2: \quad X = (\mathbb{C}^*)^2, \quad \pi(x, y) = x + y + x^{-1} y^{-1} \\
\text{Fiber } Y = \text{affine elliptic curve} 
\]

\[ 
\text{6 holomorphic discs} 
\]

Let \( B = \text{subcat of fibres category } \mathcal{F}(Y) \) with objects \( L_1, \ldots, L_m \).
\( \text{A} \subset \text{B} \) directed subcategory
\[ 
\text{So: } \text{Hom}_A (L_i, L_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \text{vs. } \text{Hom}_B (L_i, L_j) = \mathbb{C}^{\delta(i, j)} \quad \text{\( \delta \) case } i \neq j \text{ is tricky} 
\]

\[ 
\text{Ex: quiver picture of } A: \\
\begin{array}{c}
L_1 \\
\text{\( \Lambda^2 V \)} \\
\text{\( V \cong \mathbb{C}^3 \)} \\
L_2 \\
L_3 \\
\end{array} \\
\text{(compositions of the 6 discs in above picture}}}
\text{\( \equiv \text{each basis elt of } \Lambda^2 V \text{ can be written as a product in 2 ways, } e_i e_j = -e_j e_i \).}
In this case, $A$ is formal.

given push of $B$: $\pi^3 V \mapsto \pi^3 V$

(synchronization of the previous one)

Actually $H(A) \approx \Lambda^\ast(V) \otimes \Gamma$

$r = \mathbb{Z}_3$ acting diagonally

($\Lambda^\ast$'s are exterior products)

But $B$ is not formal — higher order products do in $B$

(so information is lost when forgetting "half" of $B$)

Higher product in $B$: $\pi^3 B \otimes \pi^3 B \otimes \pi^3 B \to \Lambda^0 V$

\[
\text{hom}_B(L_3, L_1) \otimes \text{hom}_B(L_2, L_3) \otimes \text{hom}_B(L_4, L_2) \to \text{hom}_B(L_4, L_1)
\]

$\forall v \in V$ basis elt, $\mu_B^3(v,v,v) = V_1 V_2 V_3$

$\mu_B^3$ is ambiguous (choice, as in tensor products), but

since $V \cap V = 0$ this one is well-defined

\[
L_1, L_2, L_3
\]

3! 4-gon with side in order $L_1, L_2, L_3, L_4$

\[\begin{align*}
D(A) &= H^0(\text{mod-}A) \approx D^b \text{Coh}(X^\ast) \\
D(B) &= H^0(\text{mod-}B) \subset D^b \text{Coh}(Y^\ast)
\end{align*}\]

full subcat. (essentially, rational to $Y^\ast$ of $\text{Coh}(X^\ast)$)

$Y^\ast = \text{Calabi-Yau hypersurf. in } X^\ast$

$Y^\ast = \{x_1 x_2 x_3 = 0\} \subset X^\ast$

Information is lost from $B$ to $A$ because $A$ doesn't remember which CY hypersurface $Y^\ast \subset X^\ast$ we were looking at.
Back to homological algebra...

- we have a short exact sequence of $A$-bimodules

\[ 0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0 \]

\[ \text{R} \]

$\text{AV}[\text{-dim}_A Y]$ (because of symmetry property of $B$ — it's a CY $A_\text{-coh}$)

There is a boundary map $B/A[-1] \cong A^\vee[-\text{dim}_A X] \rightarrow A$ (take a splitting $B \cong B/A$, it's not a module hom, but $d(\circ)$ is a module hom and lands in $A$)

- every $A$-bimodule $P$ gives rise to a convolution functor $\phi_P : D(A) \rightarrow D(A)$

  - $P = A \rightarrow \phi_P \cong \text{Id}$
  - $P = A^\vee \rightarrow \phi_P \cong \text{S} \text{ Sen functor}$

We also have, for every morphism of $A$-bimodule, a nat transform here, $B/A[-1] \rightarrow A$ induces $\phi_S : S[-\text{dim}_A X] \rightarrow \text{Id}$ (admits a natural interpretation in a (co) sheaf (filtration))

- weird symm. interpretation: $D(A) \cong \text{D}^b \text{Coh}(X^x)$

\[ S = (k_{x^x} \otimes -)[\text{dim}_A X] \]

Nat transform $S[-\text{dim}_x] \rightarrow \text{Id} \iff$ sections of $K^{-1}_x$

In our case, $\phi_S \iff x_1 x_2 x_3 \in H^0(K^{-1}_x)$

- The curved $A_\text{-coh}$ category $D$:

  $\text{Ob } D = \{ L_1, \ldots, L_n \}$

  $\text{hom}_D(L_i, L_j) = \text{hom}_A(L_i, L_j) \oplus \text{t.hom}_B(L_i, L_j)[[t]]$

  (ie. formal power series w/ $t$-coeff \in $A$)

  $\text{deg } t = 2$.
extend $\mu^k_B$ $t$-linearly to $D$ (or since $A$ subhep. $\to t^0$-terms still $\in A$)

\[ + \text{add a curvature term } \mu^0_D = t.l_x, \in \text{hom}_D^2(L,L). \]

$D$ is a curved $A_{\infty}$-category.

Symplectically, "$D$ represents the total space of the Leftkan fibration" \[ \text{Mirror-mix, } "D \text{ recap the confluence of the CY divisor } y^c \text{ inside } x^c.\]

**Thm:** \[ D(D) = D(A)/D(B) \]

\[ \text{( } A \subset B \Rightarrow \circ : D(B) \to D(A) \text{ view } B-\text{mod. as } A-\text{mod) } \]

\[ \to \text{Verdier quotient of mod. cats: } \]

\[ \text{localize along } \text{Im}(\circ). \]

**Thm:** \[ HH_x(D) = \lim_{\leftarrow} HH_x(A, (B/A)^{\otimes n}) \]

where connecting maps \[ (B/A)^{\otimes n} \to (B/A)^{\otimes n-1} \]

is induced by \[ B/A \to A \text{ on (any) one of the factors} \]

Hochschild homology: \(\text{given chains, d) collapse by inverting } \mu^k_D \)

\[ \text{extend by inverting } \mu^0_D = t.l_x. \]

**Conj:** \[ HH_x(D) = SH_x(X). \]

- **Nim:***

\[ HH_x(D) = \lim_{\leftarrow} \text{Ext}^k_{x^c_{\times X^c}}(O_\Delta, O_\Delta \otimes K^{\otimes n}) \]

\[ = \ldots \text{ (Koszul duality) } \ldots \]

\[ \cong \lim_{\to} \text{H}^{-k}(x^c, \Lambda^{-*}T_{x^c} \otimes K^{1-n}) \]

\[ \cong \Gamma(U^*, \Lambda^{-*}T_{U^*})^V \]

where \(U^* = x^c \setminus y^c.\)

(lim abus sections of $\Lambda^{-*}T$ with pole of arbitrary high order along $y^c$).
Computation on case of \( IP^2 \) →

\[
\mathcal{HH}_*(\mathcal{D}) = C[x_1^{±1}, x_2^{±1}, \theta_1, \theta_2]^V
\]

\[
SH^*(\mathcal{E}^2) = H^{-\alpha}(\mathcal{X} T^2)
\]