From Affine manifolds
to complex manifolds

Instanton corrections from tropical disks I

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Workshop on Homological Mirror Symmetry
and Applications I
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Mirror symmetry and tropical geometry
The arena of mirror symmetry is a Legendre dual pair \((B, \check{B})\) of integral affine manifolds.

The affine structure has singularities along a polyhedral complex \(\Delta \subseteq B\) with \(\text{codim}_\mathbb{R} \Delta = 2\).

\(B, \check{B}\) can be thought of as bases of dual SYZ-fibrations, but in general honest SYZ-fibrations seem to have thickened \(\Delta\).

On the complex side the affine manifold can be obtained from maximal holomorphic degenerations (GS) or by rigid-analytic methods (Kontsevich/Soibelman; \(\Delta = \emptyset\) or \(n = 2\)).

Mirror data are obtained directly from the affine geometry (e.g., \(h^{p,q}\)), or by tropical geometry on \(B\) and \(\check{B}\).

 Appropriately modified should work for any version of mirror symmetry.
In our approach $B$ and $\breve{B}$ are itself tropical — they arise from a set of integral polyhedra by facewise gluing and by specifying fan structures at the vertices.

**Tropical data:** $(B, \mathcal{P}, \varphi)$

- $\mathcal{P}$: complex of integral polyhedra ($|\mathcal{P}| = B$).
- For $\tau \in \mathcal{P}$ a fan structure $\Sigma_{\tau}$ on the normal space.
- **Polarization:** A (multivalued) integral PL-function $\varphi$ on $B$.

**Discrete Legendre transformation**

$$(B, \mathcal{P}, \varphi) \longleftrightarrow (\breve{B}, \breve{\mathcal{P}}, \varphi).$$
Fan Structures

A fan structure along $\tau \in \mathcal{P}$ is a continuous map $S_{\tau} : U_{\tau} \to \mathbb{R}^k$ with

1. $S_{\tau}^{-1}(0) = \text{Int } \tau$.
2. $\tau \leq \sigma \implies S_{\tau}|_{\text{Int } \sigma}$ is an integral affine map.
3. the collection of cones
   \[ \{ \mathbb{R}_{\geq 0} S_{\tau}(\sigma \cap U_{\tau}) \mid \sigma \in \mathcal{P} \} \]
   defines a fan $\Sigma_{\tau}$ in $\mathbb{R}^k$. 
Tropical dictionary on the complex side

Complex data: a polarized toric degeneration

- A degeneration $\pi : \mathcal{X} \to S$, $S$ a discrete valuation $k$-algebra with
  - $X_0 = \pi^{-1}(0)$ is a union of toric varieties,
  - $\pi$ is toroidal at 0-d toric strata of $X_0$.
- $L \downarrow X_0$ an ample line bundle.

Tropical data: fan and cone picture

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<th>$\mathcal{D}$</th>
<th>Fan picture</th>
<th>Cone picture</th>
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<tbody>
<tr>
<td>$\Sigma_\tau$, $\varphi :</td>
<td>\Sigma_\tau</td>
<td>\to \mathbb{R}$</td>
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The Main Theorem

Theorem (Gross/S. 2007)

Any \((B, \mathcal{P}, \varphi)\) fulfilling a maximal degeneracy condition\(^1\) arises from a polarized toric degeneration

\[
(\pi : \mathcal{X} \longrightarrow \text{Spec} \, k[[t]], L \downarrow X_0).
\]

Remarks

- Up to isomorphism \((\pi, L)\) is determined uniquely after specifying gluing data that determine \(X_0\) out of \((B, \mathcal{P}, \varphi)\).
- Any finite order deformation \(X_k \rightarrow \text{Spec} \, k[t]/(t^{k+1})\) is constructed by an explicit tropical algorithm on \((B, \mathcal{P}, \varphi)\).

\(^1\) a primitivity requirement on the local monodromy polytopes.
The Mumford construction
The Mumford Construction

As a motivating example we consider the case where \( B \subseteq \mathbb{R}^d \) is an integral, convex polyhedron \( \Delta = \emptyset, \varphi : B \to \mathbb{R} \) single-valued). In this case a toric construction produces a toric degeneration:

**The Proj construction (Mumford)**

- The upper convex hull of the graph of \( \varphi \) is an integral, unbounded polyhedron:
  \[
  \Xi := \{ (m, h) \in B \times \mathbb{R} \mid h \geq \varphi(m) \}.
  \]

- Let \( \mathfrak{X} \) be the polarized toric variety associated to \( \Xi \):
  \[
  \mathfrak{X} = \text{Proj} (\mathbb{C}[C(\Xi) \cap \mathbb{Z}^{d+2}]),
  \]
  with \( C(\Xi) = \text{cl} (\mathbb{R}_{\geq 0} \cdot (\Xi \times \{1\})) \subseteq \mathbb{R}^{d+2} \) the cone over \( \Xi \).

- Define \( \pi : \mathfrak{X} \to D \) by the action of \( \rho = (0, 1) \) on \( \Xi \).
The associated tropical manifold

...equals \((B, \mathcal{P}, \varphi)\) (cone picture):

- Toric strata of \(X_0\)
  \[=\text{faces of } \Xi \]
  \[=\text{elements of } \mathcal{P}.\]

- Near a zero-dimensional stratum
  \((\leftrightarrow \text{vertex } v \in \mathcal{P})\) the
degeneration \(\pi\) is defined by the
cone at the corresponding
vertex of \(\Xi\)
\[\implies\] the polarization equals \(\varphi.\)
Reminder: Tropical dictionary

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Alternative Viewpoint

Reduction to finite order

To find $\mathcal{X} \to D$ starting from $X_0$, it suffices to construct a compatible system of $k$-th order thickenings $X_k$ of $X_0$, for any $k > 0$:

$$X_k = \text{Proj} \left( \mathbb{C}[C(\Xi) \cap \mathbb{Z}^{d+2}] / (t^{k+1}) \right).$$

**Note:** $X_k$ can be visualized as a thickening of $\partial \Xi \subseteq \Xi$ of thickness $k$.
Primary decomposition of $X_k$

Once we have reduced to finite order, we can decompose into irreducible components (primary decomposition). There are no embedded components, so a primary component is just a thickening of an irreducible component of $X_0$:

$$X_0 = Z^1 \cup \ldots \cup Z^r \quad \implies \quad X_k = Z^1_k \cup \ldots \cup Z^r_k.$$ 

The thickenings $Z^\mu_k$ can be read off from $\varphi$!
Model rings
The General Case

Plan

The plan is now to build $X_k$ similarly from standard pieces, which are completely determined from the tropical geometry. We will soon run into troubles because of monodromy, and the whole point of the construction is to make the necessary adjustments order by order.

Open-closed decomposition

For our construction it also does not suffice to decompose into closed components. Rather we first need to go over to standard open pieces, which are then further decomposed.
Standard pieces

The rings

For each $k \in \mathbb{N}$ and inclusion $\omega \subseteq \tau$ we now define a ring $R^k_{\omega,\tau}$:

- If $\omega = \nu$ is a vertex $\varphi$ is given by a single-valued $\varphi_\nu$ on $|\Sigma_\nu| = \mathbb{R}^d$. The upper convex hull of $\varphi_\nu$ defines an affine toric variety $\text{Spec} \, k[P_\nu]$ as before. Take the $k$-th order neighbourhood of the stratum given by $\tau$ to define $R^k_{\nu,\tau}$.

- For each pair $\omega \subseteq \tau$ we can similarly define the $k$-th order neighbourhood of the $\tau$-stratum wrt. $\varphi_\omega : |\Sigma_\omega| \to \mathbb{R}$. Removes all toric strata not containing the $\omega$-stratum.

$\tau$: selects stratum, $\omega$: selects open set.

Remarks

- $R^k_{\omega,\tau}$ neglects all monomials that vanish to order $\geq k$ on any maximal cell $\sigma$ containing $\tau$.

- Monomials have directions: $P_\nu \to \Lambda_\nu \subseteq T_\nu B$, $m \mapsto \overline{m}$. 
Gluing of standard pieces
Near the discriminant locus the result of gluing depends on the choice of affine chart!

**Example**

Take $\varphi = 0$ on left triangle and with slope 1 on right triangle ($\varphi(1, 0) = 1$).

Result 1: $\mathbb{C}[x, y, w, w^{-1}, t]/(xy - t)$.

Result 2: $\mathbb{C}[x, y, w, w^{-1}, t]/(xyw^{-1} - t)$

$= \mathbb{C}[x, y, w, w^{-1}, t]/(xy - wt)$. 
Local correction

The inconsistency can be cured by introduction of a factor emanating from the singularity:

Example

**Computation 1:**
\[ \mathbb{C}[x, y, w, w^{-1}, t]/(xy(1 + w)^{-1} - t) \]

\[ = \mathbb{C}[x, y, w, w^{-1}, t]/(xy - (1 + w)t). \]

**Computation 2:**
\[ \mathbb{C}[x, y, w, w^{-1}, t]/\left(xywis\right)^{-1}(1 + w^{-1})^{-1} - t) \]

\[ = \mathbb{C}[x, y, w, w^{-1}, t]/(xy(w + 1)^{-1} - t) \]

\[ = \mathbb{C}[x, y, w, w^{-1}, t]/(xy - (1 + w)t). \]

(To do these computations we should also localize at \(1 + w\).)
Starting data
Local models for $\pi$ and log structure on $X_0$

**Situation along codimension 1 stratum $X_\rho \subset X_0$:**

$\pi : \mathcal{X} \to \text{Spec } \mathbb{k}[t]$ locally described by

$$xy - f_\rho(w_1, \ldots, w_{n-1})t^b = 0.$$ 

$f_\rho = f_{\rho,v} \in \Gamma(U, \mathcal{O}_{\mathcal{X}_\rho}^*)$ is determined uniquely if we choose $x_i = z^{m_i} \in \mathbb{k}[P_v]$ with $\overline{m}_0 = -\overline{m}_1 \in \Lambda_v \subset T_v B$.

**Note:** $f_\rho$ generalizes $1 + w$ in the previous example.

**Relation to log structures**

A compatible choice of $f_{\rho,v}$ defines a log structure on $X_0$ with a log-smooth morphism to the standard log point. If $\Delta \cap \rho \neq \emptyset$ the $f_{\rho,v}$ vanish along the singular locus of the log structure $Z \subset X_0$. (codim $Z = 2$.)

**Existence:** GS’03
Structures
Slabs

Change of vertex for $f_{\rho,v}$

If $\rho \in \mathcal{P}^{n-1}$, $v$, $v' \in \rho$ vertices then\(^2\)

\[ f_{\rho,v'} = z^{m_{v'}^\rho v} \cdot f_{\rho,v}. \]

$m_{v'}^\rho v \in \Lambda_\rho$: Monodromy for path around $\rho \cap \Delta$ through $v$, $v'$.

Definition

A slab $b$ is a convex polyhedral subset of some $\rho \in \mathcal{P}^{n-1}$ $x \in b \setminus \Delta$ with

- $f_{b,v}(x') = z^{m^\rho_{v'(x')}}^v(x) \cdot f_{b,v}(x)$ via parallel transport, where for $y \in \rho \setminus \Delta$, $v(y) \in \rho$ denotes a vertex in the same connected component.
- $f_{b,x} \equiv f_{\rho,v}(x) \mod t.$

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\(^2\)We omit factors arising from non-standard gluing data for $X_0$. 
Walls

Slabs yield perturbations of closed gluings, but these are not enough: We need to also apply automorphisms to affine open subsets. These are implemented consistently by walls.

**Definition**

A *wall* consists of an 
\((n-1)\)-dimensional polyhedral subset \( p \) of some \( \sigma \in \mathcal{P}_{\text{max}} \) of the form

\[
p = (q - \mathbb{R}_{\geq 0} \cdot \bar{m}_p) \cap \sigma,
\]

with \( p \not\subseteq \partial \sigma \), together with 
\( m_p \in P_\sigma \), \( \text{ord}(m_p) > 0 \), and 
\( c_p \in k \).

Function analogous to \( f_{b,x} \) of slabs: 
\[
f_p := 1 + c_p z^{m_p}.
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Structures

Definition

A structure $\mathcal{S}$ is a locally finite set of slabs and walls, such that

- the slabs define a polyhedral decomposition of $| \mathcal{P}_{\leq n-1} |$.
- any $\sigma \in \mathcal{P}_{\text{max}}$ is subdivided by $\{ p \in \mathcal{S} | p \subseteq \sigma \}$ into convex polyhedral subsets, called Chambers.

System of rings

For each $\omega \subseteq \tau$ and chamber $u$ with $\omega \cap u \neq \emptyset$ and $\tau \subseteq \sigma_u \in \mathcal{P}_{\text{max}}$ obtain a ring $R_{\omega, \tau}^k(u)$. 

Gluing morphisms

Change of strata

For $\omega \subseteq \omega' \subseteq \tau' \subseteq \tau$ have natural morphisms

$$R^k_{\omega, \tau}(u) \xrightarrow{\text{localiz.}} R^k_{\omega', \tau}(u) \xrightarrow{\text{quot.}} R^k_{\omega', \tau'}(u).$$

Crossing a wall or slab

Crossing $p$ or $b$ ($p, b \subseteq u \cap u'$) gives the log isomorphism

$$\theta : R^k_{\omega, \tau}(u) \in z^m \longmapsto f^{-\langle m, n \rangle} \cdot z^m \in R^k_{\omega, \tau}(u'),$$

with $f = 1 + c_p z^{m_p}$ or $f = f_{b,x}$, $x \in b \setminus \Delta$. 
Consistency and $k$-th order deformation

**Consistency**

A structure $\mathcal{S}$ is consistent to order $k$ if changing chambers cyclically around a codimension two stratum $j$ of $\mathcal{S}$ (a joint) gives the identity in $R_{\omega,\tau}^k(u)$:

$$\theta_j := \theta_r \circ \ldots \circ \theta_1 = 1.$$ 

**Theorem**

Assume $\mathcal{S}$ is consistent to order $k$. Then there exists a log-smooth deformation

$$\pi_k : X_k \rightarrow \text{Spec } \mathbb{k}[t]/(t^{k+1})$$

of $X_0$ of the required sort.

**Note:** The log structure enters via the order 0-terms in $f_b$. 
The algorithm
Step O: The initial structure

Construct $\mathcal{S}_k$, consistent to order $k$, inductively by modifying $\mathcal{S}_{k-1}$ by order $k$ stuff.

$\mathcal{S}_0$

Has only slabs $b = \rho \in \mathbb{P}^{n-1}$, $f_{b,x} = f_{\rho,v(x)}$. 

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Log automorphisms

Step I

Steps II and III

Concluding remarks
Log automorphisms

For a joint $j \subseteq u$ consider algebraic group $H_j$ of log automorphisms of $R^k_{\omega,\tau}(u)$ with Lie algebra

$$\mathfrak{h}_j = \bigoplus_{\overline{m} \notin \Lambda_j, 1 \leq \text{ord}(m) \leq k} z^m (k \otimes \mathbb{Z} (\Lambda_j^\perp \cap \overline{m}^\perp)),$$

$$[z^m \partial_n, z^{m'} \partial_{n'}] = z^{m+m'} \partial_{\langle \overline{m'}, n \rangle - \langle \overline{m}, n' \rangle} n.'$$

(Kontsevich/Soibelman).

Properties

- $\mathfrak{h}_j$ preserves holomorphic log volume form.
- $\exp(z^m \partial_n)(z^{m'}) = \exp(z^m) \langle \overline{m'}, n \rangle \cdot z^{m'}$.
- $\mathfrak{h}_j^k = \bigoplus_{\text{ord}(m) = k} z^m (k \otimes (\Lambda_j^\perp \cap \overline{m}^\perp)) \subseteq \mathfrak{h}_j$ is abelian.
Step I: Insertion of new walls

For each joint \( j \in \mathcal{I}_{k-1} \) expand \( \theta_j \) in \( H^k_j \):

\[
\theta_j = \exp \left( \sum_m c_m z^m \partial_n \right).
\]

For any \( m \) insert wall \( \left( (j - \mathbb{R}_{\geq 0} \bar{m}) \cap \sigma, c_m, m \right) \).

Remarks

- **Point:** The order of \( m \) increases along \(-\bar{m}\).
- If \( j \subseteq |\mathcal{P}_{\leq n-1}| \) we can not literally work in \( H^k_j \), because \( f_{b,x} \) contains monomials of order 0. Under the maximal degeneracy condition it still works, but if \( j \subseteq |\mathcal{P}_{\leq n-2}| \) we have to modify \( f_{b,x} \) for slabs \( b \supseteq j \) (hard!).
Steps II and III: Adjustment of slabs and normalization

**Step II: Adjustment of slabs**

The adjustment of slabs from the codimension 2 scattering causes new inconsistencies. These can be dealt with by spreading the changes along slabs contained in any \( \rho \subseteq \mathbb{P}^{n-1} \) by a homological argument.

**Step III: Normalization**

The remaining inconsistencies are *undirectional*, i.e. of the form \( ct^k \partial_n \). These can be removed by requiring the Taylor series of \( \log f_{b,x} \) at any vertex \( v \in \rho = \rho_b \) not to contain any pure \( t \)-terms.
Concluding remarks

- Local mirror symmetry computations show that $t$ should be the canonical coordinate. Depends crucially on the normalization procedure!
- Structures contain all information about the complex degeneration.
- Remaining task on complex side: Extract period data and $A_\infty$-category from structures.
- Structures on fan side $\longleftrightarrow$ tropical disks (to be explained in Mark’s talk).