

## WAM IAS Program - Graph Theory Exercises (Lecture 3)

### Exercise 1:

Let  $G$  be a graph. The *line graph*  $L(G)$  of  $G$  is the graph with vertex set  $E(G)$  and edge set  $\{ef : e, f \text{ have a common vertex}\}$ . Prove that every line graph is claw-free.

### Exercise 2:

Let  $G$  be a graph that does not contain a triangle and does not contain  $P_3$  (the path of length 3) as an induced subgraph. Prove that every component of  $G$  is a complete bipartite graph.

### Exercise 3:

Prove that for  $n \geq 5$ , every graph with  $n$  vertices and  $\lfloor n^2/4 \rfloor + 2$  edges contains two triangles with exactly one vertex in common.

### Exercise 4:

Let  $k \geq 2$  and suppose  $G$  is a graph with  $n$  vertices and no odd cycle of length less than  $2k + 1$ . Then prove that either  $G$  contains an independent set of size  $\lceil n/2 \rceil$ , or the minimum degree  $\delta(G) \leq 2n/(2k + 1)$ .

### Exercise 5:

Let  $G$  be a graph with  $2n$  vertices and minimum degree at least  $n - 1$  that does not have a Hamilton cycle (a cycle of length  $2n$ ). Then prove that  $G$  has at most  $3n^2/2 - n$  edges.

### Exercise 6:

Let  $k \geq 2$ . Let  $G$  be a connected graph with minimum degree  $k$  and at least  $2k + 1$  vertices. Prove that  $G$  contains a path with at least  $2k + 1$  vertices.