Lecture IV  Structure on $H^*_c(L)$

1. $S^1$-action. Recall $S^1 \times LX \to LX$
   $$(t, y) \mapsto y(-t).$$

   $$H_*(S^1) \otimes H_*(LX) \to H_*(S^1 \times LX) \to H_*(LX)$$

   Interesting part: $[S^1] \in H_*(S^1)$ maps $H_*(LX) \to H_{*-1}(LX)$.
   Or equiv. $H_*(LX) \to H_{*-1}(LX)$.

   Hochschild homology: Connes - Kohnen operator:
   $$B : C_*(A, A) \to C_*(A, A)$$
   $$B(a_0 \otimes \cdots \otimes a_k) = \sum_{(a)D} (-1)^{\mu} a_{d+1} \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_i$$

   Jones: These two operations correspond to each other under the isomorphism
   $$H_*(LX) \cong H_*([C_*(X), C_*(X)]).$$

   Idea of proof: $LX = \text{Maps}(|\delta_0|, X)$

   $S^1$-action: $y \in LX \mapsto [y(-t) : |\delta_0| \to |\delta_0| \to X]$

   Need to understand simplicially
   $$|\delta_0| = \bigcup_{t \geq 0} S^1 \times \Delta^2 / \sim$$
   $$\Delta^2 = \{e, e, e, \ldots, e, e\}$$

   $$3 \times \Delta^2 \to |\delta_0| \xrightarrow{+t} t \in |\delta_0|$$
   $$3 \times (t_1, \ldots, t_k) \mapsto t_i \in |\delta_0| \mapsto t_i + t \in |\delta_0|$$
\[ \mapsto S^1 \times (\Delta^2 \times \Delta^2) \mapsto \mathcal{A}_{2+1} \times \Delta^{2+1} \]

\[ (t, t_1 \leq \cdots \leq t_2, "0") \mapsto (t \leq t_1 + t \leq \cdots \leq t_2 + t, "x") \]

**CASE 0:** \( t_2 + t \leq 1 \) \quad \Rightarrow \text{CORRECT ORDERING} \]

**CASE 1:** \( t_2 + t \leq 1 < t_2 + t \) \quad \Rightarrow \frac{(t_2 + t)}{2} \text{ in front} \]

**CASE 2:** \( t \leq 1 < t_1 + 1 \)

\[ \mapsto \text{SUM OVER} \ (2+1) \ \text{COPY} \ \text{OF} \ (\mathcal{A}_{2+1} \ "0") \times \Delta^{2+1} \]

**COROLLARY:** HAVE NOW AN EXPLICIT FORMULA TO COMPUTE THE \( S^1 \)-ACTION ON \( H^*(LS^m) \).

\[ \text{EP:} \text{ OBVIOUSLY } 0 \text{ ON THE CLASSES } x \cdot x \cdot \cdots \cdot x, \text{ AND NON-ZERO ON } x \cdot x \cdot \cdots \cdot 0 \text{ ON R}. \]

**2. CHAS - SBUKAN PRODUCT**

**CAN INTERSECTION + CONCATENATION OF LOOPS DEFINE A PRODUCT? WHEN \( M = \text{CLOSED}, \) ORIENT MFD**

**THM (CHAS - SBUKAN)** \[ \iff \quad H_p(LM) \& H_q(LM) \iff H_{p+q}(LM) \]

\[ \text{ST.} \quad (H_*(M), \cdot) \overset{\text{CONC.}}{\underset{\text{INT.}}{\mapsto}} (H_*(LM), \cdot) \overset{\text{INT.}}{\underset{\text{CONC.}}{\mapsto}} (H_*(SM), \cdot) \]

**ARE RING MAPPINGS.**
Given two chains of loops, intersect the chains of basepoints in \( M \) and concatenate the loops at the intersection.

3. Algebraic Model + More

Algebraic model of \( H_\ast(LM) = HH_\ast(C^\ast(M), C^\ast(M)) \) algebra with cup product.

Recall: intersection product \( \cong \) cup product in \( H^\ast(M) \) under Poincare duality.

Digression: Frobenius Algebras.

\( H^\ast(M) + \text{cup product} + \text{PD} \Rightarrow H^\ast(M) \) "Poincare duality alg."

Def/Prop: A Frobenius Algebra

= algebra with a non-degenerate pairing
  \( A \otimes A \xrightarrow{\ast} \mathbb{R} \) s.t. \( \langle ab, c \rangle = \langle a, bc \rangle \)

= algebra which is a co-algebra

= open TFT: every topological type of surface with marked "incoming" and "outgoing" intervals defines an operation \( \otimes' : A \otimes' \to A \otimes' \)

and these are compatible under gluing.
Commutative Frobenius algebras = closed TFT

= Operations parameterized by \[
p \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix}_9
\]

**Question:** Suppose \( A \) is a (commutative) Frobenius algebra, what structure does \( C_*(A,A) \) have?

**Example:** For any algebra \( A \), have the \( B \)-operator \( C_*(A,A) \rightarrow C_{*+1}(A,A) \).

- A commutative \( \Rightarrow C_*(A,A) \) admits a product:

  \[ \text{The shuffle product} \quad \text{(models the coproduct on} \quad H^*(Lk) \quad \text{)} \]

  \[ \text{Good enough for us?} \]

  Have \( H^*(M) \) is a commutative Frobenius alg. What about \( C^*(M) \)? Some HTpy version?

**Easier:** Lambe-Richters-Stanley \[ C^*(M) \cong A_+ \text{ comm. Frobenius} \]

with \( H_*(A_+) \cong H_*(M) \) as a dg-alg.

\[ \text{As a conn. Frbg. alg.} \quad [\text{in char 0}] \]

W-Westerland

Type of algebraic structure for \( A \)

- eg: Comm, Ass., Assoc., Froeb, Poisson

\[ \text{Chain complex of natural operations on} \quad C_*(A,A) \]

\[ \text{Universal in sense} \]

[Diagram: \( \text{Machine} \rightarrow \text{Chain complex of natural operations on} \quad C_*(A,A) \)]
INPUT: FROBENIUS ALG \rightarrow OUTPUT = COMPLEX OF CHORD DIAGRAMS (RECOVERING \nTRAVERSAL-GENELIANS \PLUS SHOWING UNIVERSALITY) \\
\uparrow "H_0" (ON INPUT!)

INPUT: "OPEN TCFT" = \mathcal{A} \left( \text{MODULI OF RIEMANN SURFACES WITH MARKED INTERVALS IN THEIR BODY} \right)

\downarrow OUTPUT = "CLOSED TCFT" [RECOVERING COSTELLO + KONTSEVICH - SOIBELMAN \PLUS UNIVERSALITY]

INPUT: COMMUTATIVE FROBENIUS ALGEBRAS

\rightarrow [ANGELA KLAGA] OUTPUT = "LOOP DIAGRAMS" = STRING TOPOLOGY OPERATIONS

ROUGHLY: \forall p, q \geq 1 \exists \text{CHAIN COMPLEX } D_{p, q} \text{ WITH MAPS } D_{p, q} \otimes C_\ast(A, A)^{\otimes p} \rightarrow C_\ast(A, A)^{\otimes q}

FOR ANY COMM. ALG A, COMPATIBLE UNDER A CERTAIN COMPOSITION IN \Psi(D_{p, q})^3, AND

\[ D_{p, q} = 2 \]

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (2,0) {2};
\node (3) at (-1,1) {3};
\node (4) at (-1,-1) {4};
\draw[->] (1) to [out=30,in=150] (3);
\draw[->] (3) to [out=0,in=90] (2);
\draw[->] (2) to [out=-90,in=180] (4);
\draw[->] (4) to [out=-90,in=0] (1);
\end{tikzpicture}
\end{center}

9 CIRCLES
\begin{itemize}
\item q > 0 POINTS \neq O \in S^1 \quad k = \text{DEGREE}
\item PARTITION OF POINTS U O'S?
\item (WEIGHT ON EACH PARTITION SUBSET)
\end{itemize}

P LOOPS BASED AT SPECIAL POINTS

\text{EAST: } \exists \text{ MAPS } C_\ast (M_g, p+q) \rightarrow C_\ast (\mathcal{M}_g, p+q) \rightarrow D_{p, q} + [KAITE]