Lecture III: Hochschild Homology and
A Model for $H^*_c(U_M)$

Let $k$ be a field. (Think $k = \mathbb{Q}, \mathbb{R}$.)

**Def:** A dg-algebra is a chain complex

\[ (A = \bigoplus_{n \in \mathbb{Z}} A_n, d) \rightarrow A_{-1} \xrightarrow{d} A_0 \xrightarrow{d} A_1 \xrightarrow{d} \cdots \]

\[ d^2 = 0 \]

equipped with a multiplication

\[ m: A \otimes A \rightarrow A \text{ chain map} \]

\[ a \otimes b \mapsto ab \]

which is associative and unital ($1 \in A_0$).

**Ex:** $A = A_0 = k$ with the field multiplication.

- $A = \{x \in k[x^2] \mid \deg x = 2\}$, i.e. $x \in A_2$.

\[ = A_0 \oplus A_2 \oplus A_4 \oplus \cdots \]

\[ d = 0. \]

\[ = \{ a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n \mid a_i \in k \} \]

\[ A_2 \quad A_4 \quad \cdots \]

- $A = \langle x^2 \rangle / \langle x^3 \rangle = A_0 \oplus A_2 \quad d = 0$

- $X \rightarrow A = C^{-\bullet}(X) = \text{SINGULAR COHOMOLOGY}$

\[ C^2(X) \xrightarrow{d} C^1(X) \xrightarrow{d} C^0(X) \]

turned around so $d$ goes down

product = cup product.

- $X \rightarrow A = H^\bullet_c(X), \quad d = 0$.

same product.
NOTE: $X = S^2 \Rightarrow A = H^{-}(S^2) = k[\mathcal{A}]_{x^2}$

\[ 1x1 = -2. \]

Given a dg-algebra $(A,d,n)$, define a new chain cpx

\[(C_*(A/A) = \bigoplus_{n \geq 0} A \otimes \overline{A}^{\otimes n}, \quad d = d_A + d_H)\]

where

- $1k : \mathcal{A} \to A \to A^{\mathcal{A}} = \overline{A}$
- $l_{a_0} \otimes \ldots \otimes a_n \cdot l_{a_0} + \ldots + l_{a_n} + n \text{ (degree)}$
- $d_A(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_i \otimes \ldots \otimes a_n$
- $d_H(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{i+1} a_0 \otimes \ldots \otimes a_i \otimes \ldots \otimes a_n$

\[ (-1)^{n+1} + (-1)^n \cdot (l_{a_0} + \ldots + l_{a_{n-1}}) \]

\[ n \cdot (-1)^{n+1} \cdot a_n \cdot a_0 \otimes \ldots \otimes a_{n-1} \]

Note: signs depend on choices but are necessary!

**Facts:**

1) $d_A^2 = 0$ (Exercise)

2) $d_H^2 = 0$.

This follows from $d = \sum_{i} d_i$ for $d_i$ maps satisfying the simplicial identity.

3) $d_A d_H = d_H d_A$ (i.e., the $d_i$'s are chain maps on $A \otimes A^{\otimes n}$)

\[
d^2 = (d_A + d_H)^2 = d_A^2 + d_A d_H + d_H d_A + d_H^2 = 0
\]

Need to replace $d_H$ by $(-1)^n d_H$.
**DEF:** The Hochschild homology of $(A, d, m)$ is
\[ HH_*(A, A) = H_*(C_*(A, A), d). \]

**EX:** Suppose $A$ is an algebra $(A = A_0, d = 0)$

Then $HH_0(A, A) = \frac{\ker(d: C_0(A, A) \to 0)}{\operatorname{Im}(d: C_0(A, A) \to C_0(A, A))} = \frac{A}{[A, A]}$

A \otimes A \to A
a \otimes b \mapsto ab - ba

**RELEVANCE FOR US:**

**THM (Jones)** For $X$ 1-connected $(\pi_0 X = 0 = \pi_1(X))$

\[ HH_*(C_*(X), C_*(X)) \simeq H_*(LX) \]

**EX:** Can use this to compute $H_*(LS^m)$, $m \geq 2$

**FACT:** $S^m$ is formal: $C_*(S^m) \simeq H_*(S^m) = \mathbb{R} / \mathbb{Z}^2$

BIG! \quad \text{QUASI-ISOMORPHIC AS A DG-ALG}

$\Rightarrow HH_*(C_*(S^m), C_*(S^m)) \simeq HH_*(H_*(S^m), H_*(S^m))$

$C_*(H_*(S^m), H_*(S^m))$ generated by (as a vector space)

1, x, 1 \otimes x, x \otimes x, 1 \otimes x \otimes x, x \otimes x \otimes x, ...

deg: 0, -m, 1-m, 1-2m, 2-2m, 2-3m, ...

**DIFFERENTIAL:** $d_A = 0$, $d_H = \sum (-1)^i d_i$

C MULTIPLY TWO ENTRIES \Rightarrow 0 MOST OF THE TIME AS $\mathbb{Z}$
EXERCISE: COMPLETE THE COMPUTATION.

(ONLY POSSIBLE NON-ZERO DIFFERENTIALS FROM $d_0 \mod d_H$, WHETHER THEY CANCEL OR NOT DEPEND ON $n$ AND $m$.)

SKETCH PROOF OF JONE'S THEOREM

1. $H^*_+(C^*(X), C^*(X)) = H^*_+(\bigoplus_{n \geq 0} C^*(X)^{\otimes n+1}, d=d_A+d_H)$

   $\cong H^*_+(\bigoplus_{n \geq 0} C^*(X)^{\otimes n+1}, d=d_A+d_H)$

   $\cong H^*_+(\bigoplus_{n \geq 0} C^*(X)^{\otimes n+1}, d=d_A+d_H)$

\[\text{AW: } C^p(X) \otimes C^q(X) \xrightarrow{x} C^{p+q}(X \times X)\]

\[\text{Note: Cup product: } C^p(X) \otimes C^q(X) \xrightarrow{\text{AW}} C^{p+q}(X \times X) \xrightarrow{D^+} C^{p+q}(X)\]

$\Rightarrow d_H$ Defined using Diagonals

$\Rightarrow d_H$ Defined using cup products

$\Rightarrow \text{AW almost gives a map of double completed}$

2. $X^{*+1}$ is a cosimplicial space with

\[d^i : X^p \rightarrow X^{*+1}\]

Given by

\[d^i = x^i \times D(x^p) - \sum_{i \leq p} \]

\[d_{p+1}(x_1, \ldots, x_p) = (x_1, \ldots, x_p, 2) \]

From Exercise yesterday: $X^{*+1} \cong \text{Maps}(\mathbb{D}_0, X)$
And $|\partial_0| \cong S^1$.

**Fact:** $\text{Maps}(I/\partial_0, X) \cong \text{tot}(X^{*+1}) = \text{tot}(\text{Maps}(I, X))$

"TOTALIZATION" OF THE COSIMPPLICIAL SPACE AS IN DESCRIPTION OF LM YESTERDAY.

3. **Given a cosimplicial space** $Y^\bullet$ (think $X^{*+1}$)

- Space $\text{tot}(Y^\bullet)$
- Chain complex $(\bigoplus C^*(Y^p), d_{\text{sing}} + \varepsilon(-1)^i(d^!))_{p \geq 0}$
- Double complex

Compute THE SAME HOMOLOGY UNDER GOOD CONDITIONS

($\Rightarrow$ assumption $X$ 1-connected, by Giever).

For $Y^\bullet = X^{*+1}$, $C^*(\text{tot}(X^{*+1})) = C^*(LX)$

$\bigoplus C^*(X^{p+1}), d_{\text{sing}} + \varepsilon(-1)^i(d^!)$

is the double complex from 3.