Eulerian Polynomials and Beyond

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Based on joint work with John Shareshian

Binomial Coefficients and Eulerian numbers

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

Binomial Coefficients and Eulerian numbers

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$$

Binomial Coefficients and Eulerian numbers

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			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

				1				
			1		1			
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$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$$

$$\left\langle {n \atop j} \right\rangle = (n-j)\left\langle {n-1 \atop j-1} \right\rangle + (j+1)\left\langle {n-1 \atop j} \right\rangle$$

Binomial Coefficients and Eulerian numbers

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1		4		6		4		1

$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$$

- coeff's of polynomial $(t+1)^n = \sum_{j=0}^{n} \binom{n}{j} t^j$

				1				
			1		1			
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	1		11		11		1	
1		26		66		26		1

$$\left\langle {n \atop j} \right\rangle = (n-j) \left\langle {n-1 \atop j-1} \right\rangle + (j+1) \left\langle {n-1 \atop j} \right\rangle$$

- coeff's of Eulerian polynomial $A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j$

Binomial Coefficients and Eulerian numbers

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

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- coeff's of polynomial $(t+1)^n = \sum_{j=0}^{n} \binom{n}{j} t^j$
- rows add to 2^n

1
1 1
1 4 1
1 11 11 1
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- rows add to $n!$

Binomial Coefficients and Eulerian numbers

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

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- coeff's of polynomial $(t+1)^n = \sum_{j=0}^{n} \binom{n}{j} t^j$
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- subsets of $\{1, 2, \ldots, n\}$ of size j

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Binomial Coefficients and Eulerian numbers

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- permutations in $\mathfrak{S}_n$ with j descents

- Rows are palindromic and unimodal.

Eulerian polynomials - Euler's definition

$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

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$$\sum_{i\geq 1} t^i = \frac{t}{1-t}$$

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Leonhard Euler
(1707-1783)

Euler's triangle

1				
1	1			
1	4	1		
1	11	11	1	
1	26	66	26	1

Euler's definition

$$\sum_{i\geq 1} i^n t^i = \frac{t\, A_n(t)}{(1-t)^{n+1}}$$

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Euler's exponential generating function formula

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z}-t}$$

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$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z}-t} = \frac{(1-t)e^z}{e^{tz}-te^z}$$

Eulerian numbers - combinatorial interpretation

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1, 3, 4\}$$

Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

$$\mathrm{des}(32541) = 3$$

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Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

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Excedance set: $\mathrm{EXC}(\sigma) := \{i \in [n-1] : \sigma(i) > i\}$

$$\sigma = 32541 \qquad \mathrm{EXC}(\sigma) = \{1, 3\}$$

Define $\mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|$. So

$$\mathrm{exc}(32541) = 2$$

Eulerian numbers - combinatorial interpretation

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{des}(\sigma)} = 1+4t+t^2$$

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{exc}(\sigma)} = 1+4t+t^2$$

1
1 1
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Eulerian polynomial

$$A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

Eulerian numbers - combinatorial interpretation

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$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{exc}(\sigma)} = 1+4t+t^2$$

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1 1
1 4 1
1 11 11 1
1 26 66 26 1

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MacMahon (1905) showed equidistribution of des and exc.
Carlitz and Riordin (1955) showed equals $A_n(t)$.

Unimodality of Eulerian Polynomials

Probably not a new formula

$$A_n(t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1,\ldots,k_m \geq 2} \binom{n}{k_1 - 1, k_2, \ldots, k_m} t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$$

where

$$[k]_t := 1 + t + \cdots + t^{k-1}.$$

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where

$$[k]_t := 1 + t + \cdots + t^{k-1}.$$

Sum & Product Lemma: Let $A(t)$ and $B(t)$ be Positive, Unimodal, Palindromic with respective centers of symmetry c_A and c_B. Then

- $A(t)B(t)$ is PUP with center of symmetry $c_A + c_B$.
- If $c_A = c_B$ then $A(t) + B(t)$ is PUP with center of symmetry c_A.

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- If $c_A = c_B$ then $A(t) + B(t)$ is PUP with center of symmetry c_A.

Center of symmetry:

$$(m-1) + \sum_{i=1}^{m} \frac{k_i - 2}{2} = \frac{1}{2}(n-1).$$

Mahonian Permutation Statistics - q-analogs

Let $\sigma \in \mathfrak{S}_n$.

Inversion Number:

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \le i < j \le n, \quad \sigma(i) > \sigma(j)\}|.$$

$$\mathrm{inv}(3142) = 3$$

Major Index:

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

$$\mathrm{maj}(3142) = \mathrm{maj}(3.14.2) = 1 + 3 = 4$$

Major Percy Alexander MacMahon

(1854 - 1929)

Mahonian Permutation Statistics - q-analogs

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

Mahonian Permutation Statistics - q-analogs

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
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321	3	3

$$\sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_3} q^{\mathrm{maj}(\sigma)} = 1 + 2q + 2q^2 + q^3$$

Mahonian Permutation Statistics - q-analogs

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
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312	2	1
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$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} &= \sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{maj}(\sigma)} \\ &= 1 + 2q + 2q^2 + q^3 \\ &= (1+q+q^2)(1+q)\end{aligned}$$

Mahonian Permutation Statistics - q-analogs

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123	0	0
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213	1	1
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321	3	3

$$\sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{maj}(\sigma)} = 1+2q+2q^2+q^3 = (1+q+q^2)(1+q)$$

Theorem (MacMahon 1905)

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

q-Eulerian polynomials

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv},\mathrm{exc}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

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Theorem (Carlitz 1954)

$$\sum_{i\geq 1}[i]_q^n \ t^i = \frac{t A_n^{\mathrm{maj},\mathrm{des}}(q,t)}{\prod_{i=0}^n(1-tq^i)}$$

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$$A_n^{\mathrm{inv,des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

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Theorem (~~Carlitz 1954~~ MacMahon 1916)

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q-analogs of Euler's exp. generating function formula

Theorem (Stanley 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv,des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

where

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

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Theorem (Shareshian & MW 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj},\mathrm{exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

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q-Eulerian polynomials and q-Eulerian numbers

Theorem (Shareshian & MW 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q, tq^{-1}) \frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

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Specialization of (quasi)symmetric function identity

$$\sum_{n\geq 0}\sum_{j=0}^{n} Q_{n,j}(\mathbf{x}) t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)},$$

- the $Q_{n,j}(\mathbf{x})$ are what we call Eulerian quasisymmetric functions (a sum of certain fundamental quasisymmetric functions)
- $H(z) := \sum_{n\geq 0} h_n(\mathbf{x}) z^n$ (complete homogeneous symmetric functions)

q-Eulerian polynomials and q-Eulerian numbers

Theorem (Shareshian & MW 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q, tq^{-1})\frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}$$

From now on

$$A_n(q,t) := A_n^{\mathrm{maj,exc}}(q, tq^{-1}) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

and

$$\left\langle {n \atop j} \right\rangle_q := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}$$

Palindromicity and unimodality of the q-Eulerian numbers

$n \backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\ldots$	$6+6q+11q^2+\ldots$	$4+3q+5q^2+\ldots$	1

Theorem (Shareshian and MW)

The q-Eulerian polynomial $A_n(q,t) = \sum_{t=0}^{n-1} \left\langle {n \atop j} \right\rangle_q t^j$ is

- *palindromic in the sense that* $\left\langle {n \atop j} \right\rangle_q = \left\langle {n \atop n-1-j} \right\rangle_q$ *for* $0 \leq j \leq \frac{n-1}{2}$
- *q-unimodal in the sense that* $\left\langle {n \atop j} \right\rangle_q - \left\langle {n \atop j-1} \right\rangle_q \in \mathbb{N}[q]$ *for* $1 \leq j \leq \frac{n-1}{2}$

Palindromicity and unimodality of the q-Eulerian numbers

Proof: We use our q-analog of Euler's exponential generating function formula to prove

$$A_n(q,t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1,\dots,k_m \geq 2} \left[\begin{matrix} n \\ k_1 - 1, k_2, \dots, k_m \end{matrix} \right]_q t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t,$$

where

$$\left[\begin{matrix} n \\ k_1, \dots, k_m \end{matrix} \right]_q = \frac{[n]_q!}{[k_1]_q! \cdots [k_m]_q!}$$

Then apply the Sum & Product Lemma.

Geometric Interpretation of the Eulerian numbers

$\left(\left\langle {n \atop 0} \right\rangle, \left\langle {n \atop 1} \right\rangle, \ldots, \left\langle {n \atop n-1} \right\rangle\right)$ is the h-vector of the type A_{n-1} Coxeter complex Δ_n.

Stanley (1980): The h-vector of every simplicial convex polytope is unimodal (and palindromic).

This is part of the celebrated g-theorem of Billera, Lee and Stanley.

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Proof idea: Let P be a d-dimensional convex polytope in $\mathbb{R}^d$ with integer vertices. Let $\mathcal{V}_P$ be the toric variety associated with P.

Danilov (1978): The h-vector of P equals

$$(\dim H^0(\mathcal{V}_P), \dim H^2(\mathcal{V}_P), \ldots, \dim H^{2d}(\mathcal{V}_P))$$

Geometric Interpretation of the Eulerian numbers

$\left(\left\langle {n \atop 0} \right\rangle, \left\langle {n \atop 1} \right\rangle, \ldots, \left\langle {n \atop n-1} \right\rangle\right)$ is the h-vector of the type A_{n-1} Coxeter complex Δ_n.

Stanley (1980): The h-vector of every simplicial convex polytope is unimodal (and palindromic).

This is part of the celebrated g-theorem of Billera, Lee and Stanley.

Proof idea: Let P be a d-dimensional convex polytope in $\mathbb{R}^d$ with integer vertices. Let $\mathcal{V}_P$ be the toric variety associated with P.

Danilov (1978): The h-vector of P equals

$$(\dim H^0(\mathcal{V}_P), \dim H^2(\mathcal{V}_P), \ldots, \dim H^{2d}(\mathcal{V}_P))$$

(Hence $\left\langle {n \atop j} \right\rangle = \dim H^{2j}(\mathcal{V}_{\Delta_n})$.)

Geometric Interpretation of the Eulerian numbers

Theorem (Hard Lefschetz Theorem)

Let $\mathcal{V}$ be a "smooth" irreducible complex projective variety of (complex) dimension m. Then for some $\omega \in H^2(\mathcal{V})$ and all $i = 0, \ldots, m$, the map $H^i(\mathcal{V}) \to H^{2m-i}(\mathcal{V})$, given by multiplication by ω^{m-i} in the singular cohomology ring $H^*(\mathcal{V})$, is a vector space isomorphism.

$$H^{2j}(\mathcal{V}) \xrightarrow{\omega} H^{2(j+1)}(\mathcal{V}) \xrightarrow{\omega} \cdots \xrightarrow{\omega} H^{2m-2j}(\mathcal{V})$$

$\Longrightarrow$ $H^{2j}(\mathcal{V}) \xrightarrow{\omega} H^{2(j+1)}(\mathcal{V})$ is injective.

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Hence $A_n(t) = \sum_{j=0}^{n-1} \dim H^{2j}(\mathcal{V}_{\Delta_n}) t^j$ is unimodal and palindromic

Geometric interpretation of the q-Eulerian numbers

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There is a natural action of $\mathfrak{S}_n$ on the toric variety $\mathcal{V}_{\Delta_n}$ which induces a representation on each $H^{2j}(\mathcal{V}_{\Delta_n})$.

$$\{\text{Representations of } \mathfrak{S}_n\} \xrightarrow{\text{ch}} \Lambda_{\mathbb{Z}}^n \xrightarrow{\text{ps}_q} \mathbb{Z}[q]$$

ch: Frobenius characteristic

$\Lambda_{\mathbb{Z}}^n$: homogeneous symmetric functions over $\mathbb{Z}$ of degree n

ps_q: stable principal specialization.

$$\mathrm{ps}_q(f(x_1, x_2, \ldots)) = f(1, q, q^2, \ldots) \prod_{i=1}^{n} (1 - q^i)$$

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Theorem (Shareshian & MW)

$$\left\langle {n \atop j} \right\rangle_q = \mathrm{ps}_q(\mathrm{ch}\, H^{2j}(\mathcal{V}_{\Delta_n}))$$

Follows from

$$\sum_{n \geq 0} \sum_{j=0}^{n} Q_{n,j}(\mathbf{x}) t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)} = \sum_{n \geq 0} \sum_{j=0}^{n} \mathrm{ch}\, H^{2j}(\mathcal{V}_{\Delta_n}) t^j z^n$$

Shareshian and MW — Procesi and Stanley

Geometric interpretation of the q-Eulerian numbers

The hard Lefschetz map ω commutes with the action of $\mathfrak{S}_n$ on $\mathcal{V} = \mathcal{V}_{\Delta_n}$. This gives $\mathfrak{S}_n$-module maps for $j < \frac{m}{2}$

$$H^{2j}(\mathcal{V}) \xrightarrow{\omega} H^{2(j+1)}(\mathcal{V}) \xrightarrow{\omega} \cdots \xrightarrow{\omega} H^{2m-2j}(\mathcal{V})$$

$\Longrightarrow H^{2j}(\mathcal{V}) \xrightarrow{\omega} H^{2(j+1)}(\mathcal{V})$ is an $\mathfrak{S}_n$-module injection

$\Longrightarrow \mathrm{ch} H^{2(j+1)}(\mathcal{V}) - \mathrm{ch} H^{2j}(\mathcal{V})$ is Schur-positive

$\Longrightarrow \left\langle {n \atop j+1} \right\rangle_q - \left\langle {n \atop j} \right\rangle_q \in \mathbb{N}[q]$

$\Longrightarrow A_n(q,t)$ is q-unimodal and palindromic.

Rawlings' Mahonian permutation statistic (1981)

Let $1 \leq r \leq n$. For $\sigma \in \mathfrak{S}_n$, set

$$\begin{aligned}
\mathrm{inv}_{<r}(\sigma) &:= |\{(i,j) : 1 \leq i < j \leq n, \quad 0 < \sigma(i) - \sigma(j) < r\}|. \\
\mathrm{DES}_{\geq r}(\sigma) &:= \{i \in [n-1] : \sigma(i) - \sigma(i+1) \geq r\} \\
\mathrm{maj}_{\geq r}(\sigma) &:= \sum_{i \in \mathrm{DES}_{\geq r}} i
\end{aligned}$$

$r = 1$: $\quad \mathrm{inv}_{<1}(\sigma) = 0 \qquad \mathrm{maj}_{\geq 1}(\sigma) = \mathrm{maj}(\sigma)$

$r = n$: $\quad \mathrm{inv}_{<n}(\sigma) = \mathrm{inv}(\sigma) \qquad \mathrm{maj}_{\geq n}(\sigma) = 0$

Theorem (Rawlings, 1981)

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma) + \mathrm{inv}_{<r}(\sigma)} = [n]_q!$$

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$$A_n^{(r)}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\ge r}(\sigma)} t^{\mathrm{inv}_{<r}(\sigma)}$$

$A_n^{(r)}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma)}\, t^{\mathrm{inv}_{<r}(\sigma)}$

- $A_n^{(1)}(q,t) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$
- $A_n^{(n)}(q,t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}(\sigma)} = [n]_t!$
- $A_n^{(2)}(q,t) = ?$

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Theorem (Shareshian and MW)

$$A_n^{(2)}(q,t) = A_n(q,t)$$

Proof involves Stanley's theory of P-partitions, Gessel's theory of quasisymmetric functions, our Eulerian quasisymmetric functions.

$A_n^{(r)}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma)}\, t^{\mathrm{inv}_{<r}(\sigma)}$

So $A_n^{(r)}(1,t)$ is a generalized Eulerian polynomial and $A_n^{(r)}(q,t)$ is a generalized q-Eulerian polynomial.

r	$A_n^{(r)}(q,t)$
1	$[n]_q!$
2	$\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor} \sum_{k_1,\ldots,k_m\geq 2} \begin{bmatrix} n \\ k_1-1, k_2, \ldots, k_m \end{bmatrix}_q t^{m-1}\prod_{i=1}^{m}[k_i-1]_t$
$\vdots$	???
$n-2$	$[n]_t[n-3]_t![n-3]_t^2 + [n]_q\ t^{n-3}[n-4]_t![n-2]_t([n-3]_t+[2]_t[n-4]_t) + \begin{bmatrix} n \\ n-2, 2 \end{bmatrix}_q t^{3n-10}[n-4]![n-2]_t[2]_t$
$n-1$	$[n]_t[n-2]_t![n-2]_t + [n]_q t^{n-2}[n-2]_t!$
n	$[n]_t!$

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Conjecture (Shareshian and MW)

$A_n^{(r)}(q,t)$ is q-unimodal (and palindromic).

$A_n^{(r)}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq r}(\sigma)} t^{\mathrm{inv}_{<r}(\sigma)}$

Exercise (Stanley EC1, 1.50 f): Prove that $A_n^{(r)}(1,t)$ is palindromic and unimodal.

Solution:

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Theorem (De Mari and Shayman - 1988)

Let $\mathcal{H}_{n,r}$ be the type A_{n-1} regular semisimple Hessenberg variety of degree r. Then

$$A_n^{(r)}(1,t) = \sum_{j=0}^{d(n,r)} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$

Consequently by the hard Lefschetz theorem, $A_n^{(r)}(1,t)$ is palindromic and unimodal.

Stanley: Is there a more elementary proof of unimodality?

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Shareshian and MW: Is there a q-analog of this result?

($\mathcal{H}_{n,2}$ is the toric variety $\mathcal{V}_{\Delta_n}$)

Hessenberg Varieties (De Mari and Shayman - 1988)

Let $\mathcal{F}_n$ be the set of all flags

$$F : V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

where $\dim V_i = i$. Fix $X \in GL_n(\mathbb{C})$ with n distinct eigenvalues.

The type A regular semisimple Hessenberg variety of degree r is

$$\mathcal{H}_{n,r} := \{F \in \mathcal{F}_n \mid XV_i \subseteq V_{i+r-1} \text{ for all } i\}$$

For q-unimodality we want an action of $\mathfrak{S}_n$ on $\mathcal{H}_{n,r}$.

The Representation

- Tymoczko (2008) used a theory of Goresky, Kottwitz and MacPherson (GKM theory) to define a representation of $\mathfrak{S}_n$ on $H^{2j}(\mathcal{H}_{n,r})$.
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Conjecture (Shareshian and MW)

$$A_n^{(r)}(q,t) = \sum_{j=0}^{d(n,r)} \mathrm{ps}_q(\mathrm{ch} H^{2j}(\mathcal{H}_{n,r}))t^j$$

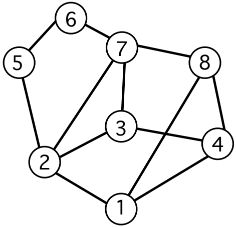
Consequently $A_n^{(r)}(q,t)$ is q-unimodal (and palindromic).

The $r = 2$ case is the toric variety case.

$$A_n^{(2)}(q,t) = A_n(q,t) = \sum_{j=0}^{n-1} \mathrm{ps}_q(\mathrm{ch} H^{2j}(\mathcal{V}_{\Delta_n}))t^j = \sum_{j=0}^{n-1} \mathrm{ps}_q(\mathrm{ch} H^{2j}(\mathcal{H}_{n,2}))t^j$$

Also true for $r = 1, n-2, n-1, n$

Chromatic symmetric functions



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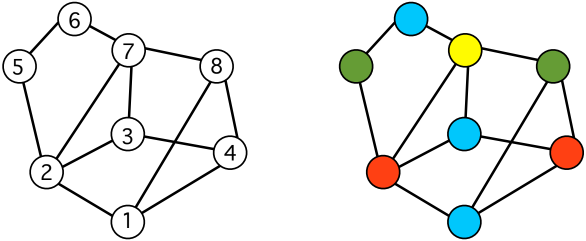

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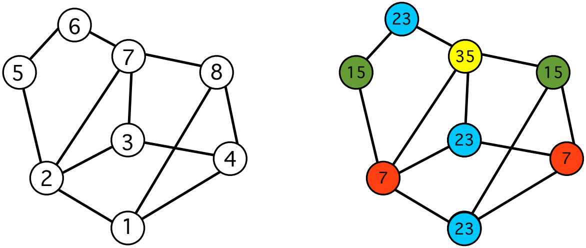
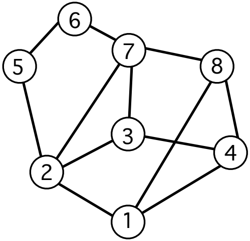
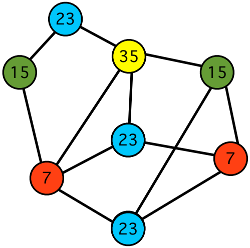


Let $V(G) = \{1, 2, \ldots, n\}$. Let $C(G)$ be set of proper colorings of G, where a proper coloring is a map $c : V(G) \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E(G)$.

Chromatic symmetric functions





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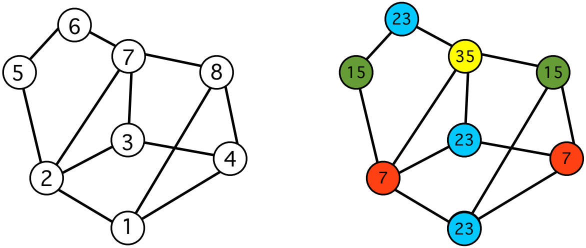
Chromatic symmetric function (Stanley, 1995)

$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

Chromatic symmetric functions



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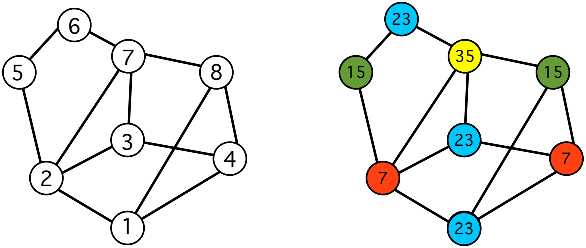
$$X_G(\underbrace{1, 1, \ldots, 1}_{m}, 0, 0, \ldots) = \chi_G(m)$$

Chromatic quasisymmetric function



Chromatic quasisymmetric function (Shareshian and MW)

$$X_G(\mathbf{x}, t) := \sum_{c \in C(G)} t^{\mathrm{des}(c)} x_{c(1)} x_{c(2)} \cdots x_{c(n)}$$

where

$$\mathrm{des}(c) := |\{\{i,j\} \in E(G) : i < j \text{ and } c(i) > c(j)\}|.$$

Chromatic quasisymmetric function

- $e_\lambda(x_1, x_2, \ldots) :=$ elementary symmetric function indexed by partition λ
- $F_\mu(x_1, x_2, \ldots) :=$ fundamental quasisymmetric function indexed by composition μ

$G =$ 

$$X_G(\mathbf{x}, t) = e_3 + (e_3 + e_{2,1})t + e_3 t^2$$

$G =$ 

$$X_G(\mathbf{x}, t) = (e_3 + F_{1,2}) + 2e_3 t + (e_3 + F_{2,1})t^2$$

Formulae

Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\{i, j\} \mid 0 < |j - i| < r\}$.

r	$X_{G_{n,r}}(\mathbf{x}, t)$
1	e_1^n
2	$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{\substack{k_1,\ldots,k_m \geq 2 \\ \sum k_i = n+1}} e_{k_1-1,k_2,\ldots,k_m} \; t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$
$\vdots$	???
$n-2$	$e_n [n]_t [n-3]_t! [n-3]_t^2 + e_{n-1,1} \; t^{n-3} [n-4]_t! [n-2]_t ([n-3]_t + [2]_t [n-4]_t)$ $+ e_{n-2,2} t^{3n-10} [n-4]! [n-2]_t [2]_t$
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Theorem (Shareshian and MW)

$$A_n^{(r)}(q,t) = \mathrm{ps}_q(\omega X_{G_{n,r}}(\mathbf{x}, t))$$

The Main Conjecture

Conjecture (Shareshian and MW)

Let G be the incomparability graph of a natural unit interval order. Then

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Theorem (Shareshian and MW - t-analog of Gasharov)

Let G be the incomparability graph of a natural unit interval order P. Then

$$X_G(\mathbf{x}, t) = \sum_{T \in \mathcal{T}_P} t^{\mathrm{inv}_G(T)} s_{\lambda(T)},$$

where $\mathcal{T}_P$ is the set of P-tableaux, inv_G is an inversion statistic on tableaux, and $\lambda(T)$ is the shape of T.

Refinement of Stanley-Stembridge Conjecture

Conjecture (Shareshian and MW)

Let G be the incomparability graph of a natural unit interval order. Then $X_G(\mathbf{x}, t)$ is e-positive and e-unimodal.

Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\{i, j\} \mid 0 < |j - i| < r\}$.

r	$X_{G_{n,r}}$
1	e_{1^n}
2	$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{\substack{k_1,\ldots,k_m \geq 2 \\ \sum k_i = n+1}} e_{k_1-1,k_2,\ldots,k_m} \; t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$
$\vdots$	???
$n-2$	$e_n [n]_t [n-3]_t! [n-3]_t^2 + e_{n-1,1} \; t^{n-3} [n-4]_t! [n-2]_t ([n-3]_t + [2]_t [n-4]_t)$ $+ e_{n-2,2} t^{3n-10} [n-4]! [n-2]_t [2]_t$
$n-1$	$e_n [n]_t [n-2]_t! [n-2]_t + e_{n-1,1} t^{n-2} [n-2]_t!$
n	$e_n [n]_t!$