4. Lecture IV: New Horizons

In this lecture we describe recent developments regarding chain enumeration and the **cd**-index which involve algebra, graph theory and topology. The first is a non-homogeneous **cd**-index for Bruhat graphs due to Billera and Brenti [8]. One motivation for studying the **cd**-index of Bruhat graphs is that the **cd**-index of the interval $[u, v]$ determines the Kazhdan–Lusztig polynomial $P_{u,v}(q)$; see [8, Section 3]. These polynomials arise out of Kazhdan and Lusztig's study of the Springer representations of the Hecke algebra of a Coxeter group [48, 49]. The Kazhdan–Lusztig polynomials have many applications, including to Verma modules and to the algebraic geometry and topology of Schubert varieties. See Section 4.1 for a further discussion.

The second recent development is the theory of balanced graphs, due to Ehrenborg and Readdy [35]. This theory relaxes the graded, poset and Eulerian requirements for chain enumeration in graded posets. Bruhat graphs are a special case of balanced graphs, and the theory simplifies the proof techniques from using quasi-symmetric theory to edge labelings in the graphs. In the case a balanced graph has a *linear edge labeling*, the authors conjecture the **cd**-index has nonnegative coefficients.

The third development is both a topological and poset theoretic generalization of flag enumeration. Ehrenborg, Goresky and Readdy have extended the theory of face incidence enumeration of polytopes, and more generally, chain enumeration in graded Eulerian posets, to that of Whitney stratified spaces and quasi-graded posets [27]. It is important to point out that, unlike the case of polytopes, the coefficients of the **cd**-index of Whitney stratied manifolds can be negative. It is hoped that by applying topological techniques to stratified manifolds, a tractable interpretation of the coefficients of the **cd**-index will emerge. This may ultimately explain Stanley's nonnegativity results for spherically shellable posets [65] and Karu's results for Gorenstein* posets [45], and settle the conjecture that non-negativity holds for regular cell complexes

4.1. **Bruhat graphs.**

Another family of Eulerian posets is formed by taking the (strong) Bruhat order on a Coxeter group [70]. Hence any interval has a **cd**-index which is homogeneous of degree one more than the length of the interval. By removing the adjacent rank assumption on the cover relation of the Bruhat order, a directed graph known as the Bruhat graph is obtained which in effect allows algebraic "short cuts" between elements.

More formally, let (W, S) be a Coxeter system, where W denotes a (finite or infinite) Coxeter group with generators S and $\ell(u)$ denotes the length of a group element u. Let T be the set of reflections, that is, $T = \{w \cdot s \cdot w^{-1}\ :$

$s \in S, w \in W\}$. The *Bruhat graph* has the group W as its vertex set and its set of labels Λ is the set of reflections T. The edges and their labeling are defined as follows. There is a directed edge from u to v labeled t if $u \cdot t = v$ and $\ell(u) < \ell(v)$. The underlying poset of the Bruhat graph is called the *(strong) Bruhat order*. It is important to note that every interval of the Bruhat order is Eulerian, that is, every interval $[x, y]$ has Möbius function given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$, where ρ denotes the rank function. For a more complete description of Coxeter systems, see Björner and Brenti's text [17].

Using the fact that the generalized Dehn–Sommerville relations hold for coefficients of polynomials arising in Kazhdan–Lusztig polynomials [19, Theorem 8.4] and quasisymmetric functions, Billera and Brenti show that the Bruhat graph has a non-homogeneous **cd**-index [8].

Theorem 4.1.1 (Billera–Brenti)**.** *For an interval $[u, v]$ in the Bruhat order, where $u < v$, the following three conditions hold:*

(i) *The interval $[u, v]$ in the Bruhat graph has a **cd**-index $\Psi([u, v])$.*
(ii) *Restricting the **cd**-index $\Psi([u, v])$ to those terms of degree $\ell(v) - \ell(u) - 1$ equals the **cd**-index of the graded poset $[u, v]$.*
(iii) *The degree of a term in the **cd**-index $\Psi([u, v])$ is less than or equal to $\ell(v) - \ell(u) - 1$ and has the same parity as $\ell(v) - \ell(u) - 1$.*

For an alternate proof using labelings of balanced graphs, see [35].

4.2. **Bruhat and balanced graphs.**

The notion of a labeled acyclic digraph was introduced in [35] in order to model poset structure in this more general setting.

Let $G = (V, E)$ be a directed, acyclic and locally finite graph with multiple edges allowed. Recall that an *acyclic graph* does not have any directed cycles and the property of a graph being *locally finite* requires that there are a finite number of paths between any two vertices. Each directed edge e has a tail and a head, denoted respectively by $\text{tail}(e)$ and $\text{head}(e)$. View each directed edge as an arrow from its tail to its head. A directed path p of length k from a vertex x to a vertex y is a list of k directed edges $(e_1, e_2, \ldots, e_k)$ such that $\text{tail}(e_1) = x$, $\text{head}(e_k) = y$ and $\text{head}(e_i) = \text{tail}(e_{i+1})$ for $i = 1, \ldots, k-1$. We denote the length of a path p by $\ell(p)$.

Since the graph is acyclic, it does not have any loops. Furthermore, the acyclicity condition implies there is a natural partial order on the vertices of G by defining the order relation $x \leq y$ if there is a directed path from the vertex x to the vertex y. It is straightforward to verify that this relation is reflexive, antisymmetric and transitive. It allows us to define the *interval* $[x, y]$ to be the set of all vertices z in $V(G)$ such that there is a directed path from x to z and a directed path from z to y. We view the interval $[x, y]$ as the vertex-induced subgraph of the digraph G, where the edges have the same labels as in the digraph G. The locally finite condition is now equivalent to that every interval $[x, y]$ in the graph has finite cardinality.

We next relax the notions of R-labeling and the **ab**-index of a poset. Let Λ be a set with a relation $\sim$, that is, there is a subset $R \subseteq \Lambda \times \Lambda$ such that for $i, j \in \Lambda$ we have $i \sim j$ if and only if $(i, j) \in R$. A *labeling of* G is a function λ from the set of edges of G to the set Λ. Let **a** and **b** be two non-commutative variables each of degree one. For a path $p = (e_1, \ldots, e_k)$ of length k, where $k \geq 1$, we define the *descent word* $u(p)$ to be the **ab**-monomial $u(p) = u_1 u_2 \cdots u_{k-1}$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } \lambda(e_i) \sim \lambda(e_{i+1}), \\ \mathbf{b} & \text{if } \lambda(e_i) \not\sim \lambda(e_{i+1}). \end{cases}$$

Observe that the descent word $u(p)$ has degree $k - 1$, that is, one less than the length of the path p. The **ab**-*index* of an interval $[x, y]$ is defined to be

$$\Psi([x, y]) = \sum_{p} u(p), \tag{4.1}$$

where the sum is over all directed paths p from x to y.

An analogue of the coalgebraic groundwork for for flag enumeration in posets holds for labeled acyclic digraphs. More specifically, the **ab**-index of a labeled acyclic digraph is a coalgebra homeomorphism from the linear span of bounded labeled acyclic digraphs to the polynomial ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$.

The following result gives three equivalent statements which imply the (non-homogeneous) **ab**-index of an acyclic digraph can be written as a (non-homogeneous) **cd**-index [35].

Theorem 4.2.1 (Ehrenborg–Readdy)**.** *For a labeled acyclic digraph G, the following three statements are equivalent:*

(i) *For every interval $[x, y]$ in the digraph G and for every non-negative integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(ii) *For every interval $[x, y]$ in the digraph G and for every* even *positive integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(iii) *The* **ab**-*index of every interval $[x, y]$ in the digraph G, where $x < y$, is a polynomial in $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.*

Definition 4.2.2. *A labeled acyclic digraph G is said to be* balanced *if it satisfies condition (i) in Theorem 4.2.1. Such a labeling is called a* balanced labeling *or B-*labeling *for short.*

An edge labeling *linear* if the underlying relation $(\Lambda, \sim)$ is that of a linear order.

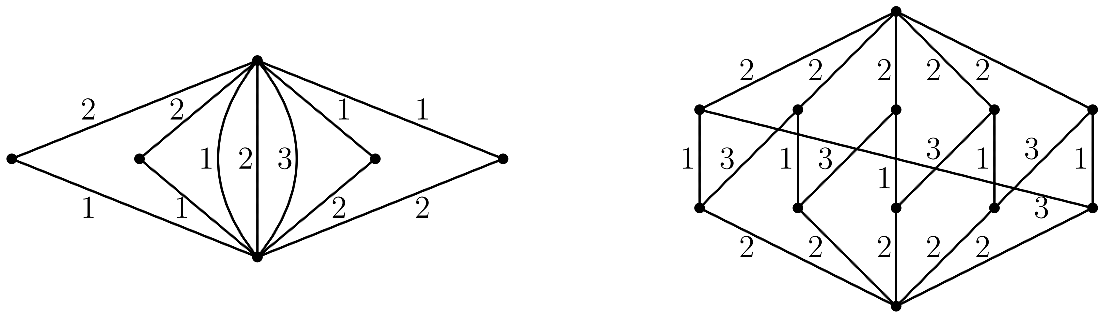

FIGURE 2. Two balanced directed graphs where the relation on the labeled set $\Lambda = \{1, 2, 3\}$ is the natural linear order. Their respective **cd**-indexes are $2 \cdot \mathbf{c} + 3$ and $5 \cdot \mathbf{d}$. These two examples show that the **cd**-index of a graph is not necessarily homogeneous and that the coefficient of the **c**-power term is not necessarily 1.

Theorem 4.2.3 (Ehrenborg–Readdy)**.** *Let u be a non-zero* **cd**-*polynomial with non-negative coefficients. Then there exists a bounded balanced labeled acyclic digraph G where the relation on the set of labels is a linear order and which satisfies $\Psi(G) = w$.*

Theorem 4.2.3 motivates the following conjecture.

Conjecture 4.2.4 (Ehrenborg–Readdy)**.** *The* **cd**-*index of a bounded labeled acyclic digraph G with a balanced linear edge labeling is non-negative.*

4.3. Euler flag enumeration of Whitney stratified spaces.

We begin with a modest example.

Example 4.3.1. Consider the non-regular CW-complex Ω consisting of one vertex v, one edge e and one 2-dimensional cell c such that the boundary of c is the union $v \cup e$, that is, boundary of the complex Ω is a one-gon. Its face poset is the four element chain $\mathscr{F}(\Omega) = \{\hat{0} < v < e < c\}$. This is not an Eulerian poset. The **ab**-index of Ω is $\mathbf{a}^2$. Note that $\mathbf{a}^2$ cannot be written in terms of **c** and **d**.

Observe that the edge e is attached to the vertex v twice. Hence it is natural to change the value of f_{01} to 2. This changes h_{01} to be 1. The **ab**-index becomes $\Psi(\Omega) = \mathbf{a}^2 + \mathbf{b}^2$ and hence its **cd**-index is $\Psi(\Omega) = \mathbf{c}^2 - \mathbf{d}$.

The *Euler characteristic* of an n-dimensional polytopal complex Δ is defined as the alternating sum of its face numbers, that is,

$$\chi(\Delta) = f_0(\Delta) - f_1(\Delta) + f_2(\Delta) - \cdots + (-1)^n \cdot f_n(\Delta).$$

This is a topological invariant, that is, any two complexes that are homotopy equivalent have the same Euler characteristic. Especially, any contractible space has Euler characteristic 1.

The motivation for the value 2 in Example 4.3.1 is best expressed in terms of the Euler characteristic of the link. The link of the vertex v in the edge e is two points whose Euler characteristic is 2. In order to view this example in the right topological setting, we review the notion of a Whitney stratification. For more details, see [23, 37, 38, 53].

A subset S of a topological space M is *locally closed* if S is a relatively open subset of its closure $\overline{S}$. Equivalently, for any point $x \in S$ there exists a neighborhood $U_x \subseteq S$ such that the closure $\overline{U_x} \subseteq S$ is closed in M. Another way to phrase this is a subset $S \subset M$ is locally closed if and only if it is the intersection of an open subset and a closed subset of M.

Definition 4.3.2. *Let W be a closed subset of a smooth manifold M which has been decomposed into a finite union of locally closed subsets*

$$W = \bigcup_{X \in \mathcal{P}} X.$$

Furthermore suppose this decomposition satisfies the condition of the frontier*:*

$$X \cap \overline{Y} \neq \emptyset \Longleftrightarrow X \subseteq \overline{Y}.$$

This implies the closure of each stratum is a union of strata, and it provides the index set $\mathcal{P}$ with the partial ordering:

$$X \subseteq \overline{Y} \Longleftrightarrow X \leq_{\mathcal{P}} Y.$$

This decomposition of W is a Whitney stratification *if*

1. *Each $X \in \mathcal{P}$ is a (locally closed, not necessarily connected) smooth submanifold of M.*
2. *If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the following inclusions hold:*

$$\text{(A)}\ T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B)}\ \ell \subseteq \tau.$$

Remark 4.3.3. For convenience we will henceforth also assume that W is pure dimensional, meaning that if $\dim(W) = n$ then the union of the n-dimensional strata of W forms a dense subset of W. Strata of dimension less than n are referred to as *singular strata*.

Whitney's conditions A and B ensure there is no fractal behavior and no infinite wiggling. A crucial result is that the links are well-defined in a Whitney stratification. See [27].

Recall the *incidence algebra* of a poset P is the set of all functions $f : I(P) \to \mathbb{C}$ where $I(P)$ denotes the set of intervals in the poset. The multiplication is given by $(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y)$ and the identity is given by the delta function $\delta(x, y) = \delta_{x,y}$, where the second delta is the usual Kronecker delta function $\delta_{x,y} = 1$ if $x = y$ and zero otherwise. The *zeta function* ζ is defined by $\zeta(x, y) = 1$ if $x \leq y$ in the poset P and 0 otherwise. The *Möbius function* μ is the inverse of the zeta function in the incidence algebra, that is, $\mu * \zeta = \zeta * \mu = \delta$.

Recall a poset is said to be *ranked* if every maximal chain in the poset has the same length. This common length is called the *rank* of the poset. A poset is said to be *graded* if it is ranked and has a minimal element $\hat{0}$ and a maximal element $\hat{1}$. For other poset terminology, we refer the reader to Stanley's text [66].

We introduce the notion of a quasi-graded poset. This extends the notion of a ranked poset.

Definition 4.3.4. *A quasi-graded poset* $(P, \rho, \bar{\zeta})$ *consists of*

(i) *a finite poset* P *(not necessarily ranked),*
(ii) *a strictly order-preserving function* ρ *from* P *to* $\mathbb{N}$*, that is,* $x < y$ *implies* $\rho(x) < \rho(y)$ *and*
(iii) *a function* $\bar{\zeta}$ *in the incidence algebra* $I(P)$ *of the poset* P*, called the* weighted zeta function*, such that* $\bar{\zeta}(x, x) = 1$ *for all elements* x *in the poset* P*.*

Observe that we do not require the poset to have a minimal element or a maximal element. Since $\bar{\zeta}(x, x) \neq 0$ for all $x \in P$, the function $\bar{\zeta}$ is invertible in the incidence algebra $I(P)$ and we denote its inverse by $\bar{\mu}$.

For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the face poset of a Whitney stratified space, define

$$\bar{\zeta}(c) = \chi(c_1) \cdot \chi(\mathrm{link}_{x_2}(x_1)) \cdots \chi(\mathrm{link}_{x_{k-1}}(x_k)),$$

where χ denotes the Euler characteristic.

The usual **ab**-index for polytopes and Eulerian posets is via the flag f- and flag h-vectors. We extend this route by introducing the flag $\bar{f}$- and flag $\bar{h}$-vectors. Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset of rank $n+1$ having a $\hat{0}$ and $\hat{1}$ such that $\rho(\hat{0}) = 0$. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset of $\{1, \ldots, n\}$, define the *flag* $\bar{f}$*-vector* by

$$\bar{f}_S = \sum_c \bar{\zeta}(c), \tag{4.2}$$

where the sum is over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}\}$ in P such that $\rho(x_i) = s_i$ for all $1 \leq i \leq k$. The *flag* $\bar{h}$*-vector* is defined by the relation (and by inclusion–exclusion, we also display its inverse relation)

$$\bar{h}_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \bar{f}_T \quad \text{and} \quad \bar{f}_S = \sum_{T \subseteq S} \bar{h}_T. \tag{4.3}$$

For a subset $S \subseteq \{1, \ldots, n\}$ define the **ab**-monomial $u_S = u_1 u_2 \cdots u_n$ by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The **ab**-index of the quasi-graded poset $(P, \rho, \bar{\zeta})$ is then given by

$$\Psi(P, \rho, \bar{\zeta}) = \sum_S \bar{h}_S \cdot u_S,$$

where the sum ranges over all subsets S. Again, in the case when we take the weighted zeta function to be the usual zeta function ζ, the flag $\bar{f}$ and flag $\bar{h}$-vectors correspond to the usual flag f- and flag h-vectors.

Definition 4.3.5. *A quasi-graded poset is said to be* Eulerian *if for all pairs of elements $x \leq z$ we have that*

$$\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y) \cdot \bar{\zeta}(y, z) = \delta_{x,z}. \tag{4.4}$$

In other words, the function $\bar{\mu}(x, y) = (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y)$ is the inverse of $\bar{\zeta}(x, y)$ in the incidence algebra. In the case $\bar{\zeta}(x, y) = \zeta(x, y)$, we refer to relation (4.4) *as the* classical Eulerian relation*.*

Generalizing the classical result of Bayer and Klapper for graded Eulerian posets, we have the analogue for quasi-graded posets.

Theorem 4.3.6 (Ehrenborg–Goresky–Readdy)**.** *For an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$ its* **ab***-index $\Psi(P, \rho, \bar{\zeta})$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$.*

Theorem 4.3.7 (Ehrenborg–Goresky–Readdy)**.** *Let M be a manifold with a Whitney stratified boundary. Then the face poset is quasi-graded and Eulerian, with*

$$\rho(x) = \dim(x) + 1$$

and

$$\bar{\zeta}(x, y) = \chi(\mathrm{link}_y(x)).$$

We now give a few examples of Whitney stratifications beginning with the classical polygon.

Example 4.3.8. Consider a two dimensional cell c with its boundary subdivided into n vertices $v_1, \ldots, v_n$ and n edges $e_1, \ldots, e_n$. There are three ways to view this as a Whitney stratification.

(1) Declare each of the $2n + 1$ cells to be individual strata. This is the classical view of an n-gon. Here the weighted zeta function is the classical zeta function, that is, always equal to 1 (assuming $n \geq 2$).
(2) Declare each of the n edges to be one stratum $e = \cup_{i=1}^{n} e_i$, that is, we have the $n + 2$ strata $v_1, \ldots, v_n, e, c$. Here the non-one values of the weighted zeta function are given by $\bar{\zeta}(\hat{0}, e) = n$ and $\bar{\zeta}(v_i, e) = 2$.

S	$\bar{f}_S$	$\bar{h}_S$	$\mathbf{c}^3$	$-\mathbf{cd}$
$\emptyset$	1	1	1	0
$\{0\}$	2	1	1	0
$\{1\}$	1	0	1	-1
$\{2\}$	1	0	1	-1
$\{0,1\}$	2	0	1	-1
$\{0,2\}$	2	0	1	-1
$\{1,2\}$	2	1	1	0
$\{0,1,2\}$	4	1	1	0

TABLE 2. The flag $\bar{f}$- and flag $\bar{h}$-vectors, **ab**-index and **cd**-index of the sphere with an edge on it. The sum of the last two columns equals the flag h column, showing the **cd**-index is $\mathbf{aaa} + \mathbf{baa} + \mathbf{abb} + \mathbf{bbb} = \mathbf{c}^3 - \mathbf{cd}$.

(3) Lastly, we can have the three strata $v = \cup_{i=1}^{n} v_i$, $e = \cup_{i=1}^{n} e_i$ and c. Now non-one values of the weighted zeta function are given by $\bar{\zeta}(\hat{0}, v) = \bar{\zeta}(\hat{0}, e) = n$ and $\bar{\zeta}(v, e) = 2$.

In contrast, we cannot have $v, e_1, \ldots, e_n, c$ as a stratification, since the link of a point p in e_i depends on the point p in v chosen.

The **cd**-index of each of the three Whitney stratifications in Example 4.3.8 are the same, that is, $\mathbf{c}^2 + (n-2) \cdot \mathbf{d}$. Hence we have the immediate corollary.

Corollary 4.3.9. *The* **cd***-index of an n-gon is given by* $\mathbf{c}^2 + (n-2) \cdot \mathbf{d}$ *for* $n \geq 1$.

The last stratification in the previous example can be be extended to any simple polytope.

Example 4.3.10. Let P be an n-dimensional simple polytope. Recall that the simple condition that implies that every interval $[x, y]$, where $\hat{0} < x \leq y$, is a Boolean algebra. We obtain a different stratification of the ball by joining all the facets together to one strata. We note that the **cd**-index does not change, since the information is carried in the weighted zeta function. We continue by joining all the subfacets together to one strata. Again the **cd**-index remains unchanged. In the end we obtain a stratification where the union of all the i-dimensional faces forms the ith strata. The face poset of this stratification is the $(n+2)$-element chain $C = \{\hat{0} = x_0 < x_1 < \cdots < x_{n+1} = \hat{0}\}$, with the rank function $\rho(x_i) = i$ and weighted zeta function $\bar{\zeta}(\hat{0}, x_i) = f_{i-1}(P)$ and $\bar{\zeta}(x_i, x_j) = \binom{n+1-i}{n+1-j}$. We have $\Psi(C, \rho, \bar{\zeta}) = \Psi(P)$.

A similar stratification can be obtained for any regular polytope.

Example 4.3.11. Consider the 2-sphere with an edge with two incident vertices on it. See Table 2 for the **cd**-index computation.

Example 4.3.12. Consider the stratification of an n-dimensional manifold with boundary, denoted $(M, \partial M)$, into its boundary ∂M and its interior M°. The face poset is $\{\hat{0} < \partial M < M^\circ\}$ with the elements having ranks 0, n and $n+1$, respectively. The weighted zeta function is given by $\bar{\zeta}(\hat{0}, \partial M) = \chi(\partial M)$, $\bar{\zeta}(\hat{0}, M^\circ) = \chi(M)$ and $\bar{\zeta}(\partial M, M^\circ) = 1$. If n is even then ∂M is an odd-dimensional manifold without boundary and hence its Euler characteristic is 0. In this case the **ab**-index is $\Psi(M) = \chi(M) \cdot (\mathbf{a} - \mathbf{b})^n$. If n is odd then we have the relation $\chi(\partial M) = 2 \cdot \chi(M)$ and hence the **ab**-index is given by $\Psi(M) = \chi(M) \cdot (\mathbf{a} - \mathbf{b})^n + 2 \cdot \chi(M) \cdot (\mathbf{a} - \mathbf{b})^{n-1} \cdot \mathbf{b}$. Passing to the **cd**-index we conclude

$$\Psi(M) = \begin{cases} \chi(M) \cdot (\mathbf{c}^2 - 2\mathbf{d})^{n/2} & \text{if } n \text{ is even,} \\ \chi(M) \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c} & \text{if } n \text{ is odd.} \end{cases}$$

The next example is a higher dimensional analogue of the one-gon in Example 4.3.1.

Example 4.3.13. Consider the subdivision Ω_n of the n-dimensional ball $\mathbb{B}^n$ consisting of a point p, an $(n-1)$-dimensional cell c and the interior b of the ball. If $n \geq 2$, the face poset is $\{\hat{0} < p < c < b\}$ with the elements having ranks 0, 1, n and $n+1$, respectively. In the case $n = 1$, the two elements p and c are incomparable. The weighted zeta function is given by $\bar{\zeta}(\hat{0}, p) = \bar{\zeta}(\hat{0}, c) = \bar{\zeta}(\hat{0}, b) = 1$, $\bar{\zeta}(p, c) = 1 + (-1)^n$, and $\bar{\zeta}(p, b) = \bar{\zeta}(c, b) = 1$. Thus the **ab**-index is

$$\Psi(\Omega_n) = (\mathbf{a}-\mathbf{b})^n + \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-1} + (\mathbf{a}-\mathbf{b})^{n-1} \cdot \mathbf{b} + (1+(-1)^n) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{b}. \tag{4.5}$$

When n is even the expression (4.5) simplifies to

$$\begin{aligned} \Psi(\Omega_n) &= \mathbf{a} \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot \mathbf{a} + \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot \mathbf{b} \\ &= \frac{1}{2} \cdot \left[(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot (\mathbf{a} - \mathbf{b}) + (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot (\mathbf{a} + \mathbf{b})\right] \\ &= \frac{1}{2} \cdot \left[(\mathbf{c}^2 - 2\mathbf{d})^{n/2} + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-2)/2} \cdot \mathbf{c}\right]. \end{aligned} \tag{4.6}$$

When n is odd the expression (4.5) simplifies to

$$\begin{aligned} \Psi(\Omega_n) &= \mathbf{a} \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot \mathbf{a} - \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot \mathbf{b} \\ &= \frac{1}{2} \cdot \left[(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot (\mathbf{a} - \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})^{n-2} \cdot (\mathbf{a} + \mathbf{b})\right] \\ &= \frac{1}{2} \cdot \left[\mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} + (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c}\right]. \end{aligned} \tag{4.7}$$

As a remark, these **cd**-polynomials played an important role in proving that the **cd**-index of a polytope is coefficient-wise minimized on the simplex, namely, $\Psi(\Omega_n) = (-1)^{n-1} \cdot \alpha_n$, where α_n are defined in [9].

Open question 4.3.14. *Find the linear inequalities that hold among the entries of the* **cd***-index of a Whitney stratified manifold.*

This expands the program of determining linear inequalities for flag vectors of polytopes. Since the coefficients may be negative, one must ask what should the new minimization inequalities be. Observe that Kalai's convolution [44] still holds. More precisely, let M and N be two linear functionals defined on the **cd**-coefficients of any m-dimensional, respectively, n-dimensional manifold. If both M and N are non-negative then their convolution is non-negative on any $(m+n+1)$-dimensional manifold.

Other inequality questions are:

Open question 4.3.15. *Can Ehrenborg's lifting technique* [26] *be extended to stratified manifolds?*

Open question 4.3.16. *What non-linear inequalities hold among the* **cd**-*coefficients?*

One interpretation of the coefficients of the **cd**-index is due to Karu [45] who, for each **cd**-monomial, gave a sequence of operators on sheaves of vector spaces to show the non-negativity of the coefficients of the **cd**-index for Gorenstein* posets [45].

Open question 4.3.17. *Is there a signed analogue of Karu's construction to explain the negative coefficients occurring in the* **cd**-*index of quasi-graded posets?*