2. Lecture II: Coalgebraic techniques and geometric operations

It will be useful to view the face structure of a polytope in terms of its *face lattice*, that is, the partially ordered set consisting of the faces of a polytope ordered by inclusion. We will see that geometric operations on polytopes correspond to poset operations on the face lattice, and hence, to operations on any "reasonable" poset. The resulting **ab**-index of the prism and pyramid operations strongly suggest an underlying coalgebraic structure, which we also introduce.

2.1. Posets, polytopes and geometric operations on them.

Recall a *partially ordered set* P, or *poset* for short, consists of a finite number of elements with a partial order $\leq$ which is

(1) reflexive: $x \leq x$ for all elements $x \in P$,
(2) antisymmetric: if $x \leq y$ and $y \leq x$ then $x = y$,
(3) transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$.

Most of the posets we will work with will have a unique minimal and maximal elements, denoted by $\hat{0}$ and $\hat{1}$ respectively. Additionally, we say a poset P with unique minimal and maximal elements is *graded* if any saturated chain of elements from $\hat{0}$ to x, that is, $c = \{\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k = x\}$ has the same length for a fixed element $x \in P$. We call this length the *rank* of x, denoted $\rho(x)$ and the rank of a graded poset is $\rho(\hat{1})$. A poset is a *lattice* if every pair of elements has a unique least upper bound and unique greatest lower bound. For more information about posets, we refer the reader to [66, Chapter 3].

There are a number of operations of posets we will need. Given posets P and Q, the Cartesian product is

$$P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}$$

with the partial order $(p, q) \leq_{P\times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$.

Example 2.1.1. The Boolean algebra. The *Boolean algebra* B_n consisting of all subsets of $\{1, \ldots, n\}$ ordered by inclusion can be realized as the product $B_n \cong \underbrace{B_1 \times \cdots \times B_1}_{n}$. As a remark, the Boolean algebra B_{n+1} is isomorphic to the face lattice of the n-simplex.

Assuming P and Q are bounded posets, that is, each has a unique minimal and maximal element, the *diamond product* is

$$P \diamond Q = (P - \{\hat{0}_P\}) \times (Q - \{\hat{0}_Q\}) \cup \{\hat{0}\}$$

and the *Stanley product* $P * Q$ consists of the elements

$$P * Q = (P - \{\hat{1}_P\}) \cup (Q - \{\hat{0}_Q\})$$

with the order relation $x \leq_{P*Q} y$ if (i) $x, y \in P$ and $x \leq_P y$, (ii) $x, y \in Q$ and $x \leq_Q y$, or (iii) $x \in P$ and $y \in Q$. Finally, the *dual* of a poset P is the poset P^* where the order relation is $x \leq_{P^*} y$ if and only if $y \leq_P x$.

In important subclass of graded posets are the *Eulerian posets*. These satisfy the condition that $\mu(x, y) = (-1)^{\rho(x,y)}$, where $\rho(x, y) = \rho(y) - \rho(x)$ and the *Möbius function* is defined by $\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$. Equivalently, in every non-trivial interval the number of elements of even rank equals the number of elements of odd rank. One important family of Eulerian posets are the face lattices of convex polytopes.

We can now discuss some geometric operations on polytopes. For P an n-polytope in $\mathbb{R}^n$, embed P in $\mathbb{R}^{n+1}$. (For instance, let the $(n+1)$st coordinate of each point in P be zero.) The *pyramid* of P is

$$\mathrm{Pyr}(P) = \mathrm{conv}(P \cup \{x\}),$$

where x is a point outside the affine hull of P. For example, $\Delta_n = \mathrm{Pyr}(\Delta_{n-1}) = \mathrm{Pyr}^n(\text{point})$. Note the dimension of $\mathrm{Pyr}(P)$ is one more than the dimension of P. The *bipyramid* of P is

$$\mathrm{Bipyr}(P) = \mathrm{conv}(P \cup \{x_+\} \cup \{x_-\}),$$

where x_+ and x_- are two points outside of the affine hull of P such that some point in the open interval (x_+, x_-) intersects the interior of P. Note that the n-dimensional cross-polytope is the repeated application of the bipyramid operation, starting with a point.

For $P \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^q$ two polytopes of dimension p and q, respectively, their *Cartesian product of polytopes* is

$$P \times Q = \{(x_1, \ldots, x_p, y_1, \ldots, y_q) = (\vec{x}, \vec{y}) : \vec{x} \in P, \vec{y} \in Q\}$$

Observe $\dim(P \times Q) = \dim(P) + \dim(Q)$. The *prism* of P is

$$\mathrm{Prism}(P) = P \times [0, 1],$$

where $[0, 1]$ is the line segment from 0 to 1. Thus the n-cube is $C_n = \mathrm{Prism}(C_{n-1})$.

The *free join* $P * Q$ of two polytopes P and Q is formed in the following manner. Embed P in the p-dimensional affine subspace of $\mathbb{R}^{p+q+1}$ as $\{(x_1, \ldots, x_p, 0, \ldots, 0) : \vec{x} \in P\}$. Embed Q in a q-dimensional affine subspace as $\{(0, \ldots, 0, y_1, \ldots, y_q, 1) : \vec{y} \in Q\}$. Then take the convex hull of these two embeddings. The resulting polytope has dimension $p + q + 1$. Geometrically the free join corresponds to putting the two polytopes P and Q in orthogonal non-intersecting affine subspaces of $\mathbb{R}^{p+q+1}$ and then taking the convex hull.

It is natural to ask how geometric operations on a polytope, such as the prism and pyramid, change the **cd**-index of the original polytope. We first consider the change on the face lattice itself [44].

Proposition 2.1.2 (Kalai)**.** *For two convex polytopes P and Q we have*

$$\begin{aligned} \mathscr{L}(P * Q) &= \mathscr{L}(P) \times \mathscr{L}(Q), \\ \mathscr{L}(P \times Q) &= \mathscr{L}(P) \diamond \mathscr{L}(Q). \end{aligned}$$

Especially,

$$\mathscr{L}(\mathrm{Pyr}(P)) = \mathscr{L}(P) \times B_1 \textit{ and } \mathscr{L}(\mathrm{Prism}(P)) = \mathscr{L}(P) \diamond B_2.$$

Definition 2.1.3. *For a graded poset P, define the pyramid and prism operations by* $\mathrm{Pyr}(P) = P \times B_1$ *and* $\mathrm{Prism}(P) = P \diamond B_2$.

An equivalent definition of the **ab**-index is as follows. Let P be a graded poset of rank $n+1$. Given a chain $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ of P, we associate a weight $w(c) = z_1 \cdots z_n$, where

$$z_i = \begin{cases} \mathbf{a} - \mathbf{b} & \text{if } i \notin \{\rho(x_1), \ldots, \rho(x_k)\}, \\ \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}. \end{cases}$$

Then the **ab**-*index* of a poset P is given by

$$\Psi(P) = \sum_c w(c),$$

where the sum is over all chains c in P. Observe that this is just a way to directly compute the flag h-vector from the flag f-vector, rather than having to compute the flag h-vector via certain alternating sums of the flag f-vector entries. For example, for the face lattice of a hexagon, we have $\Psi(\text{hexagon}) = 1(\mathbf{a}-\mathbf{b})^2 + 6\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) + 6(\mathbf{a}-\mathbf{b})\cdot\mathbf{b} + 12\mathbf{bb} = \mathbf{a} + 5\mathbf{ab} + 5\mathbf{ba} + \mathbf{bb}$.

Proposition 2.1.4. [31] *Let P be a graded poset. Then*

$$\begin{aligned} \Psi(\mathrm{Pyr}(P)) &= \frac{1}{2}\left[\Psi(P)\cdot\mathbf{c} + \mathbf{c}\cdot\Psi(P) + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x])\cdot\mathbf{d}\cdot\Psi([x, \hat{1}])\right], \\ \Psi(\mathrm{Prism}(P)) &= \Psi(P)\cdot\mathbf{c} + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x])\cdot\mathbf{d}\cdot\Psi([x, \hat{1}]). \end{aligned}$$

Proof. The first identity follows by a careful chain argument. Consider a chain c in $P \times B_1$. We have

$$c = \{(\hat{0}, \hat{0}) = (x_0, y_0) < (x_1, y_1) < \cdots < (x_k, y_k) = (\hat{1}, \hat{1})\}.$$

Let i be the smallest index such that $y_i = \hat{1}$. Let $x = x_i$. This implies $y_0 = \cdots = y_{i-1} = \hat{0}$, $y_i = \cdots = y_k = \hat{1}$ and $x_{i-1} \le x_i$. We also have the two chains $c_1 = \{\hat{0} = x_0 < x_1 < \cdots < x_{i-1} \le x\}$ in $[\hat{0}, x]$ and $c_2 = \{x < x_{i+1} < \cdots < x_k = \hat{1}\}$ in $[x, \hat{1}]$.

Three cases occur:

(1) $\hat{0} < x < \hat{1}$. Then the element $(x, \hat{0})$ may or may not be in the chain c. Let c' denote the chain $c - \{(x, \hat{0})\}$, that is, the chain without the element $(x, \hat{0})$. Similarly, let c'' denote the chain $c \cup \{(x, \hat{0})\}$, that is, the chain with the element $(x, \hat{0})$. Observe that the element $(x, \hat{1})$ belongs to both the chains c' and c'', so the weight of these chains at rank $\rho(x) + 1$ is $\mathbf{b}$. Hence we have

$$\begin{aligned} w(c') &= w_{[\hat{0},x]}(c_1) \cdot (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c'') &= w_{[\hat{0},x]}(c_1) \cdot \mathbf{b} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c') + w(c'') &= w_{[\hat{0},x]}(c_1) \cdot \mathbf{a} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2). \end{aligned}$$

(2) $x = \hat{1}$. Then the element $(\hat{1}, \hat{0})$ may or may not be in the chain c. Let c' be the chain $c - \{(\hat{1}, \hat{0})\}$ and let c'' be the chain $c \cup \{(\hat{1}, \hat{0})\}$. Then

$$\begin{aligned} w(c') &= w_P(c_1) \cdot (\mathbf{a} - \mathbf{b}), \\ w(c'') &= w_P(c_1) \cdot \mathbf{b}, \\ w(c') + w(c'') &= w_P(c_1) \cdot \mathbf{a}. \end{aligned}$$

(3) $x = \hat{0}$. Then the element $(\hat{0}, \hat{1})$ lies in the chain c, and the weight of the chain c is

$$w(c) = \mathbf{b} \cdot w_P(c_2).$$

Summing over all chains c in $P \times B_1$, we obtain

$$\Psi(P \times B_1) = \mathbf{b} \cdot \Psi(P) + \Psi(P) \cdot \mathbf{a} + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \cdot \mathbf{a} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]). \quad (2.1)$$

Applying equation (2.1) to the dual poset P^* and applying the involution * to obtain

$$\Psi(P \times B_1) = \Psi(P) \cdot \mathbf{b} + \mathbf{a} \cdot \Psi(P) + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \cdot \mathbf{b} \cdot \mathbf{a} \cdot \Psi([x, \hat{1}]). \quad (2.2)$$

Adding equations (2.1) and (2.2) gives the desired result.

The proof of the second identity, which we omit, is similar. □

2.2. **The Newtonian coalgebra of ab-polynomials.**

Proposition 2.1.4 is very suggestive that a coalgebraic structure is occurring here. We introduce these ideas in this section.

Let $\mathcal{A} = \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ be the polynomial algebra in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$ with the usual multiplication. Define the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by

$$\Delta(v_1 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n,$$

and extend by linearity. For example $\Delta(\mathbf{abba}) = 1 \otimes \mathbf{bba} + \mathbf{a} \otimes \mathbf{ba} + \mathbf{ab} \otimes \mathbf{a} + \mathbf{abb} \otimes 1$.

Formally, we write the coproduct of an element x as

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}.$$

This should be thought of as the sum over all ways of breaking the element x into the pairs $x_{(1)}$ and $x_{(2)}$. The terms $x_{(1)}$ and $x_{(2)}$ are referred to as "x Sweedler 1" and "x Sweedler 2".

A *Newtonian coalgebra* is a coalgebra with respect to the coproduct Δ and an algebra with respect to the product μ where the *Newtonian condition* holds:

$$\Delta(u \cdot v) = \Delta(u) \cdot v + u \cdot \Delta(v).$$

Equivalently, using Sweedler notation, we have

$$\Delta(x \cdot y) = \sum_x x_{(1)} \otimes x_{(2)} y + \sum_y x y_{(1)} \otimes y_{(2)}.$$

It is straightforward to check

Lemma 2.2.1. *$\mathcal{A} = \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ is a Newtonian coalgebra with a unit, but no counit.*

The Newtonian coalgebra of **ab**-polynomials has a natural grading with $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$, where $\mathcal{A}_n$ is spanned by the **ab**-monomials of degree n. We also have $\dim(\mathcal{A}_n) = 2^n$, and

$$\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j} \text{ and } \Delta(\mathcal{A}_n) \subseteq \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j.$$

Lemma 2.2.2. *Consider the coproduct $\Delta : \mathcal{A}_n \longrightarrow \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$ as a linear map. Then the kernel of Δ is one-dimensional and is spanned by the element $(\mathbf{a} - \mathbf{b})^n$.*

Note the linear map $\Delta : \mathcal{A}_n \longrightarrow \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$ is not surjective for $n \geq 2$, because $\dim(\Delta(\mathcal{A}_n)) = 2^n - 1$ and $\dim\left(\bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j\right) = n \cdot 2^{n-1} > 2^n - 1$.

2.3. **The ab-index as a coalgebra homomorphism.**

We state the following fundamental result concerning the **ab**-index [31].

Theorem 2.3.1 (Ehrenborg–Readdy)**.** *The **ab**-index is a Newtonian coalgebra homomorphism from the linear space $\mathcal{P}$ of all graded posets to the polynomial algebra $\mathcal{A} = \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$, that is, $\Psi : \mathcal{P} \to \mathcal{A}$ with $\Psi(B_1) = 1$,*

$$\Psi(P * Q) = \Psi(P) \cdot \Psi(Q),$$

and

$$\Delta(\Psi(P)) = \sum_{\substack{x\in P \\ \hat{0}<x<\hat{1}}} \Psi([\hat{0},x]) \otimes \Psi([x,\hat{1}]) \tag{2.3}$$

Equivalently,

$$\Psi \circ \mu = \mu \circ (\Psi \otimes \Psi) \text{ and } \Delta \circ \Psi = (\Psi \otimes \Psi) \circ \Delta.$$

Let us read what this theorem says on the **ab**-level. If one obtains an expression of the form

$$\sum_{\substack{x\in P \\ \hat{0}<x<\hat{1}}} B(\Psi([\hat{0},x]), \Psi([x,\hat{1}])), \tag{2.4}$$

where $B : \mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle \times \mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle \longrightarrow \mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle$ is a bilinear form, then we can evaluate (2.4) in terms of the **ab**-index of the entire poset P. That is, we have

$$\sum_{\substack{x\in P \\ \hat{0}<x<\hat{1}}} B(\Psi([\hat{0},x]), \Psi([x,\hat{1}])) = \sum_{w} B(w_{(1)}, w_{(2)}),$$

where $w = \Psi(P)$. Note this circumnavigates having to compute the **ab**-index of every subinterval in the original poset P.

Lemma 2.3.2. *The subalgebra* $\mathcal{F} = \mathbf{k}\langle\mathbf{c},\mathbf{d}\rangle$ *of* $\mathcal{A}$ *is closed under the coproduct* Δ.

Proof. This follows from $\Delta(\mathbf{c}) = \Delta(\mathbf{a}+\mathbf{b}) = 1\otimes 1 + 1\otimes 1 = 2\cdot 1\otimes 1$ and $\Delta(\mathbf{d}) = \Delta(\mathbf{ab}+\mathbf{ba}) = \mathbf{a}\otimes 1 + 1\otimes \mathbf{b} + \mathbf{b}\otimes 1 + 1\otimes\mathbf{a} = \mathbf{c}\otimes 1 + 1\otimes\mathbf{c}$. □

This Newtonian coalgebra inherits the grading from $\mathcal{A}$ in the following manner: $\mathcal{F} = \oplus_{n\geq 0}\mathcal{F}_n$ with $\dim(\mathcal{F}_0) = \dim(\mathcal{F}_1) = 1$ and $\mathcal{F}_n = \mathbf{c}\cdot\mathcal{F}_{n-1} + \mathbf{d}\cdot\mathcal{F}_{n-2}$, implying $\dim(\mathcal{F}_n) = F_n$, the nth Fibonacci number ($F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

Since the subalgebra $\mathcal{F} = \mathbf{k}\langle\mathbf{c},\mathbf{d}\rangle$ is closed under the coproduct, we have the immediate corollary to Theorem 2.3.1.

Corollary 2.3.3. *The* **cd**-*index is a coalgebra homomorphism from the linear space* $\mathcal{E}$ *of all graded Eulerian posets to the polynomial algebra* $\mathcal{A} = \mathbf{k}\langle\mathbf{c},\mathbf{d}\rangle$.

Define an involution on $\mathcal{A}$, denoted *, by reading the **ab**-monomials in reverse. That is, $(v_1\cdot v_2\cdots v_n)^* = v_n\cdots v_2\cdot v_1$ and extend it linearly to all of $\mathcal{A}$. This is also an involution on the **cd**-monomials $\mathcal{F}$, since $\mathbf{c}^* = (\mathbf{a}+\mathbf{b})^* = \mathbf{a}+\mathbf{b} = \mathbf{c}$ and $\mathbf{d}^* = (\mathbf{ab}+\mathbf{ba})^* = \mathbf{ba}+\mathbf{ab} = \mathbf{d}$. Also observe that taking the dual of a poset extends to an involution on the linear space $\mathcal{P}$.

2.4. **Derivations.**

Define a linear operator $D : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D(w) = \sum_w w_{(1)} \cdot \mathbf{d} \cdot w_{(2)}.$$

Recall that the Newtonian condition implies that D is a derivation on $\mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$. We could have defined D directly as a derivation on $\mathcal{A}$ such that $D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab} + \mathbf{ba} = \mathbf{d}$. Note that D is also a derivation on $\mathcal{F}$ since $D(\mathbf{c}) = 2 \cdot \mathbf{d}$ and $D(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$.

Combining Proposition 2.1.4 with the fact that Ψ is a Newtonian coalgebra map, we obtain:

Theorem 2.4.1 (Ehrenborg–Readdy)**.** *Let P be a graded poset. Then*

$$\begin{aligned} \Psi(\mathrm{Pyr}(P)) &= \frac{1}{2}\left[\Psi(P) \cdot \mathbf{c} + \mathbf{c} \cdot \Psi(P) + D(\Psi(P))\right], \\ \Psi(\mathrm{Prism}(P)) &= \Psi(P) \cdot \mathbf{c} + D(\Psi(P)). \end{aligned}$$

In Theorem 2.5.2 we will improve the formula for the pyramid.

As a corollary, Theorem 2.4.1 gives a new recursion formula for the **cd**-index of the n-dimensional cube C_n.

Corollary 2.4.2. *The* **cd***-index of the n-dimensional cube C_n satisfies the recursion*

$$\Psi(C_{n+1}) = \Psi(C_n) \cdot \mathbf{c} + D(\Psi(C_n)),$$

for $n \geq 0$ with $\Psi(C_0) = 1$.

This differs from Purtill's recursion obtained in [60, Corollaries 5.8 and 5.12].

2.5. **The derivation G.**

Define on the algebra $\mathcal{A}$ two derivations G and G' by letting

$$\begin{aligned} G(\mathbf{a}) &= \mathbf{ba}, & \qquad G'(\mathbf{a}) &= \mathbf{ab}, \\ G(\mathbf{b}) &= \mathbf{ab}, & \qquad G'(\mathbf{b}) &= \mathbf{ba}, \end{aligned}$$

and extending G and G' to all **ab**-polynomials by linearity and the product rule of derivations. Since $D(\mathbf{a}) = G(\mathbf{a}) + G'(\mathbf{a})$ and $D(\mathbf{b}) = G(\mathbf{b}) + G'(\mathbf{b})$, we obtain that $D(w) = G(w) + G'(w)$ for all **ab**-polynomials w, that is, $D = G + G'$.

Observe that $G(\mathbf{c}) = G(\mathbf{a} + \mathbf{b}) = \mathbf{ba} + \mathbf{ab} = \mathbf{d}$ and $G(\mathbf{d}) = G(\mathbf{a}) \cdot \mathbf{b} + \mathbf{a} \cdot G(\mathbf{b}) + G(\mathbf{b}) \cdot \mathbf{a} + \mathbf{b} \cdot G(\mathbf{a}) = \mathbf{bab} + \mathbf{aab} + \mathbf{aba} + \mathbf{bba} = \mathbf{cd}$. A similar computation gives $G'(\mathbf{c}) = \mathbf{d}$ and $G'(\mathbf{d}) = \mathbf{dc}$. Hence G and G' restrict to be derivations on $\mathcal{F}$.

Lemma 2.5.1. *For all* **ab***-monomials w, the identity*

$$w \cdot \mathbf{c} + G(w) = \mathbf{c} \cdot w + G'(w)$$

holds.

We leave the proof as an exercise. See Exercise 2.9.6.

Theorem 2.5.2 (Ehrenborg–Readdy)**.** *Let P be a graded poset. Then*

$$\begin{aligned} \Psi(\mathrm{Pyr}(P)) &= \Psi(P) \cdot \mathbf{c} + G(\Psi(P)) \\ &= \mathbf{c} \cdot \Psi(P) + G'(\Psi(P)). \end{aligned}$$

Proof. By Theorem 2.4.1 and the fact the **ab**-index is a coalgebra homomorphism we have

$$\begin{aligned} 2 \cdot \Psi(P \times B_1) &= \Psi(P) \cdot \mathbf{c} + \mathbf{c} \cdot \Psi(P) + D(\Psi(P)) \\ &= (\Psi(P) \cdot \mathbf{c} + G(\Psi(P))) + \left(\mathbf{c} \cdot \Psi(P) + G'(\Psi(P))\right). \end{aligned}$$

But by Lemma 2.5.1 the two terms are equal. Thus we have $\Psi(P \times B_1) = \Psi(P) \cdot \mathbf{c} + G(\Psi(P)) = \mathbf{c} \cdot \Psi(P) + G'(\Psi(P))$. □

2.6. **Polytopes span.**

The **cd**-monomials give a basis for the vector space of **cd**-indexes of polytopes. Conversely, can we find a spanning set of polytopes whose **cd**-indexes give all **cd**-words? The answer, due to Bayer and Billera, is yes [3].

Theorem 2.6.1 (Bayer–Billera)**.** *Let $\mathscr{B}$ be the set of n-tuples $(R_1, \ldots, R_n)$ such that each R_i is either the pyramid operation* Pyr*, or the prism operation* Prism *satisfying*

1. *R_i and R_{i+1} are not both the prism operation, and*
2. *R_1 is the pyramid operation.*

Then the set

$$\{\Psi(R_n(\cdots R_1(\mathit{point})\cdots)) \ : \ (R_1, \ldots, R_n) \in \mathscr{B}\}$$

is a basis for the **cd***-polynomials of degree n.*

Bayer and Billera's original notation was in terms of the pyramid and bipyramid operations. The proof we give here is due to Billera, Ehrenborg and Readdy [11].

Proof. Let $\mathcal{F}_i$ denote the vector space of **cd**-polynomials of degree i. We have $\mathcal{F}_0 = \langle 1 \rangle$ and thus $\mathrm{Pyr}(1) = 1 \cdot \mathbf{c} + G(1) = \mathbf{c}$, which generates $\mathcal{F}_1$, that is,

$\mathcal{F}_1 = \langle \mathbf{c} \rangle$. By the induction hypothesis, we have a spanning set of polytopes for $\mathcal{F}_n$. We will use the pyramid and prism operations to build $\mathcal{F}_{n+1}$. Since $\text{Prism} - \text{Pyr} = G'$, the derivation with $G'(\mathbf{c}) = \mathbf{d}$ and $G'(\mathbf{d}) = \mathbf{dc}$, it is enough to show

$$G'(\mathcal{F}_n) + \text{Pyr}(\mathcal{F}_n) = \mathcal{F}_{n+1}.$$

Recall by Lemma 2.5.1 we have for any **cd**-word w the relation $\mathbf{c} \cdot w + G'(w) = w \cdot \mathbf{c} + G(w)$. Thus

$$\begin{aligned} \mathbf{c} \cdot w &= w \cdot \mathbf{c} + G(w) - G'(w) \\ &= \text{Pyr}(w) - G'(w), \end{aligned}$$

implying words of the form $\mathbf{c} \cdot w$ are in the span for $w \in \mathcal{F}_n$.

Let $v \in \mathcal{F}_{n-1}$. Then $\mathbf{d} \cdot v = G'(\mathbf{c} \cdot v) - c \cdot G'(v)$. since $G'(\mathbf{c} \cdot v) \in G'(\mathcal{F}_n)$ and $\mathbf{c} \cdot G'(v)$ is in the span, we conclude that $\mathbf{d} \cdot v$ is also in the span. □

The *Minkowski sum* of two subsets X and Y of $\mathbb{R}^n$ is defined as

$$X + Y = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^n : \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

The Minkowski sum of two convex polytopes is another convex polytope. For a vector $\mathbf{x}$ denote the set $\{\lambda \cdot \mathbf{x} : 0 \leq \lambda \leq 1\}$ by $[\mathbf{0}, \mathbf{x}]$. A *zonotope* is defined to be the Minkowski sum of line segments. Observe the prism operation can be realized as the Minkowski sum with a line segment.

Theorem 2.6.2 (Billera–Ehrenborg–Readdy)**.** [11] *The* **cd***-indexes of n-dimensional zonotopes linearly span the space $\mathcal{F}_n$ of* **cd***-polynomials of degree n.*

Open question 2.6.3. *Find a basis of zonotopes which span the space $\mathcal{F}_n$ of* **cd***-polynomials of degree n.*

One such basis is conjectured by Liu [51] consisting of all BP-words of length n ending in P and having no two consecutive B's, where for a zonotope Z, the two operations $PZ = \text{Prism}(Z)$ and $BZ = M(\text{Prism}(Z))$. Here M is the Minkowski sum with a line segment taken in general direction in the same dimension.

2.7. Inequalities: a first look.

The f-vector for 3-dimensional polytopes determines the flag vector (see Exercise 1.5.3), by Steinitz' theorem all of the inequalities for flag vectors of 3-dimensional polytopes have been determined. The best-known linear inequalities for 4-dimensional polytopes are due to Bayer [2]:

Theorem 2.7.1 (Bayer)**.** *The flag f-vector of a 4-polytope satisfies*

(1) $f_{02} - 3f_2 \geq 0$
(2) $f_{02} - 3f_1 \geq 0$

(3) $f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0$
(4) $6f_1 - 6f_0 - f_{02} \geq 0$
(5) $f_0 - 5 \geq 0$
(6) $f_2 - f_1 + f_0 - 5 \geq 0$

Observe that (1) and (2) are dual, and (5) and (6) are dual, whereas (3) and (4) are self-dual.

2.8. **Notes.**

The theory of Hopf algebras is originally due to Sweedler [69]. The Newtonian coalgebra of posets is due to Ehrenborg and Hetyei (unpublished). Ehrenborg and Readdy discovered the inherent coalgebraic structure of flag vectors of polytopes and geometric operations on polytopes; see [31]

The first identity in 2.3.1 is due to Stanley [65, Lemma 1.1].

The generalized Dehn–Sommerville relations are also known as the *Bayer–Billera relations.*

The original proof of Theorem 2.6.1, due to Bayer and Billera, is long. Lou Billera described it as "Beating the beast until it was lying down".

Theorem 2.8.1 (Varchenko/Liu)**.** [11] *For an n-dimensional zonotope Z and $S = \{i_1, \ldots, i_k\}$ we have*

$$\frac{f_S(Z)}{f_{i_j}(Z)} < \binom{d - i_1}{i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k} \cdot 2^{i_k - i_1}.$$

Billera and Hetyei [12] determined the convex cone generated by all flag f-vectors of graded posets.

2.9. **Exercises.**

Exercise 2.9.1. Compute the **cd**-index of the n-dimensional simplex for $n = 1, \ldots, 5$.

Exercise 2.9.2. Compute the **cd**-index of the n-dimensional cube for $n = 1, \ldots, 5$.

Exercise 2.9.3. Compute the coefficient $\mathbf{c}^i\mathbf{d}\mathbf{c}^j$ in the **cd**-index of the Boolean algebra B_{i+j+3}.

Exercise 2.9.4. For $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, we say π has a *descent* at position j if $\pi_j > \pi_{j+1}$. Furthermore, a permutation π is an *André permutation* if π has no double descents, that is, no index j with $\pi_j > \pi_{j+1} > \pi_{j+2}$ and satisfies the more general "no double descent" condition: for all $1 < j < j' \leq n$ if $\pi_{j-1} = \max\{\pi_{j-1}, \pi_j, \pi_{j'-1}, \pi_{j'}\}$ and $\pi_{j'} = \min\{\pi_{j-1}, \pi_j, \pi_{j'-1}, \pi_{j'}\}$, then there exists a j'' with $j < j'' < j'$ such that $\pi_{j''} < \pi_{j'}$. Denote the set of

André permutations in $\mathfrak{S}_n$ by $\mathscr{A}_n$.

a. Determine the set of André permutations $\mathscr{A}_n$ for $n = 1, \ldots, 5$.
b. The noncommutative André polynomial of Foata and Schützenberger is $\sum_\pi \Omega(u_\pi)$ where the sum is over all André permutations $\pi \in \mathscr{A}_n$, u_π is the descent word of the permutation π and Ω is the map which replaces each occurrence of **ba** with a **d** and then each remaining **a** with a **c**. Compute the noncommutative André polynomials for $n = 1, \ldots, 5$.

Exercise 2.9.5. Define $\kappa : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ to be the algebra map such that $\kappa(\mathbf{a}) = \mathbf{a} - \mathbf{b}$ and $\kappa(\mathbf{b}) = 0$.
a. Prove

$$w = \kappa(w) + \sum_{w} \kappa(w_{(1)}) \cdot \mathbf{b} \cdot w_{(2)}.$$

b. What does this say about the **ab**-index of a poset?

Exercise 2.9.6. Prove Lemma 2.5.1.

Exercise 2.9.7. Let v be a vertex of a polytope P. Describe the **cd**-index of the resulting polytope when cutting off the vertex v in terms of $\Psi(P)$ and $\Psi(P/v)$. Here P/v is the vertex figure which has face lattice $[v, \hat{1}]$.

Exercise 2.9.8. Describe the **cd**-index of the bipyramid of a polytope P in terms of the **cd**-index of the original polytope P.