POLYTOPES

MARGARET A. READDY

1. Lecture I: Introduction to Polytopes and Face Enumeration

Grünbaum and Shephard [40] remarked that there were three developments which foreshadowed the modern theory of convex polytopes.

(1) The publication of Euclid's Elements and the five Platonic solids. In modern terms, these are the regular 3-polytopes.
(2) Euler's Theorem which states that that
$$v - e + f = 2$$
holds for any 3-dimensional polytope, where v, e and f denote the number of vertices, edges and facets, respectively. In modern language,
$$f_0 - f_1 + f_2 = 2,$$
where f_i, $i = 0, 1, 2$, is the number of i-dimensional faces.
(3) The discovery of polytopes in dimensions greater or equal to four by Schläfli.

We will use these as a springboard to describe the theory of convex polytopes in the 21st century.

1.1. **Examples.**

Recall a set S in $\mathbb{R}^n$ is *convex* if the line segment connecting any two points in S is completely contained in the set S. In mathematical terms, given any $x_1, x_2 \in S$, the set of all points $\lambda \cdot x_1 + (1 - \lambda)x_2 \in S$ for $0 \leq \lambda \leq 1$. A *convex polytope* or *polytope* in n-dimensional Euclidean space $\mathbb{R}^n$ is defined as the convex hull of k points $x_1, \ldots, x_k$ in $\mathbb{R}^n$, that is, the intersection of all convex sets containing these points. Throughout we will assume all of the polytopes we work with are convex.

One can also define a polytope as the bounded intersection of a finite number of half-spaces in $\mathbb{R}^n$. These two descriptions can be seen to be equivalent by Fourier-Motzkin elimination [73]. A polytope is *n-dimensional*, and thus

Date: Women and Mathematics Program, Institute for Advanced Study, May 2013.

said to be a n-polytope, if it is homeomorphic to a closed n-dimensional ball $\mathbb{B}^n = \{(x_1, \ldots, x_r) : x_1^2 + \cdots + x_n^2 \leq 1, x_{n+1} = \cdots = x_r = 0\}$ in $\mathbb{R}^r$. Given a polytope P in $\mathbb{R}^n$ with supporting hyperplane H, that is, $P \cap H \neq \emptyset$, $P \cap H_+ \neq \emptyset$ and $P \cup H_- = \emptyset$, where H_+ and H_- are the half open regions determined by the hyperplane H, then we say $P \cap H$ is a *face*. Observe that a face of a polytope is a polytope in its own right.

We now give some examples of polytopes. Note that there are many ways to describe each of these polytopes geometrically. The importance for us is that they are *combinatorially equivalent*, that is, they have the same face incidences structure though are not necessarily affinely equivalent. As an example, compare the square with a trapezoid.

Example 1.1.1. Polygons. The n-gon in $\mathbb{R}^2$ consists of n vertices, $n \geq 3$, so that no vertex is contained in the convex hull of the other $n-1$ vertices. Note the n-gon has n edges, so we encode its facial data by the f-vector $(f_0, f_1) = (n, n)$.

For the next example, we need the notion of affinely independence. A set of points $x_1, \ldots, x_n$ is *affinely independent* if

$$\sum_{1 \leq i \leq n} \lambda_i x_i = 0 \quad \text{with} \quad \sum_{1 \leq i \leq n} \lambda_i = 0 \ \text{ implies } \ \lambda_1 = \cdots = \lambda_n = 0.$$

Here $\lambda_1, \ldots, \lambda_n$ are scalars.

Example 1.1.2. The n-simplex Δ_n. *The n-dimensional simplex* or *n-simplex* is the convex hull of any $n+1$ affinely independent points in $\mathbb{R}^n$. Equivalently, it can be described as the convex hull of the $n+1$ points $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ where $\mathbf{e}_i$ is the ith unit vector $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. It is convenient to intersect this polytope with the hyperplane $x_1 + \cdots + x_{n+1} = 1$ so that the n-simplex lies in $\mathbb{R}^n$. Its f-vector has entries $f_i = \binom{n+1}{i+1}$, for $i = 0, \ldots, n-1$. The n-simplex is our second example of a *simplicial polytope*, that is, a polytope where all of its facets ($(n-1)$-dimensional faces) are combinatorially equivalent to the $(n-1)$-simplex. Our first, although trivial example, is the n-gon.

Example 1.1.3. The n-dimensional hypercube (n-cube). This is the convex hull of the 2^n points $C_n = \text{conv}\{(x_1, \ldots, x_n) : x_i \in \{0, 1\}\}$. In $\mathbb{R}^2$ this is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Observe that every vertex of the n-cube is *simple*, that is, every vertex is adjacent to exactly n edges. The f-vector has entries $f_i = \binom{n}{i} \cdot 2^{n-i}$ for $i = 0, \ldots, n$.

Example 1.1.4. The n-dimensional cross-polytope. This is the convex hull of the $2n$ points $\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \ldots, \pm\mathbf{e}_n\}$ in $\mathbb{R}^n$. In $\mathbb{R}^3$ this is the octahedron.

Consider the f-vector of the 3-cube and the octahedron. They are respectively $(8, 12, 6)$ and $(6, 12, 8)$. These two polytopes are said to be *dual* or *polar*. More formally, a polytope P is dual to a polytope P^* if there is an inclusion-reversing bijection between the faces of P and P^*.

Example 1.1.5. The permutahedron. This is the $(n-1)$-dimensional polytope defined by taking the convex hull of the $n!$ points $(\pi_1, \ldots, \pi_n)$ in $\mathbb{R}^n$, where $\pi = \pi_1 \cdots \pi_n$ is a permutation written in one-line notation from the symmetric group $\mathfrak{S}_n$ on n elements.

Example 1.1.6. The cyclic polytope. For fixed positive integers n and k the cyclic polytope $C_{n,k}$ is the convex hull of k distinct points on the moment curve $(t, t^2, \ldots, t^n)$.

Example 1.1.7. The Birkhoff polytope. The Birkhoff polytope is the set of all $n \times n$ doubly stochastic matrices, that is, all $n \times n$ matrices with non-negative entries and each row and column sum is 1. This is a polytope of dimension $(n-1)^2$. It is a nice application of Hall's Marriage Theorem that this polytope is the convex hull of the $n!$ permutation matrices.

1.2. **The face and flag vectors.**

The f-vector of a convex polytope is given by $(f_0, \ldots, f_{n-1})$, where f_i enumerates the number of i-dimensional faces in the n-dimensional polytope. It satisfies the *Euler–Poincaré relation*

$$f_0 - f_1 + f_2 - \cdots + (-1)^{n-1} \cdot f_{n-1} = 1 - (-1)^n. \tag{1.1}$$

equivalently,

$$\sum_{i=-1}^{n} (-1)^i f_i = 0, \tag{1.2}$$

where f_{-1} denotes the number of empty faces ($= 1$) and $f_n = 1$ counts the entire polytope.

In 1906 Steinitz [67] completely characterized the f-vectors of 3-polytopes.

Theorem 1.2.1 (Steinitz)**.** *For a 3-dimensional polytope, the f-vector is uniquely determined by the values f_0 and f_2. The (f_0, f_2)-vector of every 3-dimensional polytope satisfies the following two inequalities:*

$$2(f_0 - 4) \geq f_2 - 4 \text{ and } f_0 - 4 \leq 2(f_2 - 4).$$

Furthermore, every lattice point in this cone has at least one 3-dimensional polytope associated to it.

The possible f-vectors lie in the lattice cone in the $f_0 f_2$-plane with apex at $(f_0, f_2) = (4, 4)$ and two rays emanating out of this point in the direction $(1, 2)$ and $(2, 1)$. The lattice points on these extremal rays are the simple and simplicial polytopes. See Exercise 1.5.4 for cubical 3-polytopes.

For polytopes of dimension greater than three the problem of characterizing their f-vectors is still open.

Open question 1.2.2. *Characterize f-vectors of d-polytopes where $d \geq 4$.*

S	f_S	h_S	u_s	$\mathbf{c}^3$	$10 \cdot \mathbf{dc}$	$6 \cdot \mathbf{cd}$
$\emptyset$	1	1	**aaa**	1	0	0
$\{0\}$	12	11	**baa**	1	10	0
$\{1\}$	18	17	**aba**	1	10	6
$\{2\}$	8	7	**aab**	1	0	6
$\{0,1\}$	36	7	**bba**	1	0	6
$\{0,2\}$	36	17	**bab**	1	10	6
$\{1,2\}$	36	11	**abb**	1	10	0
$\{0,1,2\}$	72	1	**bbb**	1	0	0

TABLE 1. The flag f- and flag h-vectors, **ab**-index and **cd**-index of the hexagonal prism. The sum of the last three columns equals the flag h column, showing the **cd**-index of the hexagonal prism is $\mathbf{c}^3 + 10 \cdot \mathbf{dc} + 6 \cdot \mathbf{cd}$.

The f-vectors of simplicial polytopes have been completely characterized by work of McMullen [56], Billera and Lee [13] and Stanley [64]. See the lecture end-notes for further comments.

We now wish to keep track of not just the *number* of faces in a polytope, but also the face *incidences*. We encode this with the *flag f-vector* (f_S), where $S \subseteq \{0, \ldots, n-1\}$. More formally, for $S = \{s_1 < \cdots < s_k\} \subseteq \{0, \ldots, n-1\}$, define f_S to be the number of flags of faces

$$f_S = \#\{F_1 \subsetneq F_2 \cdots \subsetneq F_k\}$$

where $\dim(F_i) = s_i$. Observe that for an n-polytope the flag f-vector has 2^n entries. It also contains the f-vector data.

The *flag h-vector* $(h_S)_{S \subseteq \{0,\ldots,n-1\}}$ is defined by the invertible relation

$$h_S = \sum_{T \subseteq \{0,\ldots,n-1\}} (-1)^{|S-T|} f_T. \tag{1.3}$$

Equivalently, by the Möbius Inversion Theorem (MIT)

$$f_S = \sum_{T \subseteq \{0,\ldots,n-1\}} h_T. \tag{1.4}$$

See Table 1 for the computation of the flag f- and flag h-vectors of the hexagonal prism. Observe that the symmetry of the flag h-vector reduces the number of entries we have to keep track of from 2^3 to 2^2. This is true in general.

Theorem 1.2.3 (Stanley)**.** *For an n-polytope, and more generally, an Eulerian poset of rank n,*

$$h_S = h_{\overline{S}},$$

where $\overline{S}$ denotes the complement of S with respect to $\{0, 1, \ldots, n-1\}$.

Posets and Eulerian posets will be introduced in Lecture 2.

1.3. **The ab-index and cd-index.**

We would like to encode the flag h-vector data in a more efficient manner. The **ab**-*index* of an n-polytope P is defined by

$$\Psi(P) = \sum_{S} h_S \cdot u_S,$$

where the sum is taken over all subsets $S \subseteq \{0, \ldots, n-1\}$ and $u_S = u_0 u_1 \ldots u_{n-1}$ is the non-commutative monomial encoding the subset S by

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

Observe the resulting **ab**-index is a noncommutative polynomial of degree n in the noncommutative variables $\mathbf{a}$ and $\mathbf{b}$.

We now introduce another change of basis. Let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ be two noncommutative variables of degree 1 and 2, respectively. The following result was conjectured by J. Fine and proven by Bayer–Klapper for polytopes, and Stanley for Eulerian posets [4, 65].

Theorem 1.3.1 (Bayer–Klapper, Stanley)**.** *For the face lattice of a polytope, and more generally, an Eulerian poset, the* **ab**-*index* $\Psi(P)$ *can be written uniquely in terms of the noncommutative variables* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$, *that is,* $\Psi(P) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.

The resulting noncommutative polynomial is called the **cd**-*index*.

Bayer and Billera proved that the **cd**-index removes *all* of the linear redundancies holding among the flag vector entries [3]. Hence the **cd**-monomials form a natural basis for the vector space of **ab**-indexes of polytopes. These linear relations, known as the *generalized Dehn-Sommerville relations*, are given by

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}} = (1 - (-1)^{k-i-1}) \cdot f_S, \tag{1.5}$$

where $i \leq k-2$, the elements i and k are elements of $S \cup \{-1, n\}$, and the subset S contains no integer between i and k. These are all the linear relations holding among the flag f-vector entries. Observe that Euler-Poincaré follows if we take $S = \emptyset$, $i = -1$ and $k = n$.

The **cd**-index did not generate very much excitement in the mathematical community until Stanley's proof of the nonnegativity of its coefficients, which we state here.

Theorem 1.3.2 (Stanley)**.** *The* **cd**-*index of the face lattice of a polytope, more generally, the augmented face poset of any spherically-shellable regular CW-sphere, has nonnegative coefficients*

Stanley's result opened the door to the following question.

Open question 1.3.3. *Give a combinatorial interpretation of the coefficients of the* **cd**-*index.*

One interpretation of the coefficients of the **cd**-index is due to Karu, who, for each **cd**-monomial, gave a sequence of operators on sheaves of vector spaces to show the non-negativity of the coefficients of the **cd**-index for Gorenstein* posets [45]. See Exercise 2.9.4 for Purtill's combinatorial interpretation of the **cd**-index coefficients for the n-simplex and the n-cube.

1.4. **Notes.**

For general references on polytopes, we refer the reader to the second edition of Grünbaum's treatise [39], Coxeter's book on regular polytopes [21] and Ziegler's text [73].

See [73] for more information on the Fourier–Motzkin algorithm.

Euler's formula that $v - e + f = 2$ was mentioned in a 1750 letter Euler wrote to Goldbach, and proved by Descartes about 100 years earlier [22]. The "scissor" proof is due to von Staudt in 1847 [71]. Poincaré's proof of the more general Euler–Poincaré–Schläfli formula for polytopes required him to develop homology groups and algebraic topology. This discussion can be found in H.S.M. Coxeter's Regular Polytopes, Chapter IX, pages 165–166 [21]. Sommerville's 1929 proof of Euler–Poincaré–Schläfli was not correct as he assumed polytopes could be built facet by facet in an inductive manner which controls the homotopy type of the cell complex at each stage. This problem was rectified in 1971 when Bruggesser and Mani proved that polytopes are *shellable*. The concept of shellability has proven to be very powerful as it allows controlled inductive arguments for results on polytopes. See Lecture 3 for further discussion.

For any finite polyhedral complex C with Betti numbers given by the reduced integer homology $\beta_i = \mathrm{rank}(\widetilde{H_i}(C, \mathbb{Z}))$, the Euler–Poincaré formula is

$$f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} = 1 = \beta_0 - \beta_1 + \beta_2 - \cdots + (-1)^{d-1} \beta_{d-1}. \quad (1.6)$$

This holds for more general cell complexes. See [15].

Shortly after McMullen and Shephard's book [57] was published, the Upper and Lower Bound Theorems were proved [1, 55].

Theorem 1.4.1 (Upper Bound and Lower Bound Theorems)**.** *(a)* [McMullen] *For fixed nonnegative integers n and k, the maximum number of j-dimensional faces in an n-dimensional polytope P with k vertices is given by the cyclic polytope $C(n, k)$, that is,*

$$f_j(P) \leq f_j(C(n,k)), \quad \textit{for } 0 \leq j \leq n.$$

(b) [Barnette] *For an n-dimensional simplicial polytope P with $n \geq 4$,*

$$f_j(\text{Stack}(n,k)) \leq f_j(P), \quad \textit{for } 0 \leq j \leq n,$$

where Stack(n,k) *is any n-dimensional polytope on k vertices formed by repeatedly adding a pyramid over the facet of a simplicial n-polytope, beginning with the n-simplex.*

The g-theorem which characterizes f-vectors of simplicial polytopes involved a geometric construction of Billera and Lee [13] for the sufficiency proof, and tools from algebraic geometry for Stanley's necessity proof [64]. In particular, this required the Hard Lefschetz Theorem.

For convenience and those who are interested, we include Björner's reformulation of the g-theorem as stated in [39, section 10.6]:

Theorem 1.4.2 (The g-theorem)**.** (Billera–Lee; Stanley)
The vector $(1, f_0, \ldots, f_{d-1})$ is the f-vector of a simplicial d-polytope if and only if it is a vector of the form $\mathbf{g} \cdot M_d$, where M_d is the $([d/2]+1) \times (d+1)$ matrix with nonnegative entries given by

$$M_d = \left(\binom{d+1-j}{d+1-k} - \binom{j}{d+1-k} \right)_{0 \leq j \leq d, 0 \leq k \leq d}, \tag{1.7}$$

*and $\mathbf{g} = (g_0, \ldots, g_{[d/2]})$ is an M-*sequence*, that is, a nonnegative integer vector with $g_0 = 1$ and $g_{k-1} \geq \partial^k(g_k)$ for $0 < k \leq d/2$. The upper boundary operator ∂^k is given by*

$$\partial^k = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-2} \tag{1.8}$$

where the unique binomial expansion of n is

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}, \tag{1.9}$$

with $a_k > a_{k-1} > \cdots a_i \geq i > 0$. For a given polytope P the vector $g = g(P)$ is determined by the f-vector, respectively h-vector, as $g_k = h_k - h_{k-1}$ for $0 < k \leq d/2$ with $g_0 = 1$.

The motivating question in Purtill's dissertation was to prove nonnegativity of the coefficients of the **cd**-index for convex polytopes. He did prove nonnegativity in the case of the n-cube and the n-simplex by giving a combinatorial interpretation of the coefficients using André and signed André permutations [60]. Stanley [65] proved nonnegativity for spherically-shellable posets, of which face lattice of polytopes are examples.

1.5. **Exercises.**

Exercise 1.5.1. Build Platonic solids and other polytopes from nets.

Exercise 1.5.2. Prove Steinitz' theorem.
Hint: Every face is at least a triangle, so what are the inequalities on the vectors?
Hint': Every vertex is at least incident to 3 edges, again inequalities?
This will be the foundation for the Kalai product in Lecture II
Hint'': What happens to f_0 and f_2 after cutting off a simple vertex?

Exercise 1.5.3. Show that for 3-dimensional polytopes, the entries of the flag f-vector are determined by the values f_0 and f_2.

Exercise 1.5.4. We call a polytope *cubical* if all of its faces are combinatorial cubes. For example, the facets of 3-dimensional cubical polytopes are squares.
a. Show that a 3-dimensional cubical polytope satisfies $f_0 - f_2 = 2$ and $f_0 \geq 8$.
b. Show there is no 3-dimensional cubical polytope with $(f_0, f_2) = (9, 7)$.
c. Show that any other lattice point on the line $f_0 - f_2 = 2$ for $f_0 \geq 8$ comes from a cubical polytope.

Exercise 1.5.5. a. Starting with the cube, cut off a vertex and compute the **cd**-index.
b. Repeat part a. with the 4-cube.

Exercise 1.5.6. a. What is the **cd**-index of the n-gon?
b. What is the **cd**-index of the prism of the n-gon?
c. What is the **cd**-index of the pyramid of the n-gon?