Williams

IAS Lecture 3

A <u>polytope</u> in $\mathbb{R}^d$ is higher dim'l analogue of a polygon. Can describe in 2 ways:

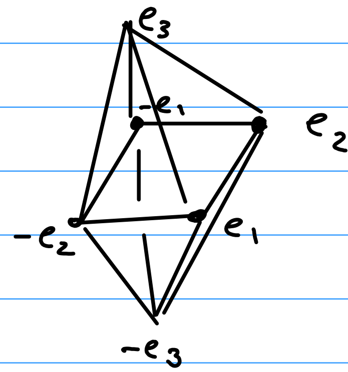
1. Vertices

A polytope is the "convex hull" of a finite set of points $V = \{v_1, \dots, v_n\}$ in $\mathbb{R}^d$.

$$P = \operatorname{conv}(V) = \left\{ x = \sum_{i=1}^{n} a_i v_i \;\middle|\; a_i \geq 0,\ \sum a_i = 1 \right\}$$

eg $\operatorname{conv}(\pm e_1, \pm e_2, \pm e_3)$ is

octahedron



Think: snap a rubber band (rubber suit?) around V. This is the smallest convex set containing V.

2. Inequalities

A polytope is a bounded intersection of halfspaces defined by linear inequalities.

$$P = \{ x \in \mathbb{R}^d \mid a_1 \cdot x \leq z_1, \dots, a_m \cdot x \leq z_m \}$$

Here $a_i, z_i \in \mathbb{R}^d$ and $\cdot$ is the dot product

octahedron: $x_1 + x_2 + x_3 \leq 1$

$x_1 - x_2 + x_3 \leq 1$

$\vdots$

$\pm x_1 \pm x_2 \pm x_3 \leq 1$ (8 facets)

Theorem: A subset $S \subseteq \mathbb{R}^d$ is a convex hull of a finite set of points iff it is a bounded intersection of halfspaces.

Cor:

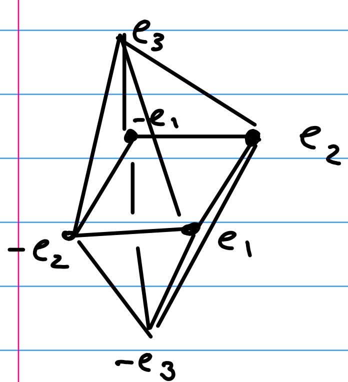
- polytope $\cap$ polytope = polytope (facet description)
- projection of polytope = polytope (vertex desc)

To define a face F of polytope P one "points in its direction": for $w \in \mathbb{R}^d$,

$$F_w = \{x \in P \mid w \cdot x \text{ is maximum}\}$$

(a polytope of smaller dim)

Faces of octahedron.
1 3D, 8 2D, 12 1D, 6 0D,
(1 (-1)D)



Ex: The face w/ vertices $\{e_1, e_2, -e_3\}$ equals

$$F_{(1,1,-1)} = \left\{(x_1,x_2,x_3) \in P \;\middle|\; \begin{array}{l}(1,1,-1)\cdot(x_1,x_2,x_3) \\ = x_1+x_2-x_3 \text{ is max}\end{array}\right\}$$

Note: $F_{(1,1,-1)} = F_{(a,a,-a)}$ for any $a \in \mathbb{R}_{>0}$

Ex: The face w/ vertices $\{e_1, e_2\}$ equals $F_{(1,1,0)}$.
Also equals $F_{(c,c,b)}$ where $0 \leq b < c$

Ex: The face w/ unique vertex $\{-e_3\}$ is $F_{(0,0,-1)}$
Lots of other ways to express face!

Recall: A matroid $M=(E,\mathcal{I})$ where $\mathcal{I}$ is the collection of <u>independent</u> subset of E.
A <u>basis</u> of M is a max'l element of $\mathcal{I}$.
All bases of M have the same size.

Theorem 1: Let $\mathcal{B}$ be a set of subsets of a finite set E. Then $\mathcal{B}$ is the collection of bases of a matroid on E iff $\mathcal{B}$ satisfies:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) If B_1 and $B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$ then $\exists\, y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

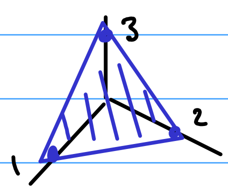
Def: The <u>matroid polytope</u> P_M of M (or matroid basis polytope) is

$$P_M := \operatorname{conv}\left(\{ e_{b_1} + \dots + e_{b_r} \mid \{b_1, \dots, b_r\} \text{ a basis of } M \right)$$

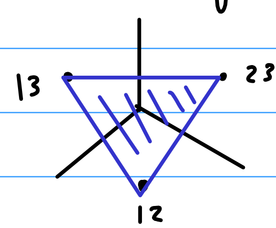
in $\mathbb{R}^E$.

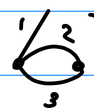
Let $E = \{1,2,3\}$.

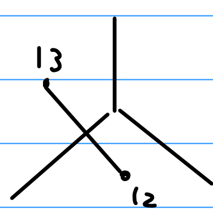
Ex 1: $\mathcal{I} = \{1,2,3\}$.



Ex: $M(A)$ for $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$



Ex: $M($  $)$



Def: Given matroid $M = (E, \mathcal{I})$, the **rank function** is r: Subsets of $E \to \mathbb{Z}_{\geq 0}$ defined by $r(S) =$ size of largest indep set contained in E.

rank of $M = r(E)$.

Can define matroids using rank function:

Thm: Given finite set E and function r: subsets of $E \to \mathbb{Z}_{\geq 0}$, r is the rank function of a matroid iff:

- For any $A, B \subset E$, $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$. (submodular)
- For any $A \subset E$ and $x \in E$, $r(A) \leq r(A \cup \{x\}) \leq r(A) + 1$.

(won't prove this)

Thm:
$$P_M = \left\{ x \in \mathbb{R}^E \;\middle|\; \begin{array}{ll} ① \; x_i \geq 0 & \forall\, i \in E \\ ② \; \sum_{i \in S} x_i \leq r(S) & \forall\, S \subset E \\ ③ \; \sum_{i \in E} x_i = r(M) & \end{array} \right\}$$

Pf: (only $\subset$)

Suffices to show that each vertex v_A of P_M satisfies these inequalities.

Each $v_A \in \{0,1\}^E$ w/ exactly $r(M)$ coordinate $= 1$.

$\therefore$ ① & ③ true.

$\sum_{i \in S} (v_A)_i = \#(S \cap A) \leq r(S)$ since A is indep. set. ✓

Next time we will prove:

Theorem (Gelfand - Goresky - MacPherson - Serganova, '87):
Let $\mathcal{B}$ be a collection of k-subsets of $E = \{1, 2, \dots, n\}$. Let $P_{\mathcal{B}} = \mathrm{conv}(v_B : B \in \mathcal{B})$.
$(E, \mathcal{B})$ is a matroid $\iff$ every edge of $P_{\mathcal{B}}$ is of the form $e_i - e_j$.

Exercises

1. Given the rank function $r: E \to \mathbb{Z}_{\geq 0}$ of a matroid, how should we define the independent sets?

2. The <u>uniform matroid</u> $U_{k,n}$ is the matroid on ground set $[n] = \{1, 2, .., n\}$ whose bases consist of <u>all</u> k-element subsets of $[n]$. Draw the matroid polytope associated to $U_{2,4}$ + express each face in the form F_ω.
Also write down the vectors corresponding to the edges of the polytope.

3. (Symmetric Basis exchange)
If $\mathcal{B}$ is the collection of bases of a matroid M on E then for any $B_1, B_2 \in \mathcal{B}$, if $b_1 \in B_1 - B_2$ $\exists$ $b_2 \in B_2 - B_1$ s.t. $(B_1 - b_1 \cup b_2)$ and $(B_2 - b_2 \cup b_1)$ are also bases of M.