A polytope in $\mathbb{R}^d$ is higher dim'l analogue of a polygon. Can describe in 2 ways:

1. **Vertices**

   A polytope is the "convex hull" of a finite set of points $V = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^d$.

   
   $\text{P} = \text{conv}(V) = \{x = \sum_{i=1}^{n} a_i v_i \mid a_i \geq 0, \sum a_i = 1\}$

   
   eg. $\text{conv}(\pm e_1, \pm e_2, \pm e_3)$ is an octahedron.

   Think: Snap a rubber band (rubber suit?) around $V$. This is the smallest convex set containing $V$.

2. **Inequalities**

   A polytope is a bounded intersection of halfspaces defined by linear inequalities.

   
   $\text{P} = \{x \in \mathbb{R}^d \mid a_i \cdot x \leq z_i, \ldots, a_m \cdot x \leq z_m\}$

   Here $a_i, z_i \in \mathbb{R}^d$ and $\cdot$ is the dot product.

   Octahedron:

   
   $x_1 + x_2 + x_3 \leq 1$

   $x_1 - x_2 + x_3 \leq 1$

   $\vdots$

   $\pm x_1 \pm x_2 \pm x_3 \leq 1 \quad (8 \text{ facets})$
Theorem: A subset $S \subseteq \mathbb{R}^d$ is a convex hull of a finite set of points iff it is a bounded intersection of halfspaces.

Cor:
- Polytope $\cup$ Polytope = Polytope (fact described)
- Projection of Polytope = Polytope (vertex desc)

To define a face $F$ of polytope $P$ one "points in its direction": for $w \in \mathbb{R}^d$,
$$F_w = \{ x \in P \mid w \cdot x \text{ is maximum} \}$$
(a polytope of smaller dim)

Faces of octahedron:
- 3D, 8 2D, 12 1D, 6 0D,
- (1 (-1) 0)

**Ex:** The face w/ vertices \{ $e_1$, $e_2$, $-e_3$ \} equals
$$F_{(1,1,-1)} = \{ (x_1, x_2, x_3) \in P \mid (1,1,-1) \cdot (x_1, x_2, x_3) = x_1 + x_2 - x_3 \text{ is max} \}$$

Note: $F_{(1,1,-1)} = F_{(a, a, -a)}$ for any $a \in \mathbb{R}_{>0}$

**Ex:** The face w/ vertices \{ $e_1$, $e_2$, $e_3$ \} equals $F_{(1,1,0)}$.
Also equals $F_{(c, c, b)}$ where $0 \leq b < c$

**Ex:** The face w/ unique vertex \{-$e_3$\} is $F_{(0,0,-1)}$

Lots of other ways to express face!
Recall: A matroid $\mathcal{M}=(E, I)$ where $I$ is the collection of independent subset of $E$. A basis of $\mathcal{M}$ is a maximal element of $I$. All bases of $\mathcal{M}$ have the same size.

**Theorem 1**: Let $B$ be a set of subsets of a finite set $E$. Then $B$ is the collection of bases of a matroid on $E$ iff $B$ satisfies:

- (B1) $B \neq \emptyset$.
- (B2) If $B_1$ and $B_2 \in B$ and $x \in B_1 - B_2$ then $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in B$.

**Def**: The matroid polytope $P_M$ of $\mathcal{M}$ (or matroid basis polytope) is:

$$P_M := \text{conv}(\{e_{b_1} + \ldots + e_{b_r} | \{b_1, \ldots, b_r\} \text{ a basis of } \mathcal{M}\})$$

in $\mathbb{R}^E$.

Let $E = \{1, 2, 3\}$.

**Ex 1**: $I = \{1, 2, 3\}$. \(\mathcal{M}(A)\) for $A = (1, 0, 1)$

**Ex**: $\mathcal{M}(\bar{E}^2)$
Def: Given matroid \( M = (E, \mathcal{I}) \), the rank function is \( r: \text{subset of } E \rightarrow \mathbb{Z}_{\geq 0} \)
defined by \( r(s) = \text{size of largest independent set contained in } E \).

rank of \( M = r(E) \).

Can define matroids using rank function:

Thm: Given finite set \( E \) and function \( r: \text{subset of } E \rightarrow \mathbb{Z}_{\geq 0} \),
\( r \) is the rank function of a matroid iff:

- For any \( A, B \subseteq E \), \( r(A \cup B) + r(A \cap B) \leq r(A) + r(B) \) (submodular)
- For any \( A \subseteq E \) and \( x \in E \), \( r(A) \cup \{x\} \leq r(A \cup \{x\}) \leq r(A) + 1 \).

(we will prove this)

Thm: \( P_M = \left\{ x \in \mathbb{R}^E \mid \begin{array}{l}
\sum_{i \in S} x_i = 0 \quad \forall S \in \mathcal{I} \\
\sum_{i \in E} x_i \leq r(S) \quad \forall S \subseteq E \\
\sum_{i \in E} x_i = r(M)
\end{array} \right\} \)

Pf; (only \( \leq \))

Suffices to show that each vertex \( v_A \) of \( P_M \) satisfies these inequalities.

Each \( v_A \in \{0, 1\}^E \) with exactly \( r(M) \) coordinates = 1.

\( \sum_{i \in S} (v_A)_i = \#(S \cap A) \leq r(S) \) since \( A \) is independent set.
Next time we will prove:

**Theorem (Gelfand - Goresky - MacPherson - Serganova, '87):**
Let \( B \) be a collection of \( k \)-subsets of 
\( E = \{1, 2, \ldots, n\} \). Let 
\[ P_B = \text{conv} \left( v_\beta : \beta \in B \right), \]

\((E, B)\) is a matroid \( \iff \) every edge of \( P_B \)
is of the form \( e_i - e_j \).
Exercises

1. Given the rank function $r: E \to \mathbb{Z}_{\geq 0}$ of a matroid, how should we define the independent sets?

2. The uniform matroid $\mathcal{U}_{k, n}$ is the matroid on ground set $[n] = \{1, 2, \ldots, n\}$ whose bases consist of all $k$-element subsets of $[n]$. Draw the matroid polytope associated to $\mathcal{U}_{2, 4}$ and express each face in the form $F_w$. Also write down the vectors corresponding to the edges of the polytope.

3. (Symmetric Basis exchange) If $\mathcal{B}$ is the collection of bases of a matroid $M$ on $E$ then for any $B_1, B_2 \in \mathcal{B}$, if $b_1 \in B_1 - B_2$ then there exists $b_2 \in B_2 - B_1$ such that $(B_1 - b_1 \cup b_2)$ and $(B_2 - b_2 \cup b_1)$ are also bases of $M$. 