IAS Lecture 2

Def: A matroid M is a pair $(E, \mathcal{I})$ consisting of a finite set E and a collection of subsets of E satisfying

(I1) $\phi \in \mathcal{I}$

(I2) $\mathcal{I}$ is hereditary: if $I \in \mathcal{I}$ and $J \subset I$ then $J \in \mathcal{I}$.

(I3) If X and Y in $\mathcal{I}$ and $|X| = |Y| + 1$, then $\exists\ x \in X - Y$ s.t. $Y \cup \{x\} \in \mathcal{I}$.

There are many other equivalent (cryptomorphic) ways to define matroids!

Def: A set in a matroid that is not indep is called dependent.

A maximal (by inclusion) independent set is called a basis.

A minimal (by inclusion) dependent set is called a circuit.

Rk: Since the independent sets $\mathcal{I}$ of a matroid are hereditary, a matroid is determined by its bases.

Prop: All the bases of a matroid have the same size.

Pf: Suppose not, say $|B_1| < |B_2|$ are bases. By (I3), $\exists\ b \in B_2$ s.t. $B_1 \cup \{b\}$ is indep. $\Rightarrow\!\Leftarrow$ -

Def: The <u>rank</u> $r(M)$ is the size of any basis.

Theorem 1: Let $\mathcal{B}$ be a set of subsets of a finite set E. Then $\mathcal{B}$ is the collection of bases of a matroid on E iff $\mathcal{B}$ satisfies:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) If B_1 and $B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$ then $\exists$ $y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$. (basis exchange)

Pf: ($\Rightarrow$) Let $\mathcal{B}$ be collection of bases of $(E, \mathcal{I})$.

(B1) There are indep. sets so there are max'l ones.

(B2) Apply (I3) to $B_1 - \{x\}$ and B_2.

($\Leftarrow$) (Sketch)

Suppose $\mathcal{B}$ satisfies (B1), (B2)

<u>Step 1</u>: Show the sets in $\mathcal{B}$ have same size.

Take $B \in \mathcal{B}$ w/ $|B|$ minimal.

Goal: Each other $A \in \mathcal{B}$ has $|A| = |B|$.

Induct on $|A - B|$...

<u>Step 2</u>: Let $\mathcal{I} = \{I \subseteq E : I \subseteq B \text{ for some } B \in \mathcal{B}\}$.

<u>Claim</u>: $(E, \mathcal{I})$ is a matroid.

(I1): Let $B \in \mathcal{B}$. $\emptyset \subset B \Rightarrow \emptyset \in \mathcal{I}$. ✓

(I2): Suppose $I \subset J$, $J \in \mathcal{I} \Rightarrow J \subset B$ for a $B \in \mathcal{B} \Rightarrow I \subset B \Rightarrow I \in \mathcal{I}$. ✓

(I3): Given $I, J \in \mathcal{I}$, $|I| < |J|$, need ($\exists$ $j \in J$ with $I \cup j \in \mathcal{I}$).

Pf of I3: Induct on $r - |J|$ ...

$r = |B|$ for any $B \in \mathcal{B}$

Rk: (1) We could have defined a matroid to be a pair $(E, \mathcal{B})$ satisfying (B1) & (B2)!

(2) For $M[A]$ where A is a matrix, the notion of basis coincides w/ the linear algebra def'n.

(3) For $M(G)$ where G a connected graph, the bases of $M(G)$ are the max'l sets of edges that do not contain a cycle — these are the edge sets of spanning trees of G!

(4) If G connected w/ m vertices, what is $\operatorname{rank}(M(G))$?
Any spanning tree has $m-1$ edges, so $\operatorname{rank}(M(G)) = m-1$.

(5) If G not connected, bases of $M(G) \leftrightarrow$ spanning forests (spanning tree in each connected component).

Given the indep sets $\mathcal{I}$ of a matroid M the bases $\mathcal{B}$ are the max'l indep sets. Given the bases $\mathcal{B}$ of M, we get $\mathcal{I}$ by taking $\mathcal{B}$ together w/ all proper subsets of all bases.

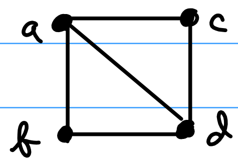
Enumerative Diversion: the Matrix-Tree Thm.

Q: How many spanning trees does a graph $G=(V,E)$ have?

A: Form a $V \times V$ matrix L "Laplacian".

Let $L_{v,v} = \deg(v)$

$L_{u,v} = -$ (# edges from u to v).

Ex: 

$$L = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 3 & -1 & -1 & -1 \\ b & -1 & 2 & 0 & -1 \\ c & -1 & 0 & 2 & -1 \\ d & -1 & -1 & -1 & 3 \end{array}$$

The eigenvalues of L are $0, \lambda_1, \lambda_2, \ldots, \lambda_{|V|-1}$.

↑ why?

Matrix-Tree Theorem (Kirchoff) The number of spanning trees of G is $\frac{1}{|V|}\lambda_1\lambda_2 \ldots \lambda_{n-1}$.

Can also compute this by deleting the i^{th} row & column of L & taking the determinant.

Do example!

Another way to characterize matroids uses circuits.

Theorem 2: Let M be a matroid & $\mathcal{C}$ its collection of circuits. Then $\mathcal{C}$ satisfies:

(C1): $\emptyset \notin \mathcal{C}$

(C2): If $C \in \mathcal{C}$ and $D \subsetneq C$ then $D \notin \mathcal{C}$.

(C3): If C_1 and C_2 are in $\mathcal{C}$ and distinct, and $x \in C_1 \cap C_2$, then $(C_1 \cup C_2) - \{x\}$ contains a member of $\mathcal{C}$.

Conversely, suppose $\mathcal{C}$ is a collection of subsets of E satisfying (C1), (C2), (C3). Let $\mathcal{I}$ be those subsets of E that contain no member of $\mathcal{C}$. Then $(E, \mathcal{I})$ is a matroid having $\mathcal{C}$ as its collection of circuits.

<u>Proof</u> (only the $(\Rightarrow)$)

Let $M = (E, \mathcal{I})$ be a matroid and $\mathcal{C}$ its collection of circuits (min'l dep set).

(C1), ϕ is indep so its not a circuit.

(C2) If $C_1 \subset C_2$ were circuits, C_2 wouldn't be min'l. ✓

(C3) Suppose not, Then $(C_1 \cup C_2) - x$ is indep.

Take $y \in C_1 - C_2$.

Then $C_1 - y$ indep, $C_1 \cup C_2 - x$ indep,

& $|C_1 - y| < |C_1 \cup C_2 - x|$.

$\Rightarrow C_1 \cup a - y$ indep for some $a \in C_2$

Q: $|C_1 \cup a - y| < |C_1 \cup C_2 - x|$?

If so, $C_1 \cup a \cup b - y$ indep for $b \in C_2$

Q: $|C_1 \cup a \cup b - y| < |C_1 \cup C_2 - x|$?

$\vdots$

This only stops with $C_1 \cup C_2 - y$ indep.

But $\underset{\uparrow \text{ dep}}{C_2} \subseteq \underbrace{C_1 \cup C_2}_{\text{indep.}} - y \quad \Rightarrow\Leftarrow$.

New matroids from old: Fix $M = (E, \mathcal{I})$

- If $S \subset E$, the restriction of M to S, denoted $M|_S$, is the matroid w/ indep sets $\{ I \in \mathcal{I},\ I \subset S \}$.

- If $T \subset E$, T indep subset of E, the contraction of M by T, denoted M/T, is matroid on ground set $E - T$ w/ indep sets $\{ I \in \mathcal{I} \mid I \cup T \in \mathcal{I} \}$.

Def: A minor N of matroid M is a matroid that can be obtained from M by sequence of restriction & contraction operations.

Def: A regular matroid is one representable over every field.

Rk: A graphic matroid is representable by a 0-1 matrix & hence is regular.

Given $G = (V, E)$, the cocycle matroid has ground set E, & its bases are complements of spanning trees.

Tutte's Theorem (1958): A regular matroid is graphic iff it does not contain as a minor the cocycle matroid of either K_5 or $K_{3,3}$.

Exercises

1. Use the Matrix-Tree theorem to count the number of spanning trees of the complete graph K_n on n vertices.
(in other words, this is the number of trees on vertices $\{1, 2, \ldots, n\}$)

2. Given a matroid $M = (E, \mathcal{I})$ & a function $w: E \to \mathbb{R}$.
We want to find a basis B of minimum weight, where $wt(B) := \sum_{b \in B} wt(b)$
Prove that the following greedy algorithm works:
   - Start with $J = \emptyset$,
   - Add to J a cheapest element $e \in E$ s.t. $J \cup \{e\} \in \mathcal{I}$.
   - Repeat until you have a basis.

   Rk: Relates to "real-life" problem!
   Given graph $G = (V, E)$ & a weight on each edge, find a minimum weight spanning tree of G.

3. Fill in details of proof of theorem 1