IAS Lecture 1

Matroids: • Introduced by Whitney 1937
• Generalize notion of dependence in linear algebra & graph theory
• Connections to polytopes, optimization, geometry...

Overview of my lectures
1. Examples & def's from vector config's & graphs
2. Bases & circuits of matroid
3. Brief intro to polytopes; matroid polytopes
4. Edges of matroid polytopes; Gelfand Goresky MacPherson Serganova

Example: Consider the matrix

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Let $E = \{1, 2, \ldots, 7\}$ be the column labels of A & let $\mathcal{I}$ be the collection of subsets $I \subset E$ whose corresponding columns are linearly independent over $\mathbb{C}$.

What is $\mathcal{I}$?

Easier to describe the subsets **not** in $\mathcal{I}$:
• Any subset of size > 3
• Any subset containing 7
• Any subset containing 5 and 6
• $\{1,2,4\}$, $\{2,3,5\}$, and $\{2,3,6\}$

All other subsets are in $\mathcal{I}$.
The pair $(E, \mathcal{I})$ is an example of matroid.
E called ground set,
$\mathcal{I}$ called independent sets of matroid.

Here are some properties of $\mathcal{I}$:
(I1) $\phi \in \mathcal{I}$
(I2) $\mathcal{I}$ is hereditary: if $I \in \mathcal{I}$ and $J \subset I$ then $J \in \mathcal{I}$.
(I3) If X and Y in $\mathcal{I}$ and $|X| = |Y|+1$, then $\exists\ x \in X - Y$ s.t. $Y \cup \{x\} \in \mathcal{I}$.

Def: A matroid M is a pair $(E, \mathcal{I})$ consisting of a finite set E and a collection of subsets of E satisfying (I1), (I2), (I3).

Example: What are the possible matroids w/ ground set $E = \{1,2\}$?
$\mathcal{I}$ may be equal to:
$\{\phi\}$, $\{\phi, \{1\}\}$, $\{\phi, \{2\}\}$, $\{\phi, \{1\}, \{2\}\}$, $\{\phi, \{1\}, \{2\}, \{1,2\}\}$.
M_1 M_2 M_3 M_4 M_5
But M_2 and M_3 have the same structure...
they are isomorphic.

Def: Two matroids $M = (E, \mathcal{I})$ and $M' = (E', \mathcal{I}')$ are 'isomorphic if $\exists$ bijection $\phi: E \to E'$ s.t. $I \in \mathcal{I}$ iff $\phi(I) \in \mathcal{I}'$.

Need to show that matrices actually give rise to matroids.

Thm: Let A be a matrix over field $\mathbb{F}$ (eg $\mathbb{R}$). Let E be the set of column labels of A, let $\mathcal{I}$ be the collection of subsets I of E s.t. the corresponding subset of columns of the matrix are linearly independent. Then $(E, \mathcal{I})$ is a matroid.

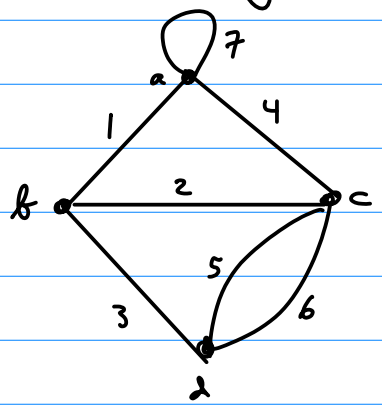
Proof: Clearly (I1) and (I2) are satisfied. Let's check (I3):
Let $X \neq Y$ be linearly indep subsets of E s.t. $|X| = |Y| + 1$.
Let W be vect. space spanned by $X \cup Y$.
Then $\dim(W) \geq |X|$.
Suppose $Y \cup \{x\}$ is lin. dependent $\forall\, x \in X - Y$.
Then Y is a spanning set for W so $\dim(W) \leq |Y|$.
Then $|X| \leq \dim(W) \leq |Y|$ but $|X| = |Y| + 1$. $\Rightarrow\Leftarrow$

The matroid obtained from A in thm is denoted $M[A]$ & is called the <u>vector matroid</u> of A. Such a matroid is called <u>representable</u> or <u>realizable</u> over $\mathbb{F}$.

Q: Is every matroid realizable over some field? (no... can you find counterex?)

Matroids from graphs

Ex:



Consider graph G on vertices $\{a, b, c, d\}$ w/ edges as shown.

Let $E = \{1, 2, \ldots, 7\}$, the edge set of G.
Let $\mathcal{I}$ be the collection of subsets of
(*) E that do not contain all the edges of any <u>cycle</u> of G.

The cycles of G have edge sets:
$\{7\}$, $\{5, 6\}$, $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{2, 3, 6\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$.

Compare $\mathcal{I}$ here to the $\mathcal{I}$ associated to the 3×7 matrix A we saw earlier — same!

<u>Def/Thm</u>: Let G be a finite graph. Let E be set of edges & define $\mathcal{I}$ as in (*). Then $(E, \mathcal{I})$ is a matroid called the <u>cycle matroid</u> $M(G)$. A matroid that arises in this way is called <u>graphic</u>.

To prove the thm, we first define the vertex-edge incidence matrix of a graph $G = (V, E)$.

This is a $|V| \times |E|$ matrix $(A_{v,e})$ w/ rows & columns indexed by V and E, defined by:

$$A_{v,e} = \begin{cases} 1 & \text{if } v \text{ and } e \text{ are incident, i.e. } v \text{ is a vertex of edge } e, \text{ and } e \text{ is not a loop} \\ 0 & \text{otherwise} \end{cases}$$

Ex: For the previous graph, we get

	1	2	3	4	5	6	7
a	1	0	0	1	0	0	0
b	1	1	1	0	0	0	0
c	0	1	0	1	1	1	0
d	0	0	1	0	1	1	0

Pf of Thm: Let G be finite graph & let A be its incidence matrix.

In order to show that $M(G)$ is a matroid, it suffices to show that $M(G) = M[A]$ over the finite field $\mathbb{F}_2 = \{0, 1\}$, since we proved that the vector matroid $M[A]$ is a matroid.

The ground set of both $M(G)$ & $M[A]$ is the edges E of G.

Need to show: For any subset $I \subseteq E$, I contains a cycle of G iff the columns of A corresp to I are dependent.

Suffices to show: C is a cycle of G iff the columns corresp to C are min'l dependent.

Suppose C is cycle. If it's a loop, the corresp column is the 0-vector. ✓

When C is not a loop, each vertex which is met by C is met by exactly 2 edges of C. ∴ the sum of these vectors $\equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ mod 2.
Hence dependent. (Recall: using $\mathbb{F}_2$ here)

Conversely, suppose we have set X of lin. dependent columns. Choose a subset $D \subset X$ which is min'l & still lin. depend; ie. all its proper subsets are indep.

If D is the 0-column, the corresp vector is a loop hence a cycle. ✓

Otherwise, since D is min'l dependent, & we're working over $\mathbb{F}_2$, the sum of the vectors in D is 0. (only kind of dependence relation over $\mathbb{F}_2$).
It follows that D corresponds to a set of edges.

Exercises

1. Show that if $\mathcal{I}$ is a non-empty hereditary set of subsets of a finite set E, then $(E, \mathcal{I})$ is a matroid iff for all $X \subseteq E$, all max'l members of $\{I : I \in \mathcal{I} \text{ and } I \subseteq X\}$ have the same cardinality.

2. (a) Let $E = \{1, 2, 3\}$. Show there are exactly 8 non-isomorphic matroids on E.
   (b) How many non-isomorphic matroids are there on a 4-element set.

3. Use a graph-theoretic argument to show that if G is a graph, $M(G)$ is a matroid.

4. Find an example of a matroid that is not realizable over $\mathbb{R}$.