Exercise 1 Follow the outline below to prove the following "Baby model": every degree n polynomial p with complex coefficient has at least one complex root y.

- let P denote the space of deg n poly and let the root $x \in Y$ (a cx number to begin with);
- consider the subset $Z = \{(p, x) \in P \times Y \mid (p(x) = 0\} = \mathrm{ev}^{-1}(0)$ where

$$\begin{array}{ccc} P \times Y & \xrightarrow{\mathrm{ev}} & \mathbb{C} \\ \downarrow{\scriptstyle \pi} & & \\ P & & \end{array} \qquad\qquad \begin{array}{ccc} (p, x) & \xrightarrow{\mathrm{ev}} & p(x) \\ \downarrow{\scriptstyle \pi} & & \\ p & & \end{array} \tag{1}$$

- Show that $Z \neq \emptyset$ is a smooth, oriented manifold and that $\pi : Z \to P$ is generically a covering map
- arrange that is also a compact manifold Z (hint: this may require replacing P by a another compact model)
- conclude that $\pi : Z \to P$ is a degree n map (hint: look at $\pi_* : H_{top}(Z) \to H_{top}(P)$)
- conclude that the generic polynomial will have n roots, and any polynomial will have at least one root

EC: What changes if we replace $\mathbb{C}$ by $\mathbb{R}$ in the above argument?

Exercise 2 Now consider the situation discussed in the lecture: $P = \mathcal{J}$ the space of almost complex structures on X compatible with ω and $Y = \mathcal{M}aps(C, X)$ the space of (smooth) maps from fixed, smooth domain C to X .

- discuss how to put a manifold structures on these; calculate the tangent spaces $T_f\mathcal{M}aps(C, X)$ and $T_J\mathcal{J}$
- regard the holomorphic map equation $s(f, J) = \overline{\partial}_J f$ as a section s in a bundle E over $P \times Y$ – remember that $\overline{\partial}_J f$ is the anti-complex linear part of df, so it is a (0,1)-form on C;
- the moduli space is $\mathcal{M}(X) = s^{-1}(0)$ where 0 is the zero section of the bundle E;
- calculate the linearization $D_f\xi$ (i.e. the vertical part of the differential of the section s) and show it has the form
$$D_f\xi = \overline{\partial}\xi + \frac{1}{2}\nabla_\xi J \circ df \circ j$$
where ∇J is *some* connection, not necessarily torsion free;

Exercise 3 Show that the space of almost complex structures on X compatible with ω is contractible; discuss how they vary as we vary ω (EC: what happens if J is only tamed?) Show that the Chern classes of TX are symplectic deformation invariants.

Exercise 4 Assume C is smooth, and $f : C \to X$ is holomorphic where X is a closed, 2 dim Riemann surface. For simplicity, you could take $X = \mathbb{CP}^1$;

- Show that f is either constant or else it is a degree d brach cover of X;
- show that up to finite information (e.g. monodromy), a branch cover is determined by the branch points (i.e. the critical values of f); in particular, the complex structure on C is determined by that on X by pullback;
- show that the (cx) dimension of the moduli space (where we allow C to vary) in this case is precisely the number of branch points, counted with multiplicity (Riemann-Hurwitz formula): $b = 2d + 2g - 2$; you can use the fact that the generic branch cover has only ramification index one points (the ramification index k point of the cover f is a point for which the local model is $f(z) = z^k$);
- for the experts only: the branch covers are multiple covers, but are cut transversely in this case (why?). What happens if we fix the complex structure on the domain?

Exercise 5 Assume $X = \mathbb{CP}^2$ and $A = l$ the class of the line. Calculate by hand GW_A in genus zero; how about when $A = 2l$ or even $3l$? (you can use the fact that the standard integrable J is "generic" ie. regular value).

Exercise 6 Consider the product complex structure on $X = S^2 \times T^{2n-2}$ where T^{2n-2} is a torus $(\mathbb{C}^{2n-2}/\Lambda)$;

- describe the moduli space $\mathcal{M}(X)$ of holomorphic maps from genus zero curve to X representing $A = S^2 \times pt$ (hint consider the projection onto one of the two factors)
- prove (or take it as a fact) that the moduli space is compact in this case (for top reasons);
- consider the evaluation map $\mathrm{ev} : \mathcal{M}(X) \to X$. Show that this is a degree 1 map;
- conclude that $GW(pt) = 1$; here $pt \in H^0(X)$ is poincare dual to the class represented by a point in $H_0(X)$;
- conclude that thorough any point $p \in X$ and for any J compatible almost complex structure on X there is at least one holomorphic curve passing though X (this was one step in the proof of Gromov's non squeezing)

Exercise 7 (hard) Show that a J-holomorphic map $f : C \to X$ (with smooth, but possibly disconnected domain) is either simple, or else it has multiply covered components. On these f is either constant, or else it factors as $g \circ \varphi$ where $\varphi : C \to \Sigma$ is a degree > 1 map and $g : \Sigma \to X$ is simple.

Exercise 8 Consider X a symplectic manifold which is homeomorphic to $\mathbb{CP}^2$, and pick J a compatible almost complex structure; Let $\mathcal{M}(X)$ denote the moduli space of J-holo maps from a genus zero curve with 2 marked points to X and which represent $A = [l] \in H^2(X)$ the (positive) generator in homology. Assume $\mathcal{M}(X)$ is nonempty;

- Show that for any two fixed, distinct points $p, q \in X$ there is at most one map in $\mathcal{M}(X)$ passing though both;

- Prove (or take it as a fact) that the moduli space is compact;
- conclude that any f in the moduli space is smooth and embedded (hint: use adjunction formula)
- show that any two distinct f intersect transversally in precisely one point;
- by considering the evaluation map at the two marked points, show that if the moduli space is nonempty, there precisely one f in the moduli space passing though any two distinct points of X;
- use this to conclude that X is diffeomorphic to $\mathbb{CP}^2$;

Exercise 9 Construct (by hand) the (stable map) limit of the following sequences of genus zero maps to $\mathbb{CP}^2$: (a) $f_n(z) = \left[z^2, z, \frac{1}{n}\right]$; (b) $f_n(z) = \left[z(z - \frac{1}{n}), z, \frac{1}{n}\right]$ (c) $f_n(z) = \left[z^2 - \frac{1}{n^2}, z - \frac{1}{n^2}, \frac{1}{n}\right]$; (d) $f_n(z) = [z, z^n, 1]$.
EC: Do the same for the sequence $f_n : \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by $f(z) = \left[z^4 - 1, (z - \frac{1}{n})^4 - 1\right]$;