Symplectic Techniques:

The moduli spaces of Holomorphic Maps

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Brief outline of the analytical set-up

Key ingredients:

- the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ (describes how the domains vary);
- Gromov compactness and the moduli of stable maps $\overline{\mathcal{M}}_{A,g,n}(X)$;
- Fredhom theory (done on each stratum):
  transversality, index and orientations;
- gluing: describes how the strata fit together
  (the normal bundle to each stratum);

Upshot: moduli space is stratified by the top type of the domain,
each nodal stratum is constructed from lower dim moduli spaces

Basic properties of J-holomorphic curves

Assume $f : C \to X$ J-holomorphic, with domain C fixed, smooth;

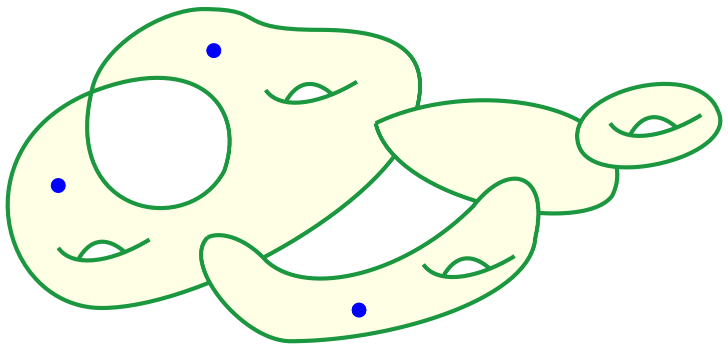
$$\frac{1}{2}(df + J(f) \circ df \circ j) = 0 \qquad \text{elliptic equation;}$$

$$D_f \xi = \overline{\partial}\xi + \frac{1}{2}\nabla_\xi J \circ df \circ j \qquad \text{its linearization in } f;$$

- the energy $E(f) = \frac{1}{2}\int_C |df|^2 dvol_C$ is topological $E(f) = \int_C f^*\omega$;
- f minimizes both energy and area in its homology class;
- singularities of f are 'modeled' on those for actual holo curves;
- elliptic regularity, unique continuation principle;

use Sobolev $W^{k,p}$ norms with $kp > 2$, $p \geq 1$ ($W^{1,2}$ borderline);

The Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves



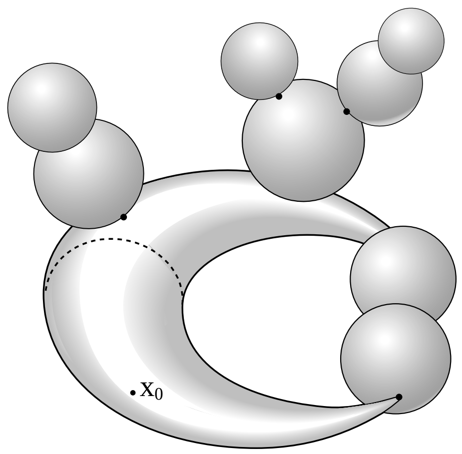
$\overline{\mathcal{M}}_{g,n} =$ moduli space of stable curves C:

- C nodal, 'genus' g, n marked points;
- C stable i.e. Aut$C$ finite;

only equivalence classes up to isomorphism;

- $\overline{\mathcal{M}}_{g,n}$ is a smooth, projective of complex dimension $3g - 3 + n$;
- universal curve $\pi : \overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ whose fiber* at $C \in \overline{\mathcal{M}}_{g,n}$ is C;
- $\overline{\mathcal{M}}_{g,n}$ comes with a natural stratification by the top type of C:
  - top stratum is $\mathcal{M}_{g,n}$ the moduli space of smooth stable curves;
  - the boundary strata are complex codimension ≥ 1;

Gromov Compactness Theorem



Let $f_n : C_n \to X$ be J_n holo, with $J_n \to J$ in C^∞;
Assume f_n has unif. bounded energy and topol;
(i.e. C_n bounded topology and $\omega(A) < K$);
Then, after reparametrization, f_n has a
'Gromov-convergent' sequence where:

- the limit $f : C \to X$ is a stable J-holo map;
- $C_n \to \mathrm{st}(C)$ in $\overline{\mathcal{M}}_{g,n}$ while $f_n \to f$ in C^0 (Hausdorff distance).
- $f_n \to f$ uniformly in C^∞ on compacts away from the nodes of C;

Brief outline:

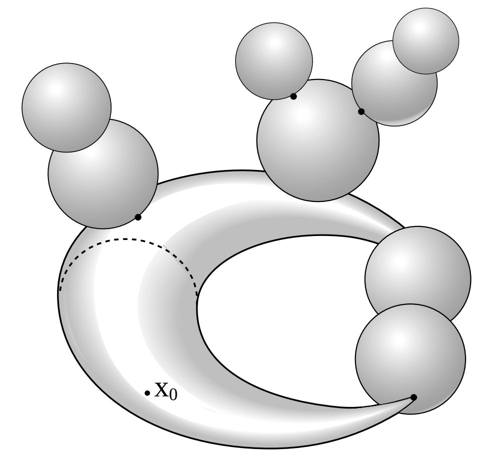
• ($\overline{\mathcal{M}}_{g,n}$ compact) C_n converge to a limit C_0 in $\overline{\mathcal{M}}_{g,n}$;

• bounded energy $\Longrightarrow$ $\exists$ only finitely many (blow-up) points where energy accumulates;

• $f_n \to f_0$ uniformly in C^∞ on compacts away from the blow-up points and the nodes of C_0;

• rescale around each blow-up point p to catch a bubble;

(top/analysis differs if p smooth or node)



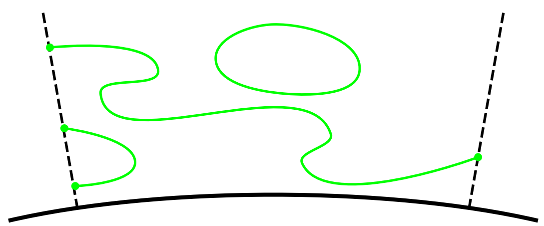
• (remov. sing.) any bounded energy J-holo map extends across punctures;

• (lower energy bd) $E(f) \geq \alpha_X > 0$ for any non-constant J-holo $f : S^2 \to X$;

• (isoperimetric ineq) implies nodes connect (limit is continuous at nodes).

Analytical set-up of the moduli space of stable maps

Work on each stratum where the domain has fixed topology;
assemble them at the end using Gromov topology (and gluing)



$$\begin{array}{ccc} \mathcal{M}_\Sigma(X) & \longrightarrow & \mathcal{U}\mathcal{M}_\Sigma(X) \\ & & \downarrow \pi \\ & & \mathcal{J}(X) \end{array}$$

- transversality: the full linearization $\mathbf{L}_f$ is *onto* (at simple maps),
  so the universal moduli space is smooth* modeled on Ker $\mathbf{L}_f$;
- (Sard-Smale) for generic parameter the moduli space $\mathcal{M}(X)$ is smooth*,
  modeled by Ker L_f at f (and Coker L_f=0);
- Index theory: both the dimension and the orientability of $\mathcal{M}(X)$ are topological!
- strata are codimension ≥ 2 so carries a (virtual) fundamental cycle;