EXERCISES FOR DAY 1

Lecture topics. Recursions, generating functions, rationality

Exercise 1.1. (a) Find the generating function for the three-term Fibonacci sequence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ with initial terms $1, 1, 1$.
(b) Same for $a_n = a_{n-1} + 6a_{n-2}$ with initial terms $0, 1$.

Exercise 1.2. Consider the recursion $2a_n = a_{n-1} - a_{n-2}$. Play around with some different initial terms. Under what conditions is this an integer sequence? Find the generating function.

Exercise 1.3. Find the power series whose coefficient sequence is n^2. Also do n^3. Sketch out how to get arbitrary quasipolynomials.

Exercise 1.4. Show that any eventually constant sequence has rational growth. (Use the explicit example of the sequence $1, 3, 5, 10, 2, 4, 4, 4, 4, 4, 4, \ldots$ as a guide.)

Exercise 1.5. The *cyclotomic polynomials* $\phi_n(x) \in \mathbb{Z}[x]$ are defined as the (unique) irreducible polynomials such that $\phi_n(x) \mid x^n - 1$, but which divide no smaller $x^k - 1$. (In other words, they're the minimal polynomials satisfied by the roots of unity: $\phi_n(x) = \prod (x - {\zeta_n}^k)$ over $k \le n$ relatively prime to n, where $\zeta_n = e^{2\pi i/n}$ is the primitive nth root of unity.) The first few cyclotomic polynomials are

$$\phi_1(x) = x - 1, \quad \phi_2(x) = x + 1, \quad \phi_3(x) = x^2 + x + 1, \quad \phi_4(x) = x^2 + 1$$
$$\phi_5(x) = x^4 + x^3 + x^2 + x + 1, \quad \phi_6(x) = x^2 - x + 1.$$

Note that if p is prime, we have $\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$.

Find a formula for $\phi_{2p}(x)$ (valid for $p \ge 3$).

Exercise 1.6. Find the generating function for the quasipolynomial $q(n) = n + (-1)^n$.

Exercise 1.7. The *Eulerian numbers* $\left\langle {n \atop k} \right\rangle$ are defined by, for fixed n,

$$\sum_m m^n x^m = \frac{\sum_k \left\langle {n \atop k} \right\rangle x^k}{(1-x)^{n+1}}.$$

Find the Eulerian numbers $\left\langle {4 \atop k} \right\rangle$.

Exercise 1.8. Show that when rational functions have poles inside the unit circle, then the associated sequences grow exponentially. Show on the other hand that if there is no pole in the unit circle, then the growth is bounded above by a polynomial.

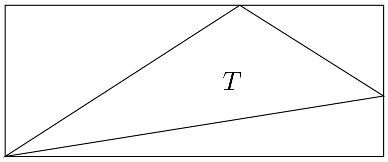
Exercises for Day 2

Lecture topics. Gauss circle problem, Pick's theorem, Ehrhart polynomials

Exercise 2.1. Compute $G_{\mathbb{D}}(n)$ for $n = 10$ and compare it to πn^2 as follows: If $G_{\mathbb{D}}(n) = \pi n^2 + E(n)$ and $P(n) = \log_n |E(n)|$, find $E(10)$ and $P(10)$.

Exercise 2.2. Verify Pick's Theorem for the trapezoid in the plane with vertices at $(0,0), (0,8), (2,5),$ and $(6,5)$.

Exercise 2.3. (a) Prove Pick's Theorem for lattice triangles, with the help of this suggestive picture:



(b) Consider a lattice polygon P and a lattice triangle T, with one edge in common. Assume that each of P and T separately satisfies the conclusion of Pick's theorem ($A = i + \frac{1}{2}b - 1$). Show that it also holds for the polygon P' obtained by fusing T with P.
(c) Deduce Pick's theorem.

Exercise 2.4. Consider the polygon Q with rational coordinates $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right), (0,1),$ and $(0,-1)$. Find $G_Q(n)$ and express it as a quasi-polynomial. Repeat with the polygon Q' with coordinates $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right), \left(0, \frac{2}{3}\right),$ and $\left(0, -\frac{2}{3}\right)$.

Exercise 2.5. Compute the Ehrhart polynomials $G_\Omega(n)$ for the following polyhedra:

- the interval $[-1,1]$ in $\mathbb{R}$
- the square with vertices at $\pm(1,0), \pm(0,1)$ in $\mathbb{R}^2$
- the hexagon with vertices at $\pm(1,0), \pm(0,1)$, and $\pm(1,1)$ in $\mathbb{R}^2$
- the octagon with vertices at $(\pm 2, \pm 1)$ and $(\pm 1, \pm 2)$ in $\mathbb{R}^2$
- the octahedron with vertices at $\pm(1,0,0), \pm(0,1,0)$, and $\pm(0,0,1)$ in $\mathbb{R}^3$

Exercise 2.6. Give the generating functions $\sum_n G_\Omega(n) \cdot x^n$ for these expressions, as rational functions.

EXERCISES FOR DAY 3

Lecture topics. Groups—abelian, hyperbolic, nilpotent—Cayley graphs, decision problems

Exercise 3.1. Draw the Cayley graph for $\mathbb{Z}$, but instead of the standard generators, use the generators $\pm\{2,3\}$. What are β_n and σ_n for this generating set?

Exercise 3.2. For the Heisenberg group $H(\mathbb{Z})$ with standard generators $\{a,b\}^{\pm}$, use matrices or a normal form to work out σ_n for $n \le 4$. Draw the corresponding portion of the Cayley graph.

Exercise 3.3. Find all the relations between the elementary generators of the 5-parameter Heisenberg group. What is the minimal relator set R needed to write $H_5(\mathbb{Z}) = N_{2,4}/R$?

Exercise 3.4. In one of the exercises from yesterday, you computed the Ehrhart polynomials $G_\Omega(n)$ for the following polyhedra:

- the interval with endpoints ± 1 in $\mathbb{R}$
- the square with vertices at $\pm(1,0), \pm(0,1)$ in $\mathbb{R}^2$
- the hexagon with vertices at $\pm(1,0), \pm(0,1)$, and $\pm(1,1)$ in $\mathbb{R}^2$
- the octagon with vertices at $(\pm 2, \pm 1)$ and $(\pm 1, \pm 2)$ in $\mathbb{R}^2$
- the octahedron with vertices at $\pm(1,0,0), \pm(0,1,0)$, and $\pm(0,0,1)$ in $\mathbb{R}^3$

Now consider the corresponding generating sets for the free abelian groups $\mathbb{Z}^d$ $(d = 1,2,3)$. In which cases are the group growth functions equal to the Ehrhart polynomials?

Exercise 3.5. Let (X, d_X) and (Y, d_Y) be metric spaces. A surjective map $f : X \longrightarrow Y$ is a bi-Lipschitz equivalence if there exists a constant $K > 0$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1,x_2) \le d_Y(f(x_1), f(x_2)) \le K d_X(x_1,x_2).$$

Let G be a finitely generated group, and let S, S' be two generating sets for G whose associated word metrics are d_S, $d_{S'}$, respectively. Show that (G, d_S) and $(G, d_{S'})$ are bi-Lipschitz equivalent.